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ADVANCED SIMPLE AND DOUBLE INTEGRAL INEQUALITIES WITH THREE-PARAMETER RATIO-MINIMUM KERNELS

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ABSTRACT. Integral inequalities play a crucial role in various areas of mathematics, both in theoretical analysis and practical applications. The discovery of new forms of such inequalities remains an important and ongoing area of research. This article is a contribution in this sense. We present new integral inequalities involving three-parameter ratio-minimum weight functions or kernel functions. In particular, we establish simple integral inequalities of the weighted Hölder type and double integral inequalities of the Hardy-Hilbert type. The arctangent function plays a crucial role in defining the upper bounds. These results extend the classical inequalities by incorporating additional parameters, thereby increasing their flexibility and applicability. Detailed proofs are provided to ensure clarity and facilitate further research in this area.

1. INTRODUCTION

Integral inequalities are one of the most useful tools in mathematical analysis. Classically, they provide bounds that are essential for solving various integral-type problems. Famous simple and double integral inequalities, such as the Hölder, Hardy or Hilbert integral inequalities, have been extensively studied. They are used mainly because of their wide applicability and deep connections to functional spaces and operator theory. We refer the reader to the books [7, 3, 15, 2, 5]. In recent years, there has been a growing interest in generalizing these classical inequalities by introducing additional parameters and auxiliary functions. We refer to the survey [4] and the books [16, 17] for a focus on double integral inequalities. In this article, we propose new simple and double integral inequalities using three-parameter ratio-minimum weight functions or kernel functions. These parametric functions offer greater flexibility and precision in determining upper bounds. By adjusting the parameters, we can obtain more accurate and general results. This approach thus improves the adaptability and sharpness of bounds in various applications, especially in

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functional analysis, partial differential equations, optimization and mathematical physics. Before proceeding, a brief overview of the topic is proposed in the subsection below.

1.1. **Overview.** Our main motivation stems from a double integral inequality of the Hardy-Hilbert type established by W.T. Sulaiman in 2008. It is referred to as [11, Theorem 2]. The double integral under consideration is defined with a kernel function of the one-parameter ratio-minimum type, given by

$$\mathcal{K}(x,y) = \frac{1}{[x+y+\min(x,y)]^{\upsilon}},$$

where υ is the parameter. It is clearly homogeneous of degree $-\upsilon$, which means that, for any $\lambda > 0$, we have $\mathcal{K}(\lambda x, \lambda y) = \lambda^{-\upsilon}\mathcal{K}(x,y)$. A formal statement of [11, Theorem 2] is given below, followed by a discussion. Let $p > 1$, $q = p/(p-1)$ and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions. Then the double integral inequalities below apply, distinguishing the cases $\upsilon > 1$, $\upsilon = 1$ and $\upsilon \in (0,1)$.

For any $\upsilon > 1$, we have

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{[x+y+\min(x,y)]^{\upsilon}} f(x)g(y)dxdy \\ &\leq \frac{1}{2(\upsilon-1)}\left(1+3^{1-\upsilon}\right)\left[\int_0^{+\infty} x^{1-\upsilon} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{1-\upsilon} g^q(y)dy\right]^{1/q},\end{aligned}$$

where the integrals on the right-hand side must converge. This inequality informs on how the parameter υ affects the constant factor, i.e., $(1+3^{1-\upsilon})/[2(\upsilon-1)]$, and the weight function of the integral norms of f and g, i.e., $x^{1-\upsilon}$.

For $\upsilon = 1$, we have

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{x+y+\min(x,y)} f(x)g(y)dxdy = \iint_0^{+\infty} \frac{1}{[x+y+\min(x,y)]^{\upsilon}} f(x)g(y)dxdy \\ &\leq 2\sqrt{2}\arctan\left[\sqrt{2}\right]\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q},\end{aligned}$$

where the integrals on the right-hand side must converge. It is interesting to note that the arctangent function is naturally contained in the constant factor. We mention the following approximation for this constant: $2\sqrt{2}\arctan\left[\sqrt{2}\right] \approx 2.70204$.

For any $\upsilon \in (0,1)$, we have

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{[x+y+\min(x,y)]^{\upsilon}} f(x)g(y)dxdy \\ &\leq \frac{1}{2^{\upsilon/2}}\left[B\left(\frac{\upsilon}{2},\frac{\upsilon}{2}\right) + \int_{1/2}^{2} \frac{t^{\upsilon/2-1}}{(1+t)^{\upsilon}}dt\right] \times \\ &\left[\int_0^{+\infty} x^{(1-\upsilon/2)p-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(1-\upsilon/2)q-1} g^q(y)dy\right]^{1/q},\end{aligned}$$

where $B(a,b)$ is the beta function at $a, b > 0$ defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, and the integrals on the right-hand side must converge. In this case, a mathematical error in the original formulation in [11, Theorem 2] has been corrected: the constant factor 1 has been replaced exactly by $1/2^{\upsilon/2}$. The complexity of this constant factor, with a non-closed integral term, is a drawback for further analysis. However, the following applies:

$$\int_{1/2}^{2} \frac{t^{\upsilon/2-1}}{(1+t)^{\upsilon}} dt \leq \frac{1}{(1+1/2)^{\upsilon}} \int_{1/2}^{2} t^{\upsilon/2-1} dt = \frac{2^{\upsilon+1}}{3^{\upsilon}\upsilon} \left(2^{\upsilon/2} - 2^{-\upsilon/2}\right).$$

With this result, we can therefore deal with the problem of complexity by replacing the integral term with an upper bound, thus simplifying the analysis at the expense of some precision. The presence of the υ parameter also provides an interesting dimension of flexibility. However, it should be noted that the minimum of the variables in the kernel function cannot be modulated. Consequently, this kernel function does not recover the classical Hardy-Hilbert kernel function defined by

$$\mathcal{K}_{\star}(x,y) = \frac{1}{x+y}.$$

In addition, we observe that the variables x and y contribute symmetrically within the minimum. This symmetry suggests a natural direction for generalization, considering a weighted or modulated minimum, such as $\min(x, \omega y)$, where ω acts as an adjustable parameter, or exploring more sophisticated kernel function structures. This perspective opens the door to more flexible kernel structures and has motivated further research, as outlined in the next subsection.

