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ARCTANGENT IDENTITIES INVOLVING THE JACOBSTHAL AND JACOBSTHAL-LUCAS NUMBERS

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Abstract. This study presents novel arctangent identities that establish connections between the Jacobsthal and Jacobsthal-Lucas numbers. These findings contribute to the understanding of the interplay between trigonometric functions and number theory, particularly in relation to well-known mathematical sequences and constants.

1. Introduction

The Jacobsthal and Jacobsthal-Lucas numbers have garnered significant attention in the field of number theory and combinatorics due to their fascinating properties and wide-ranging applications. These sequences, denoted respectively as A001045 and A014551 in the OEIS database [24], are defined by simple recurrence relations yet exhibit deep connections to various mathematical concepts, including graph theory, cryptography, and algorithmic design. The Jacobsthal sequence $\{J_n\}_{n\geq 0}$ is defined by the initial conditions $J_0 = 0$ and $J_1 = 1$, along with the recurrence relation:

$$J_{n+2} = J_{n+1} + 2J_n, \qquad n \geq 0. \tag{1}$$

The first few terms of this sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341$. Similarly, the Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ is defined by the initial values $j_0 = 2$ and $j_1 = 1$, and follows the recurrence:

$$j_{n+2} = j_{n+1} + 2j_n, \qquad n \geq 0. \tag{2}$$

with its initial terms being $2, 1, 5, 7, 17, 31, 65, 127, 257, 511$. These sequences have been extensively studied in the literature, as seen in works such as [2, 4, 5, 10, 15, 17, 3, 25, 21, 22, 23, 18, 7, 8, 9, 12, 14, 16, 20, 19].

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The Jacobsthal numbers, in particular, have found applications in various fields. For instance, they appear in the study of tiling problems, where they count the number of ways to tile a $3 \times n$ rectangle with 1×1 and 2×2 squares. They also play a role in the analysis of certain algorithms, particularly those involving divide-and-conquer strategies, where the recurrence relations of Jacobsthal numbers often emerge naturally. In cryptography, Jacobsthal numbers have been used in the design of pseudorandom number generators due to their rapid growth and combinatorial properties. Similarly, Jacobsthal-Lucas numbers, which are closely related to Jacobsthal numbers, have applications in the study of graph colorings and network design, where their recurrence relations help model complex structures.

Both sequences are governed by the characteristic equation:

$$x^2 - x - 2 = 0,$$

whose roots are $x_1 = 2$ and $x_2 = -1$. These roots satisfy the relationships:

$$x_1 + x_2 = 1, \quad x_1 - x_2 = 3, \quad \text{and} \quad x_1 x_2 = -2.$$

The Binet formulas for the Jacobsthal and Jacobsthal-Lucas sequences are given by:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{3}$$

and

$$j_n = 2^n + (-1)^n. \tag{4}$$

These formulas, along with various interrelationships such as [13, 6]:

$$2^n = J_{n+1} + J_n$$

$$(-1)^n = 2J_{n-1} - J_n$$

$$J_n j_n = J_{2n}$$

$$j_n^2 + 9J_n^2 = 2j_{2n}$$

$$J_n + j_n = 2J_{n+1}$$

$$3J_n + j_n = 2^{n+1}$$

$$j_n = J_{n+1} + 2J_{n-1}$$

$$9J_n = j_{n+1} + 2j_{n-1}$$

$$J_m j_n + J_n j_m = 2J_{m+n}$$

$$j_m j_n + 9J_m J_n = 2j_{m+n}$$

$$j_{n+1} + j_n = 3.2^n$$

$$j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) = 3(-1)^{n+1}$$

highlight the intricate connections between these sequences and their combinatorial properties.