We end this overview with the following key references for double integral inequalities of the Hardy-Hilbert type involving the minimum or maximum of variables: [14, 8, 1, 9, 12, 13, 10]. They make significant contributions to the field, but none of them take our research direction into account.

1.2. **Contributions.** Motivated by [11, Theorem 2] and the above considerations, this article develops new integral inequalities for both simple and double integrals. It is divided into three main parts. The first part is devoted to four lemmas that provide integral formulas. They form the analytical basis from which our inequalities are constructed. In the second part, simple integral inequalities of the weighted Hölder type are presented and proved. These results modify the classical Hölder integral inequality by incorporating three-parameters weight functions. For example, one of the weight functions considered is of the form

$$\mathcal{W}(x) = \frac{1}{1+\theta x + \tau \min(x, \omega)},$$

where τ, θ and ω are the parameters. Note that the parameters τ and ω modulate the minimum term in two different ways. Unlike the framework of [11, Theorem 2], we do not introduce an exponent parameter υ to maintain analytical tractability. In total, four different weighted Hölder-type integral inequalities are established. In the last part, we derive double integral inequalities of the Hardy-Hilbert type. These results extend the classical Hardy-Hilbert integral inequality by introducing three-parameter-kernel functions. One such kernel function is given by

$$\mathcal{K}_{\diamond}(x,y) = \frac{1}{x+\theta y + \tau \min(x, \omega y)},$$

where τ, θ and ω are the parameters. This leads to a flexible functional structure, where τ modulates the minimum term, and ω weights the contribution of the variable y within this minimum. As in the second part, we avoid the use of an exponent parameter to maintain analytical tractability. In total, five Hardy-Hilbert-type integral inequalities are established.

To the best of our knowledge, all the results presented in this article are new to the literature. They provide a flexible framework for further developments in integral inequalities, and their applications in analysis. Complete proofs are given in full detail, with careful attention to the role of the parameters.

1.3. **Organization.** The article is organized as follows: Section 2 presents the main integral formulas. Section 3 applies these results to derive simple integral inequalities of the weighted Hölder type. Section 4 is devoted to the double integral inequalities of the Hardy-Hilbert type. Detailed proofs are given throughout the text. We conclude with a summary, supplemented by remarks on possible extensions and applications in Section 5.

2. Some integral formulas

This section presents the new integral formulas that we will need for the proofs of our main results. They have the feature of being dependent on three adjustable parameters and involving a minimum term. The first formula is given below.

Lemma 2.1. *For any $\tau, \theta, \omega \in \mathbb{R}$ such that $\theta + \tau > 0$, $(\theta + \tau)\omega \geq 0$, $(1 + \tau\omega)\theta > 0$ and $(1 + \tau\omega)/(\theta\omega) \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{1 + \theta x + \tau \min(x, \omega)} dx$$
$$= 2\left\{ \frac{1}{\sqrt{\theta + \tau}} \arctan[\sqrt{(\theta + \tau)\omega}] + \frac{1}{\sqrt{(1 + \tau\omega)\theta}} \arctan\left[\sqrt{\frac{1 + \tau\omega}{\theta\omega}}\right] \right\}.$$

The case $\omega \to 0$ yields

$$\int_0^{+\infty} \frac{x^{-1/2}}{1 + \theta x} dx = \frac{\pi}{\sqrt{\theta}}.$$

Proof. Changing the variables as $x = \omega y$, we find that

$$\int_0^{+\infty} \frac{x^{-1/2}}{1 + \theta x + \tau \min(x, \omega)} dx = \int_0^{+\infty} \frac{(\omega y)^{-1/2}}{1 + \theta\omega y + \tau \min(\omega y, \omega)} (\omega dy)$$
$$= \sqrt{\omega} \int_0^{+\infty} \frac{y^{-1/2}}{1 + \theta\omega y + \tau\omega \min(y, 1)} dy.$$

Decomposing this integral by the Chasles rule at the threshold value $y = 1$, taking into account the term $\min(y, 1)$ and using standard arctangent formulas, we get

$$\begin{aligned}
&\sqrt{\omega}\int_0^{+\infty} \frac{y^{-1/2}}{1+\theta\omega y+\tau\omega\min(y,1)}dy\\
&= \sqrt{\omega}\left\{\int_0^1 \frac{y^{-1/2}}{1+\theta\omega y+\tau\omega\min(y,1)}dy + \int_1^{+\infty} \frac{y^{-1/2}}{1+\theta\omega y+\tau\omega\min(y,1)}dy\right\}\\
&= \sqrt{\omega}\left\{\int_0^1 \frac{y^{-1/2}}{1+\theta\omega y+\tau\omega\times y}dy + \int_1^{+\infty} \frac{y^{-1/2}}{1+\theta\omega y+\tau\omega\times 1}dy\right\}\\
&= \sqrt{\omega}\left\{\int_0^1 \frac{y^{-1/2}}{1+[\sqrt{(\theta+\tau)\omega y}]^2}dy + \frac{1}{1+\tau\omega}\int_1^{+\infty} \frac{y^{-1/2}}{1+[\sqrt{\theta\omega y/(1+\tau\omega)}]^2}dy\right\}\\
&= \sqrt{\omega}\left\{\left[\frac{2}{\sqrt{(\theta+\tau)\omega}}\arctan[\sqrt{(\theta+\tau)\omega y}]\right]_{y=0}^{y=1}\right.\\
&\left.+\frac{1}{1+\tau\omega}\left[2\sqrt{\frac{1+\tau\omega}{\theta\omega}}\arctan\left[\sqrt{\frac{\theta\omega y}{1+\tau\omega}}\right]\right]_{y=1}^{y\to+\infty}\right\}\\
&= \sqrt{\omega}\left\{\frac{2}{\sqrt{(\theta+\tau)\omega}}\arctan[\sqrt{(\theta+\tau)\omega}]\right.\\
&\left.+\frac{2}{\sqrt{(1+\tau\omega)\theta\omega}}\left\{\frac{\pi}{2}-\arctan\left[\sqrt{\frac{\theta\omega}{1+\tau\omega}}\right]\right\}\right\}\\
&= \sqrt{\omega}\left\{\frac{2}{\sqrt{(\theta+\tau)\omega}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{2}{\sqrt{(1+\tau\omega)\theta\omega}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}\\
&= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\min(x,\omega)}dx\\
&= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}.
\end{aligned}$$

For the case $\omega \to 0$, using the limit result $\arctan(t) \to \pi/2$ when $t \to +\infty$, we obtain

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x}dx = 2\left\{0+\frac{1}{\sqrt{\theta}}\times\frac{\pi}{2}\right\} = \frac{\pi}{\sqrt{\theta}}.$$

This ends the proof of Lemma 2.1. □

We see that the arctangent function plays a central role in the final expression. The same will apply to the new simple and double inequalities presented in Sections 3 and 4.

As a side note, using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, the main formula implies that

$$\int_0^{+\infty} \frac{x^{-1/2}}{2(1+\theta x)+\tau[x+\omega-|x-\omega|]}dx$$
$$= \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right].$$

A polynomial-absolute value term has replaced the minimum term in a sense.

The second integral formula is given below. Compared to the previous lemma, the parameter ω now weights the variable x in the minimum, and the threshold value is set to 1. The proof is mainly based on Lemma 2.1.

Lemma 2.2. *For any $\tau, \theta, \omega \in \mathbb{R}$ such that $\theta + \tau\omega > 0$, $\theta/\omega + \tau \geq 0$, $(1+\tau)\theta > 0$ and $(1+\tau)\omega/\theta \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau \min(\omega x, 1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

Proof. Using a basic property of the minimum, we can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau \min(\omega x, 1)}dx = \int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\omega \min(x, 1/\omega)}dx.$$

Applying Lemma 2.1 with "$\tau\omega$" instead of "τ" and "$1/\omega$" instead of "ω", we get

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\omega \min(x, 1/\omega)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta+\tau\omega}{\omega}}\right] + \frac{1}{\sqrt{(1+(\tau\omega)/\omega)\theta}} \arctan\left[\sqrt{\frac{1+(\tau\omega)/\omega}{\theta/\omega}}\right]\right\}$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

So we have

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau \min(\omega x, 1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

This ends the proof of Lemma 2.2. □

The role of the arctangent function in this formula is once again of crucial importance.

Using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, we also have

$$\int_0^{+\infty} \frac{x^{-1/2}}{2(1+\theta x)+\tau[\omega x+1-|\omega x-1|]}dx$$
$$= \frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right].$$

The third integral formula is given below, with a change to the previous formula concerning the effect of θ. The proof is again based on Lemma 2.1.

Lemma 2.3. *For any $\tau, \theta, \omega \in \mathbb{R}$ such that $\theta+\tau>0$, $(\theta+\tau)\omega \geq 0$, $(1+\tau\omega)\theta > 0$ and $(1+\tau\omega)/(\theta\omega) \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\min(\omega x,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}.$$

Proof. It follows from Lemma 2.1 that

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\min(x,\omega)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}.$$

Changing the variables as $x = 1/y$, the integral can be expressed as

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\min(x,\omega)}dx = \int_{+\infty}^{0} \frac{(1/y)^{-1/2}}{1+\theta(1/y)+\tau\min(1/y,\omega)}\left(-\frac{1}{y^2}dy\right)$$
$$= \int_0^{+\infty} \frac{y^{-1/2}}{y+\theta+\tau y\min(1/y,\omega)}dy = \int_0^{+\infty} \frac{y^{-1/2}}{y+\theta+\tau\min(1,\omega y)}dy$$
$$= \int_0^{+\infty} \frac{y^{-1/2}}{y+\theta+\tau\min(\omega y,1)}dy.$$

Uniformizing the notation, we therefore have

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\min(\omega x,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}.$$

This concludes the proof of Lemma 2.3. □

Using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, this integral formula gives

$$\int_0^{+\infty} \frac{x^{-1/2}}{2(x+\theta)+\tau[\omega x+1-|\omega x-1|]}dx$$
$$= \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right].$$

The last formula is given below. The proof is mainly based on Lemma 2.3.