In this study, we explore the interplay between the Jacobsthal numbers and trigonometric functions, particularly focusing on arctangent identities. Drawing inspiration from the works of Adegoke [1] and Guo and Chu [11], who derived arctangent identities involving the golden ratio, Fibonacci numbers, and Lucas numbers, we aim to establish novel identities connecting the roots of the Jacobsthal characteristic equation (-1 and 2) with the Jacobsthal and Jacobsthal-Lucas sequences. Additionally, we employ fundamental trigonometric identities such as:

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right), \quad xy < 1 \tag{5}$$

$$\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{1+xy}\right), \quad xy > -1 \tag{6}$$

$$\arctan\left(\frac{1}{x}\right) + \arctan(x) = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases} \tag{7}$$

to derive new results that extend the existing literature on arctangent series and their applications in number theory.

Our work not only contributes to the theoretical understanding of Jacobsthal numbers but also opens new avenues for exploring BBP-type formulas and their connections to mathematical constants, as demonstrated in the context of the golden ratio by Adegoke [1]. By combining telescoping methods with Cassini-like formulae, as seen in Guo and Chu [11], we aim to present closed-form evaluations of arctangent series involving Jacobsthal and Jacobsthal-Lucas numbers, thereby enriching the growing body of research on these sequences.

2. The reciprocal Jacobsthal and reciprocal Jacobsthal-Lucas numbers in arctangent formulae

Theorem 2.1. *For positive integers m, we have*

$$\arctan(2^{m-\frac{1}{2}}) = 2\arctan(1) - \frac{1}{2}\arctan\left(\frac{2^{m+\frac{1}{2}}}{j_{2m-1}}\right). \tag{8}$$

Proof. Taking $x = \dfrac{1}{2^{m-\frac{1}{2}}} = y$ in Eq. (5) and using Eq. (4), we arrive

$$\begin{aligned} 2\arctan\left(\frac{1}{2^{m-\frac{1}{2}}}\right) &= \arctan\left(\frac{\frac{2}{2^{m-\frac{1}{2}}}}{\frac{2^{2m-1}-1}{2^{2m-1}}}\right) \\ &= \arctan\left(\frac{2^{m+\frac{1}{2}}}{2^{2m-1}-1}\right) \\ &= \arctan\left(\frac{2^{m+\frac{1}{2}}}{j_{2m-1}}\right) \end{aligned}$$

Therefore, combining the above result with Eq. (7), the proof is concluded. □

We can illustrate the Eq. (8) of the theorem above with an example as follows: For $m = 1$, we have

$$\arctan\left(2^{\frac{1}{2}}\right) + \arctan\left(2^{\frac{3}{2}}\right) = \pi - \arctan\left(2^{\frac{1}{2}}\right).$$

Taking the tangent of both sides of this equality, we obtain

$$\tan\left(\arctan\left(2^{\frac{1}{2}}\right) + \arctan\left(2^{\frac{3}{2}}\right)\right) = \tan\left(\pi - \arctan\left(2^{\frac{1}{2}}\right)\right).$$

Therefore, applying the tangent addition formula, we reach

$$\begin{aligned}\frac{\tan(\arctan(2^{\frac{1}{2}})) + \tan(\arctan(2^{\frac{3}{2}}))}{1 - \tan(\arctan(2^{\frac{1}{2}}))\tan(\arctan(2^{\frac{3}{2}}))} &= \frac{\tan(\pi) - \tan(\arctan(2^{\frac{1}{2}}))}{1 + \tan(\pi)\tan(\arctan(2^{\frac{1}{2}}))} \\ \frac{2^{\frac{1}{2}} + 2^{\frac{3}{2}}}{1 - 4} &= \frac{0 - 2^{\frac{1}{2}}}{1 + 0} \\ -2^{\frac{1}{2}} &= -2^{\frac{1}{2}}.\end{aligned}$$

Similar methods are used to verify the other identities for small positive values of m, and their verification is left to the reader.

Lemma 2.1. *For positive integers k, we obtain*

$$4\arctan(1) - 2\arctan(2^k) = \arctan\left(\frac{J_{k+2} + J_{k+1}}{j_{2k}}\right). \tag{9}$$

Proof. The proof follows by taking $m = k + \frac{1}{2}$ in Eq. (8) in Theorem 2.1. □

Theorem 2.2. *For positive integers n, the following identities are valid:*

i . $3\arctan(2^{-2n}) + \arctan(2^{2n}) = 2\arctan\left(\frac{j_{2n}}{3J_{2n}}\right)$,

ii . $3\arctan(2^{-(2n+1)}) + \arctan(2^{2n+1}) = 2\arctan\left(\frac{3J_{2n+1}}{j_{2n+1}}\right)$,

iii . $\arctan(2^{2n}) - \arctan(2^{-2n}) = 2\arctan\left(\frac{3J_{2n}}{j_{2n}}\right)$,

iv . $\arctan(2^{2n+1}) - \arctan(2^{-(2n+1)}) = 2\arctan\left(\frac{j_{2n+1}}{3J_{2n+1}}\right)$.

Proof. i. Taking $x = \dfrac{1}{2^{2n}}$ and $y = 1$ in Eq. (5) and using Eqs. (7), (3) and (4), we arrive

$$\begin{aligned}\arctan(2^{-2n}) + \arctan(1) &= \arctan\left(\frac{2^{-2n} + 1}{1 - 2^{-2n}}\right) \\ \arctan(2^{-2n}) + \frac{\pi}{4} &= \arctan\left(\frac{2^{2n} + 1}{2^{2n} - 1}\right) \\ 2\arctan(2^{-2n}) + \frac{\pi}{2} &= 2\arctan\left(\frac{2^{2n} + (-1)^{2n}}{2^{2n} - (-1)^{2n}}\right) \\ 3\arctan(2^{-2n}) + \arctan(2^{2n}) &= 2\arctan\left(\frac{j_{2n}}{3J_{2n}}\right).\end{aligned}$$

ii. Taking $x = \dfrac{1}{2^{2n+1}}$ and $y = -1$ in Eq. (6) and using Eq. (7), (3) and (4), we arrive

$$\begin{aligned}
\arctan(2^{-(2n+1)}) - \arctan(-1) &= \arctan\left(\frac{2^{-(2n+1)}+1}{1-2^{-(2n+1)}}\right)\\
\arctan(2^{-(2n+1)}) + \frac{\pi}{4} &= \arctan\left(\frac{2^{(2n+1)}+1}{2^{(2n+1)}-1}\right)\\
2\arctan(2^{-(2n+1)}) + \frac{\pi}{2} &= 2\arctan\left(\frac{2^{2n+1}-(-1)^{2n+1}}{2^{2n+1}+(-1)^{2n+1}}\right)\\
3\arctan(2^{-(2n+1)}) + \arctan(2^{2n+1}) &= 2\arctan\left(\frac{3J_{2n+1}}{j_{2n+1}}\right).
\end{aligned}$$

iii. It can be proven similarly to the proof of i.
iv. It can be proven similarly to the proof of ii.

□

Theorem 2.3. *For non-zero integers m, the following identities are valid:*

i.

$$\arctan\left(2^{m+\frac{1}{2}}\right) = 2\arctan(1) + \frac{1}{2}\arctan\left(\frac{2^{m-\frac{1}{2}}}{j_{2m}}\right) - \frac{1}{2}\arctan\left(\frac{3.2^{m-\frac{1}{2}}}{J_{2m}}\right), \quad (10)$$

ii.

$$\arctan\left(2^{m-\frac{1}{2}}\right) = 2\arctan(1) - \frac{1}{2}\arctan\left(\frac{2^{m-\frac{1}{2}}}{j_{2m}}\right) - \frac{1}{2}\arctan\left(\frac{3.2^{m-\frac{1}{2}}}{J_{2m}}\right). \quad (11)$$