Lemma 2.4. *For any $\tau, \theta, \omega \in \mathbb{R}$ such that $\theta + \tau\omega > 0$, $\theta/\omega + \tau \geq 0$, $(1+\tau)\theta > 0$ and $(1+\tau)\omega/\theta \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\min(x,\omega)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

Proof. We can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\min(x,\omega)}dx = \int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\omega\min(x/\omega,1)}dx.$$

Applying Lemma 2.3 with "$\tau\omega$" instead of "τ" and "$1/\omega$" instead of "ω", we find that

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\omega\min(x/\omega,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta+\tau\omega}{\omega}}\right] + \frac{1}{\sqrt{(1+(\tau\omega)/\omega)\theta}} \arctan\left[\sqrt{\frac{1+(\tau\omega)/\omega}{\theta/\omega}}\right]\right\}$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

We therefore have

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\theta+\tau\min(x,\omega)}dx$$
$$= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}.$$

This ends the proof of Lemma 2.4. □

Using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, we also derive

$$\int_0^{+\infty} \frac{x^{-1/2}}{2(x+\theta)+\tau[x+\omega x-|x-\omega|]}dx$$
$$= \frac{1}{\sqrt{\theta+\tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}} \arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right].$$

These lemmas are new, and in particular are not included in the reference book [6]. For the purposes of this article, we will use them as key tools in establishing both simple and double integral inequalities. They are, of course, of independent interest and can be used in a variety of other analytical contexts. In particular, we have in mind operator theory or the formulation of new integral transforms.

3. Simple integral inequalities of the weighted Hölder-type

The theorem below contains four simple integral inequalities of the weighted Hölder type. As mentioned earlier, the corresponding weight functions can be described as three-parameter ratio-minimum functions. The proof is based on the Hölder integral inequality and the previous lemmas.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that $\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx < +\infty$. Then the four simple integral inequalities below, called Simple integral inequality 1, 2, 3 and 4, hold.*

Simple integral inequality 1: *For any $\tau, \theta, \omega \geq 0$ such that $\theta + \tau > 0$, $(\theta + \tau)\omega \geq 0$, $(1 + \tau\omega)\theta > 0$ and $(1 + \tau\omega)/(\theta\omega) \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[1 + \theta x + \tau \min(x, \omega)]^{1/p}} f(x)dx$$
$$\leq 2^{1/p} \left\{ \frac{1}{\sqrt{\theta + \tau}} \arctan[\sqrt{(\theta + \tau)\omega}] + \frac{1}{\sqrt{(1 + \tau\omega)\theta}} \arctan\left[\sqrt{\frac{1 + \tau\omega}{\theta\omega}}\right] \right\}^{1/p} \times$$
$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Simple integral inequality 2: *For any $\tau, \theta, \omega \geq 0$ such that $\theta + \tau\omega > 0$, $\theta/\omega + \tau \geq 0$, $(1 + \tau)\theta > 0$ and $(1 + \tau)\omega/\theta \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[1 + \theta x + \tau \min(\omega x, 1)]^{1/p}} f(x)dx$$
$$\leq 2^{1/p} \left\{ \frac{1}{\sqrt{\theta + \tau\omega}} \arctan\left[\sqrt{\frac{\theta}{\omega} + \tau}\right] + \frac{1}{\sqrt{(1 + \tau)\theta}} \arctan\left[\sqrt{\frac{(1 + \tau)\omega}{\theta}}\right] \right\}^{1/p} \times$$
$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Simple integral inequality 3: *For any $\tau, \theta, \omega \geq 0$ such that $\theta + \tau > 0$, $(\theta + \tau)\omega \geq 0$, $(1 + \tau\omega)\theta > 0$ and $(1 + \tau\omega)/(\theta\omega) \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[x + \theta + \tau \min(\omega x, 1)]^{1/p}} f(x)dx$$
$$\leq 2^{1/p} \left\{ \frac{1}{\sqrt{\theta + \tau}} \arctan[\sqrt{(\theta + \tau)\omega}] + \frac{1}{\sqrt{(1 + \tau\omega)\theta}} \arctan\left[\sqrt{\frac{1 + \tau\omega}{\theta\omega}}\right] \right\}^{1/p} \times$$
$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Simple integral inequality 4: *For any $\tau, \theta, \omega \geq 0$ such that $\theta+\tau\omega > 0$, $\theta/\omega+\tau \geq 0$, $(1+\tau)\theta > 0$ and $(1+\tau)\omega/\theta \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[x+\theta+\tau\min(x,\omega)]^{1/p}} f(x)dx$$

$$\leq 2^{1/p}\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}^{1/p}\times$$

$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q},$$

where the integrals on the right-hand side must converge.

For each of these inequalities, we can eventually apply $\omega \to 0$.

Proof. For the sake of redundancy, we will only give the details for Simple integral inequality 1. The other proofs follow the same lines; each uses the Hölder integral inequality and one of the lemmas established in the previous section.

Simple integral inequality 1: Using $1 = x^{-1/(2p)}x^{1/(2p)}$, which leads to an appropriate product decomposition of the integrand, and the Hölder integral inequality with the parameters p and q, we get

$$\int_0^{+\infty} \frac{1}{[1+\theta x+\tau\min(x,\omega)]^{1/p}} f(x)dx = \int_0^{+\infty} \frac{x^{-1/(2p)}}{[1+\theta x+\tau\min(x,\omega)]^{1/p}} x^{1/(2p)} f(x)dx$$

$$\leq \left[\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\min(x,\omega)}dx\right]^{1/p}\left[\int_0^{+\infty} x^{q/(2p)} f^q(x)dx\right]^{1/q}.$$

It follows from Lemma 2.1 and the identity $p = q/(q-1)$ that

$$\left[\int_0^{+\infty} \frac{x^{-1/2}}{1+\theta x+\tau\min(x,\omega)}dx\right]^{1/p}\left[\int_0^{+\infty} x^{q/(2p)} f^q(x)dx\right]^{1/q}$$

$$= 2^{1/p}\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}]+\frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}^{1/p}\times$$

$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.$$

We therefore have

$$\int_0^{+\infty} \frac{1}{[1+\theta x+\tau\min(x,\omega)]^{1/p}} f(x)dx$$

$$\leq 2^{1/p}\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}]+\frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}^{1/p}\times$$

$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.$$

The desired result is obtained.