Proof. By selecting $x = \frac{1}{2^{m+\frac{1}{2}}}$ and $x = \frac{1}{2^{m-\frac{1}{2}}}$ in Eq. (5), it is simple to prove that

$$\begin{aligned}
\arctan\left(\frac{1}{2^{m+\frac{1}{2}}}\right) + \arctan\left(\frac{1}{2^{m-\frac{1}{2}}}\right) &= \arctan\left(\frac{\frac{1}{2^{m+\frac{1}{2}}}+\frac{1}{2^{m-\frac{1}{2}}}}{1-\frac{1}{2^{2m}}}\right)\\
&= \arctan\left(\frac{2^{m-\frac{1}{2}}+2^{m+\frac{1}{2}}}{2^{2m}-1}\right)\\
&= \arctan\left(\frac{3.2^{m-\frac{1}{2}}}{J_{2m}}\right).
\end{aligned}$$

Therefore, using Eq. (7, we obtain

$$\arctan\left(2^{m+\frac{1}{2}}\right) + \arctan\left(2^{m-\frac{1}{2}}\right) = \pi - \arctan\left(\frac{3.2^{m-\frac{1}{2}}}{J_{2m}}\right). \quad (12)$$

By selecting $x = \frac{1}{2^{m+\frac{1}{2}}}$ and $x = \frac{1}{2^{m-\frac{1}{2}}}$ in Eq. (6) and using Eq. (7, we get

$$\arctan\left(2^{m+\frac{1}{2}}\right) - \arctan\left(2^{m-\frac{1}{2}}\right) = \arctan\left(\frac{2^{m-\frac{1}{2}}}{j_{2m}}\right). \quad (13)$$

Eq. (10) is obtained by adding Eqs. (12) and (13), while Eq. (11) is obtained by subtracting Eq. (13) from Eq. (12). □

Lemma 2.2. *For non-zero integers n, the following identities are valid:*

i.

$$\arctan(2^{2k+1}) = 2\arctan(1) + \frac{1}{2}\arctan\left(\frac{j_{2k+2} - j_{2k}}{j_{4k+1}}\right) - \frac{1}{2}\arctan\left(\frac{3(J_{2k+1} + J_{2k})}{J_{4k+1}}\right), \tag{14}$$

ii.

$$\arctan(2^{2k-1}) = 2\arctan(1) - \frac{1}{2}\arctan\left(\frac{j_{2k+1} - j_{2k-1}}{j_{4k-1}}\right) - \frac{1}{2}\arctan\left(\frac{3(J_{2k} + J_{2k-1})}{J_{4k-1}}\right). \tag{15}$$

Proof. The proof follows by taking $m = 2k + \frac{1}{2}$ and $m = 2k - \frac{1}{2}$ in Eqs. (10) and (11), respectively. □

Remark 1. *Observe that the telescoping summation provided by Eq. (13) may be used to demonstrate that*

$$\arctan\left(2^{n+\frac{1}{2}}\right) = \arctan\left(2^{-\frac{1}{2}}\right) + \sum_{m=1}^{n}\arctan\left(\frac{2^{m-\frac{1}{2}}}{j_{2m}}\right) \tag{16}$$

which can be written as

$$\arctan\left(2^{n+\frac{1}{2}}\right) = \arctan(1) + \frac{1}{2}\arctan\left(\frac{1}{2^{\frac{3}{2}}}\right) + \sum_{m=1}^{n}\arctan\left(\frac{2^{m-\frac{1}{2}}}{j_{2m}}\right) \tag{17}$$

by using

$$\arctan\left(2^{\frac{1}{2}}\right) = \arctan(1) + \frac{1}{2}\arctan\left(\frac{1}{2^{\frac{3}{2}}}\right) \quad \textit{(from } n = 1 \textit{ in Theorem 2.1)} \tag{18}$$

Theorem 2.4. *For non-negative integers n, the following identities are valid:*

i.

$$\arctan\left(2^{2n-\frac{1}{2}}\right) = 3\arctan(1) - \frac{1}{2}\arctan\left(\frac{1}{2^{\frac{3}{2}}}\right) - \arctan\left(\frac{2^{\frac{1}{2}}J_{2n-1}}{J_{2n}}\right), \tag{19}$$

ii.

$$\arctan\left(2^{2n-\frac{3}{2}}\right) = 3\arctan(1) - \frac{1}{2}\arctan\left(\frac{1}{2^{\frac{3}{2}}}\right) - \arctan\left(\frac{2^{\frac{1}{2}}J_{2n-2}}{J_{2n-1}}\right). \tag{20}$$