Simple integral inequality 2, 3 and 4: To prove Simple integral inequality 2, 3 and 4, it is sufficient to proceed as we did for Simple integral inequality 1, but using Lemmas 2.2, 2.3 and 2.4 instead of Lemma 2.1, respectively.

This concludes the proof of Theorem 3.1. □

By considering the weight function

$$\mathcal{W}(x) = \frac{1}{1+\theta x+\tau \min(x,\omega)},$$

the first simple integral inequality reads as follows:

$$\int_0^{+\infty} \mathcal{W}^{1/p}(x) f(x)dx \leq C_{ons} \left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q},$$

where

$$C_{ons} = 2^{1/p}\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}^{1/p}.$$

This can also be written as

$$\left[\int_0^{+\infty} \mathcal{W}^{1/p}(x) f(x)dx\right]^q \leq C_{ons}^q \int_0^{+\infty} x^{(q-1)/2} f^q(x)dx.$$

This is a new weighted Hölder integral inequality in the literature. Similar formulations can be presented for the other integral inequalities.

The main applications of these simple inequalities can be found in the analysis of partial differential equations, estimates in harmonic analysis, and the study of function spaces where weighted norms are involved.

As a side note, using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, Simple integral inequality 1 implies that

$$\int_0^{+\infty} \frac{1}{[2(1+\theta x)+\tau[x+\omega-|x-\omega|]]^{1/p}} f(x)dx$$
$$\leq \left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}^{1/p} \times$$
$$\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.$$

The setting can therefore go beyond the strict inclusion of a minimum term.

4. Double integral inequalities of the Hardy-Hilbert type

New double integral inequalities of the Hardy-Hilbert type are given in the theorem below. They are mainly concerned with a three-parameter ratio-minimum kernel function involving the product of the variables, i.e., xy.

Theorem 4.2. *Let $p > 1$, $q = p/(p-1)$ and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$. Then the four double integral inequalities below, called Double integral inequality 1, 2, 3 and 4, hold.*

Double integral inequality 1: *For any $\tau, \theta, \omega \geq 0$ such that $\theta+\tau > 0$, $(\theta+\tau)\omega \geq 0$, $(1+\tau\omega)\theta > 0$ and $(1+\tau\omega)/(\theta\omega) \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{1+\theta xy+\tau\min(xy,\omega)} f(x)g(y)dxdy$$

$$\leq 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}\times$$

$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Double integral inequality 2: *For any $\tau, \theta, \omega \geq 0$ such that $\theta+\tau\omega > 0$, $\theta/\omega+\tau \geq 0$, $(1+\tau)\theta > 0$ and $(1+\tau)\omega/\theta \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{1+\theta xy+\tau\min(\omega xy,1)} f(x)g(y)dxdy$$

$$\leq 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times$$

$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Double integral inequality 3: *For any $\tau, \theta, \omega \geq 0$ such that $\theta+\tau > 0$, $(\theta+\tau)\omega \geq 0$, $(1+\tau\omega)\theta > 0$ and $(1+\tau\omega)/(\theta\omega) \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{xy+\theta+\tau\min(\omega xy,1)} f(x)g(y)dxdy$$

$$\leq 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}\times$$

$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Double integral inequality 4: *For any $\tau, \theta, \omega \geq 0$ such that $\theta+\tau\omega > 0$, $\theta/\omega+\tau \geq 0$, $(1+\tau)\theta > 0$ and $(1+\tau)\omega/\theta \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{xy+\theta+\tau\min(xy,\omega)} f(x)g(y)dxdy$$

$$\leq 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right] + \frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times$$

$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

For each of these inequalities, we can eventually apply $\omega \to 0$.

Proof. For the sake of redundancy, we will only give the details for the proof of Double integral inequality 1. The other proofs follow the same lines; each uses the Hölder integral inequality and one of the lemmas established in Section 2.

Double integral inequality 1: Using $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and the identity $1/p + 1/q = 1$, which lead to a suitable product decomposition of the integrand, and the Hölder integral inequality at the parameters p and q, we get

$$\begin{aligned}
&\iint_0^{+\infty} \frac{1}{1+\theta xy+\tau\min(xy,\omega)} f(x)g(y)dxdy \\
&= \iint_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{[1+\theta xy+\tau\min(xy,\omega)]^{1/p}} f(x) \times \frac{x^{-1/(2q)}y^{1/(2p)}}{[1+\theta xy+\tau\min(xy,\omega)]^{1/q}} g(y)dxdy \\
&\leq \mathfrak{A}^{1/p}(\tau,\theta,\omega)\mathfrak{B}^{1/q}(\tau,\theta,\omega), \qquad (1)
\end{aligned}$$

where

$$\mathfrak{A}(\tau,\theta,\omega) = \iint_0^{+\infty} \frac{x^{p/(2q)}y^{-1/2}}{1+\theta xy+\tau\min(xy,\omega)} f^p(x)dxdy$$

and

$$\mathfrak{B}(\tau,\theta,\omega) = \iint_0^{+\infty} \frac{x^{-1/2}y^{q/(2p)}}{1+\theta xy+\tau\min(xy,\omega)} g^q(y)dxdy.$$

Let us determine the expressions of $\mathfrak{A}(\tau,\theta,\omega)$ and $\mathfrak{B}(\tau,\theta,\omega)$, one after the other.