Proof. By selecting $x = \frac{1}{2^{\frac{1}{2}}}$ and $y = \frac{2^{\frac{1}{2}}J_{m-1}}{J_m}$ in Eq. (5), we obtain

$$\arctan\left(\frac{1}{2^{\frac{1}{2}}}\right) - \arctan\left(\frac{2^{\frac{1}{2}}J_{m-1}}{J_m}\right) = (-1)^{m-1}\arctan\left(\frac{1}{2^{m-\frac{1}{2}}}\right).$$

From $k = 1$ in Theorem 2.1, and using Eq. (7), we have

$$(-1)^m\arctan\left(\frac{1}{2^{m-\frac{1}{2}}}\right) = (2(-1)^m + 1)\arctan(1) - \frac{1}{2}\arctan\left(\frac{1}{2^{\frac{3}{2}}}\right) - \arctan\left(\frac{2^{\frac{1}{2}}J_{m-1}}{J_m}\right). \tag{21}$$

Setting $m = 2n$ in Eq. (21), we get identity (19), while $m = 2n - 1$ in Eq. (21) gives identity (20). □

Theorem 2.5. *For non-negative integers n, the following identities are valid:*

i.

$$\arctan\left(2^{2n-\frac{1}{2}}\right) = \frac{1}{2}\arctan\left(2^{\frac{3}{2}}\right) + \arctan\left(\frac{J_{2n}}{2^{\frac{1}{2}}J_{2n-1}}\right), \tag{22}$$

ii.

$$\arctan\left(2^{2n-\frac{3}{2}}\right) = -\frac{1}{2}\arctan\left(2^{\frac{3}{2}}\right) - \arctan\left(\frac{J_{2n-1}}{2^{\frac{1}{2}}J_{2n-2}}\right). \tag{23}$$

Proof. By selecting $x = \frac{1}{2^{\frac{1}{2}}}$ and $x = \frac{J_m}{2^{\frac{1}{2}}J_{m-1}}$ in Eq. (5), we obtain

$$\arctan\left(\frac{1}{2^{\frac{1}{2}}}\right) + \arctan\left(\frac{J_m}{2^{\frac{1}{2}}J_{m-1}}\right) = (-1)^m \arctan\left(2^{m-\frac{1}{2}}\right).$$

From $m = 1$ in Theorem 2.1, and using Eq. (7), we have

$$\frac{1}{2}\arctan\left(2^{\frac{3}{2}}\right) + \arctan\left(\frac{J_m}{2^{\frac{1}{2}}J_{m-1}}\right) = (-1)^m \arctan\left(2^{m-\frac{1}{2}}\right). \tag{24}$$

Setting $m = 2n$ in Eq. (24), we get identity (22), while $m = 2n - 1$ in Eq. (24) gives identity (23). □

Theorem 2.6. *For $n \in \mathbb{N}$, we have*

$$2\arctan\left(\frac{1}{2^k}\right) = \arctan\left(\frac{2^{k+1}}{3J_{2k}}\right). \tag{25}$$

Proof. The selection of $x = \dfrac{1}{2^k} = y$ in Eqs. (3) and (5) leads to Eq. (25). □

Theorem 2.7. *For non-zero integers k, the following identities are valid:*

i.

$$2\arctan\left(\frac{1}{2^k}\right) = \arctan\left(\frac{J_{k+1}+J_k}{3J_{2k+1}}\right) + \arctan\left(\frac{j_{k+1}+j_k}{j_{2k+1}}\right), \tag{26}$$

ii.

$$2\arctan\left(\frac{1}{2^{k+1}}\right) = \arctan\left(\frac{j_{k+1}+j_k}{j_{2k+1}}\right) - \arctan\left(\frac{J_{k+1}+J_k}{3J_{2k+1}}\right). \tag{27}$$

Proof. By selecting $x = \frac{1}{2^k}$ and $x = \frac{1}{2^{k+2}}$ in Eq. (5), it is simple to prove that

$$\arctan\left(\frac{1}{2^k}\right) - \arctan\left(\frac{1}{2^{k+1}}\right) = \arctan\left(\frac{32^k}{2^{2k+1}-1}\right). \tag{28}$$

Similarly, selecting $x = \frac{1}{2^k}$ and $x = \frac{1}{2^{k+1}}$ in Eq. (6), we have

$$\arctan\left(\frac{1}{2^k}\right) - \arctan\left(\frac{1}{2^{k+1}}\right) = \arctan\left(\frac{2^k}{2^{2k+1}+1}\right). \tag{29}$$

Eq. (26) is obtained by adding Eqs. (28) and (29), while Eq. (27) is obtained by subtracting Eq. (29) from Eq. (28). □

Remark 2. *Utilizing the telescoping summation that is requested by Eq. (29), we get*

$$\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{2^{n+1}}\right) = \sum_{k=1}^{n}\arctan\left(\frac{J_{k+1}+J_k}{J_{2k+1}}\right).$$

We get the formula

$$\arctan\left(\frac{1}{2}\right)=\sum_{k=1}^{\infty}\arctan\left(\frac{J_{k+1}+J_k}{J_{2k+1}}\right)$$

by taking limit $n\to\infty$.

Theorem 2.8. *For non-zero integers n, the following identities are valid:*

i.

$$\arctan\left(\frac{3(J_{2n-1}+J_{2n-2})}{J_{4n-3}}\right)=\arctan\left(\frac{j_{2n}-j_{2n-2}}{j_{4n-3}}\right)+\arctan\left(\frac{j_{2n+1}-j_{2n-1}}{j_{4n-1}}\right)+\arctan\left(\frac{3(J_{2n}+J_{2n-1})}{J_{4n-1}}\right),$$

ii.

$$\arctan\left(\frac{J_{2n+1}+J_{2n}}{j_{4n-2}}\right)=\arctan\left(\frac{j_{2n+1}-j_{2n-1}}{j_{4n-1}}\right)+\arctan\left(\frac{3(J_{2n}+J_{2n-1}}{J_{4n-1})}\right),$$

iii.

$$\arctan\left(\frac{3(J_{2n+1}+J_{2n})}{J_{4n+1)}}\right)=\arctan\left(\frac{J_{2n+3}+J_{2n+2}}{J_{4n+2}}\right)+\arctan\left(\frac{j_{2n+2}-j_{2n}}{j_{4n+1}}\right),$$

iv.

$$\arctan\left(\frac{j_{n+1}+j_n}{j_{2n+1}}\right)=\arctan\left(\frac{J_{n+1}+J_n}{3J_{2n+3}}\right)+\arctan\left(\frac{j_{n+2}+j_{n+1}}{j_{2n+3}}\right)+\arctan\left(\frac{J_{n+1}+J_n}{3J_{2n+1}}\right).$$

Proof. *i.* By setting $k=n-1$ in Eq. (14) and $k=n$ in Eq. (15), it may be proven.

ii. By setting $k=2n-1$ in Eq. (9) and $k=n$ in Eq. (15), it may be proven.

iii. By setting $k=2n+1$ in Eq. (9) and $k=n$ in Eq. (14), it may be proven.

iv. By setting $k=n+1$ in Eq. (26) and $k=n$ in Eq. (27), it may be proven. □

3. Conclusion

In this study, we have established novel arctangent identities that connect the Jacobsthal and Jacobsthal-Lucas numbers. By leveraging telescoping techniques, we derived closed-form evaluations of sums involving the products of two arctangent functions. These results extend existing arctangent identities related to Jacobsthal and Jacobsthal-Lucas numbers, further enriching the interplay between number theory and trigonometric functions.

These identities not only contribute to theoretical advancements but also hold potential applications in mathematical analysis and special function theory. Future research may explore further generalizations to other recurrence sequences and investigate potential applications in analytical number theory and mathematical physics.

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