For $\mathfrak{A}(\tau,\theta,\omega)$, exchanging the order of integration, which is possible by the Fubini-Tonelli integral theorem, changing the variables as $u = xy$ with respect to y, using the identity $q = p/(p-1)$ and applying Lemma 2.1, we obtain

$$\begin{aligned}
\mathfrak{A}(\tau,\theta,\omega) &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x)\left[\int_0^{+\infty} \frac{(xy)^{-1/2}}{1+\theta xy+\tau\min(xy,\omega)} xdy\right]dx \\
&= \int_0^{+\infty} x^{p/2-1} f^p(x)\left[\int_0^{+\infty} \frac{u^{-1/2}}{1+\theta u+\tau\min(u,\omega)} du\right]dx \\
&= \int_0^{+\infty} x^{p/2-1} f^p(x)\times \\
&\left[2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}\right]dx \\
&= 2\left\{\frac{1}{\sqrt{\theta+\tau}}\arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}}\arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\}\times \\
&\int_0^{+\infty} x^{p/2-1} f^p(x)dx. \qquad (2)
\end{aligned}$$

For $\mathfrak{B}(\tau,\theta,\omega)$, we proceed as for $\mathfrak{A}(\tau,\theta,\omega)$, but changing the variables as $v = xy$ with respect to x. We find that

$$\mathfrak{B}(\tau,\theta,\omega) = \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{(xy)^{-1/2}}{1+\theta xy + \tau \min(xy,\omega)} y dx \right] dy$$

$$= \int_0^{+\infty} y^{q/2-1} g^q(y) \left[\int_0^{+\infty} \frac{v^{-1/2}}{1+\theta v + \tau \min(v,\omega)} dv \right] dy$$

$$= \int_0^{+\infty} y^{q/2-1} g^q(y) \times$$

$$\left[2 \left\{ \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}} \right] \right\} \right] dy$$

$$= 2 \left\{ \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}} \right] \right\} \times$$

$$\int_0^{+\infty} y^{q/2-1} g^q(y) dy. \tag{3}$$

Substituting the expressions of $\mathfrak{A}(\tau,\theta,\omega)$ and $\mathfrak{B}(\tau,\theta,\omega)$ determined in Equations (2) and (3) into Equation (1), and using the identity $1/p + 1/q = 1$, we get

$$\iint_0^{+\infty} \frac{1}{1+\theta xy + \tau \min(xy,\omega)} f(x) g(y) dx dy$$

$$\leq \left[2 \left\{ \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}} \right] \right\} \times \right.$$

$$\left. \int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \times$$

$$\left[2 \left\{ \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}} \right] \right\} \times \right.$$

$$\left. \int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}$$

$$= 2 \left\{ \frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}} \right] \right\} \times$$

$$\left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}.$$

The desired inequality is established.

Double integral inequality 2, 3 and 4: For proving Double integral inequality 2, 3 and 4, it is sufficient to proceed as above, but using Lemmas 2.2, 2.3 and 2.4 instead of Lemma 2.1, respectively.

This concludes the proof of Theorem 4.2. □

As a special case, applying $\omega \to 0$, we obtain $\min(x,\omega) \to 0$, and Theorem 4.2 gives

$$\iint_0^{+\infty} \frac{1}{1+\theta xy} f(x)g(y)dxdy \leq \frac{\pi}{\sqrt{\theta}} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This is a well-known variant of the Hardy-Hilbert integral inequality. See, for example, [16]. The other cases leading to new double integral inequalities.

Using the identity $\min(a,b) = (1/2)[a+b-|a-b|]$ with $a,b \in \mathbb{R}$, the double integral of Double integral inequality 1 implies that

$$\begin{aligned} &\iint_0^{+\infty} \frac{1}{2(1+\theta xy)+\tau[xy+\omega-|xy-\omega|]} f(x)g(y)dxdy \\ &\leq \left\{\frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\} \times \\ &\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}. \end{aligned}$$

Similar expressions can be obtained for Double integral inequality 2, 3 and 4. These reformulations show the relativity of using the minimum term and the flexibility of our results.

The theorem below proposes a general framework that has the property of unifying the Hardy-Hilbert integral inequality and [11, Theorem 2].

Theorem 4.3. *Let $p > 1$, $q = p/(p-1)$, $\tau, \theta, \omega \geq 0$ such that $(1+\tau)\theta > 0$, $\theta/[(1+\tau)\omega] \geq 0$, $\theta + \tau\omega > 0$ and $\omega/(\theta+\tau\omega) \geq 0$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$. Then we have*

$$\begin{aligned} &\iint_0^{+\infty} \frac{1}{x+\theta y+\tau\min(x,\omega y)} f(x)g(y)dxdy \\ &\leq 2\left\{\frac{1}{\sqrt{\theta+\tau}} \arctan[\sqrt{(\theta+\tau)\omega}] + \frac{1}{\sqrt{(1+\tau\omega)\theta}} \arctan\left[\sqrt{\frac{1+\tau\omega}{\theta\omega}}\right]\right\} \times \\ &\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}, \end{aligned}$$

where the integrals on the right-hand side must converge. Note that we can eventually apply $\omega \to 0$.

Proof. Using $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and the identity $1/p + 1/q = 1$, which lead to an appropriate product decomposition of the integrand, and the Hölder integral inequality at p and q, we obtain

$$\begin{aligned} &\iint_0^{+\infty} \frac{1}{x+\theta y+\tau\min(x,\omega y)} f(x)g(y)dxdy \\ &= \iint_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{[x+\theta y+\tau\min(x,\omega y)]^{1/p}} f(x) \times \frac{x^{-1/(2q)}y^{1/(2p)}}{[x+\theta y+\tau\min(x,\omega y)]^{1/q}} g(y)dxdy \\ &\leq \mathfrak{C}^{1/p}(\tau,\theta,\omega)\mathfrak{D}^{1/q}(\tau,\theta,\omega), \qquad (4) \end{aligned}$$

where

$$\mathfrak{C}(\tau,\theta,\omega)=\iint\limits_0^{+\infty}\frac{x^{p/(2q)}y^{-1/2}}{x+\theta y+\tau\min(x,\omega y)}f^p(x)dxdy$$

and

$$\mathfrak{D}(\tau,\theta,\omega)=\iint\limits_0^{+\infty}\frac{x^{-1/2}y^{q/(2p)}}{x+\theta y+\tau\min(x,\omega y)}g^q(y)dxdy.$$

Let us determine the expressions of $\mathfrak{C}(\tau,\theta,\omega)$ and $\mathfrak{D}(\tau,\theta,\omega)$, one after the other.

For $\mathfrak{C}(\tau,\theta,\omega)$, exchanging the order of integration, which is possible by the Fubini-Tonelli integral theorem, changing the variables as $u=y/x$ with respect to y, using the identity $q=p/(p-1)$ and applying Lemma 2.2, we obtain

$$\begin{aligned}\mathfrak{C}(\tau,\theta,\omega)&=\int_0^{+\infty}x^{p/(2q)-1/2}f^p(x)\left[\int_0^{+\infty}\frac{(y/x)^{-1/2}}{1+\theta(y/x)+\tau\min(1,\omega y/x)}\times\frac{1}{x}dy\right]dx\\&=\int_0^{+\infty}x^{p/2-1}f^p(x)\left[\int_0^{+\infty}\frac{u^{-1/2}}{1+\theta u+\tau\min(1,\omega u)}du\right]dx\\&=\int_0^{+\infty}x^{p/2-1}f^p(x)\times\\&\left\{2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\right\}dx\\&=2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times\\&\int_0^{+\infty}x^{p/2-1}f^p(x)dx.\end{aligned}\tag{5}$$

For $\mathfrak{D}(\tau,\theta,\omega)$, we proceed as for $\mathfrak{C}(\tau,\theta,\omega)$, but with the change of variables $v=x/y$ with respect to x, and Lemma 2.4. We find that

$$\begin{aligned}\mathfrak{D}(\tau,\theta,\omega)&=\int_0^{+\infty}y^{q/(2p)-1/2}g^q(y)\left[\int_0^{+\infty}\frac{(x/y)^{-1/2}}{x/y+\theta+\tau\min(x/y,\omega)}\times\frac{1}{y}dx\right]dy\\&=\int_0^{+\infty}y^{q/2-1}g^q(y)\left[\int_0^{+\infty}\frac{v^{-1/2}}{v+\theta+\tau\min(v,\omega)}dv\right]dy\\&=\int_0^{+\infty}y^{q/2-1}g^q(y)\times\\&\left\{2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\right\}dy\\&=2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times\\&\int_0^{+\infty}y^{q/2-1}g^q(y)dy.\end{aligned}\tag{6}$$

Substituting the expressions of $\mathfrak{C}(\tau,\theta,\omega)$ and $\mathfrak{D}(\tau,\theta,\omega)$ obtained in Equations (5) and (6) into Equation (4), and using the identity $1/p+1/q=1$, we obtain

$$\begin{aligned}
&\iint_0^{+\infty} \frac{1}{x+\theta y+\tau\min(x,\omega y)} f(x)g(y)dxdy \\
&\leq \left[2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times \right.\\
&\left.\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p} \times \\
&\left[2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times \right.\\
&\left.\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q} \\
&= 2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times \\
&\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Theorem 4.3. □

In the framework of this theorem, if we take $\tau=1$, $\theta=1$ and $\omega\to 0$, then the double integral simplifies to:

$$\iint_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy$$

and the constant factor becomes

$$2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}=\pi.$$

The corresponding inequality thus reduces the Hardy-Hilbert integral inequality.

As another important special case, if we take $\tau=1$, $\theta=1$ and $\omega=1$, then the double integral simplifies to:

$$\iint_0^{+\infty} \frac{1}{x+y+\min(x,\omega y)} f(x)g(y)dxdy$$

and the constant factor becomes

$$\begin{aligned}
&2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\\
&=2\left\{\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2}\right]\right\}=2\sqrt{2}\arctan\left[\sqrt{2}\right].
\end{aligned}$$

The corresponding inequality thus reduces to [11, Theorem 2].

Theorem 4.3 therefore unifies these two key results and adds a greater degree of flexibility thanks to the parameters τ, θ and ω.

Just activating the parameter ω, so by taking $\tau = 1$ and $\theta = 1$, then the double integral simplifies to:

$$\iint_0^{+\infty} \frac{1}{x+y+\min(x,\omega y)} f(x)g(y)dxdy$$

and the constant factor becomes

$$2\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}$$
$$=2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\},$$

leading to the following double integral inequality:

$$\iint_0^{+\infty} \frac{1}{x+y+\min(x,\omega y)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}\times$$
$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

This result captures the complexity of the problem and shows how a simple modulation of the variable y affects the constant factor in a sophisticated way. For numerical illustration, three examples of this result are given below.

Numerical example 1: If we take $p = 2$, $f(x) = e^{-x}$, $g(y) = e^{-y}$ and $\omega = 2$, then we have

$$\iint_0^{+\infty} \frac{1}{x+y+\min(x,\omega y)} f(x)g(y)dxdy = \iint_0^{+\infty} \frac{1}{x+y+\min(x,2y)} e^{-x-y}dxdy$$
$$\approx 0.766238,$$

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}$$
$$=2\left\{\frac{1}{\sqrt{3}}\arctan\left[\sqrt{\frac{1}{2}+1}\right]+\frac{1}{\sqrt{2}}\arctan(2)\right\}$$
$$\approx 2.58889,$$

$$\int_0^{+\infty} x^{p/2-1}f^p(x)dx = \int_0^{+\infty} e^{-2x}dx = 0.5$$

and

$$\int_0^{+\infty} y^{q/2-1}g^q(y)dy = \int_0^{+\infty} e^{-2y}dy = 0.5,$$

so that

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}\times$$
$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}$$
$$\approx 2.58889\times 0.5^{1/2}\times 0.5^{1/2}=1.294445.$$

The inequality is illustrated since $0.766238 < 1.294445$.

Numerical example 2: If we take $p=2$, $f(x)=(1/x^2)e^{-1/x}$, $g(y)=(1/y^2)e^{-1/y}$ and $\omega=2$, then we have

$$\iint_0^{+\infty}\frac{1}{x+y+\min(x,\omega y)}f(x)g(y)dxdy$$
$$=\iint_0^{+\infty}\frac{1}{x+y+\min(x,2y)}\times\frac{1}{x^2y^2}e^{-1/x-1/y}dxdy$$
$$\approx 0.240245,$$

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}$$
$$=2\left\{\frac{1}{\sqrt{3}}\arctan\left[\sqrt{\frac{1}{2}+1}\right]+\frac{1}{\sqrt{2}}\arctan(2)\right\}$$
$$\approx 2.58889,$$

$$\int_0^{+\infty} x^{p/2-1}f^p(x)dx=\int_0^{+\infty}\frac{1}{x^4}e^{-2/x}dx=0.25$$

and

$$\int_0^{+\infty} y^{q/2-1}g^q(y)dy=\int_0^{+\infty}\frac{1}{y^4}e^{-2/y}dy=0.25,$$

so that

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}\times$$
$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}$$
$$\approx 2.58889\times 0.25^{1/2}\times 0.25^{1/2}=0.6472246.$$

We obviously have $0.240245 < 0.6472246$, supporting the theory.

Numerical example 3: If we take $p=2$, $f(x)=e^{-x^2}$, $g(y)=e^{-y^2}$ and $\omega=\pi$, then we have

$$\iint_0^{+\infty}\frac{1}{x+y+\min(x,\omega y)}f(x)g(y)dxdy=\iint_0^{+\infty}\frac{1}{x+y+\min(x,\pi y)}e^{-x^2-y^2}dxdy$$
$$\approx 0.811612,$$

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}$$
$$=2\left\{\frac{1}{\sqrt{1+\pi}}\arctan\left[\sqrt{\frac{1}{\pi}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\pi}\right]\right\}$$
$$\approx 2.52415,$$

$$\int_0^{+\infty} x^{p/2-1}f^p(x)dx=\int_0^{+\infty} e^{-2x^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}\approx 0.626657$$

and

$$\int_0^{+\infty} y^{q/2-1}g^q(y)dy=\int_0^{+\infty} e^{-2y^2}dy=\frac{1}{2}\sqrt{\frac{\pi}{2}}\approx 0.626657,$$

so that

$$2\left\{\frac{1}{\sqrt{1+\omega}}\arctan\left[\sqrt{\frac{1}{\omega}+1}\right]+\frac{1}{\sqrt{2}}\arctan\left[\sqrt{2\omega}\right]\right\}\times$$
$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}$$
$$\approx 2.52415\times 0.626657^{1/2}\times 0.626657^{1/2}=1.581776.$$

As expected, we have $0.811612 < 1.581776$.

As a side note, using the identity $\min(a,b)=(1/2)[a+b-|a-b|]$ with $a,b\in\mathbb{R}$, the double integral inequality in Theorem 4.3 implies that

$$\iint_0^{+\infty}\frac{1}{2(x+\theta y)+\tau[x+\omega y-|x-\omega y|]}f(x)g(y)dxdy$$
$$\leq\left\{\frac{1}{\sqrt{\theta+\tau\omega}}\arctan\left[\sqrt{\frac{\theta}{\omega}+\tau}\right]+\frac{1}{\sqrt{(1+\tau)\theta}}\arctan\left[\sqrt{\frac{(1+\tau)\omega}{\theta}}\right]\right\}\times$$
$$\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

This alternative formulation may be of interest when dealing with a double integral where there is a ratio term with an absolute value.

5. Conclusion

This article introduces new integral inequalities involving three-parameter ratio-minimum weight or kernel functions. In particular, we derive simple inequalities of the weighted Hölder type and double integral inequalities of the Hardy-Hilbert type. The arctangent function turns out to be a key component in characterizing the upper bounds. These generalizations extend the classical results by introducing greater flexibility through adjustable parameters. The detailed proofs provide a solid foundation for further mathematical exploration. The limitation of the study remains the mathematical complexity of the inequalities, but it remains well balanced with their degree of adaptability. Future research may focus on extending these inequalities to other functional settings or applying them to the analysis of differential and integral equations.

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