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ON COMMON FIXED POINT THEOREMS FOR (ψ, ϕ)-WEAK CONTRACTION IN BRANCIARI TYPE GENERALIZED METRIC SPACES

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Abstract. This paper investigates fixed-point theorems for mappings satisfying (ψ, ϕ)-weak contraction conditions within the framework of Branciari-type generalized metric spaces. These spaces extend the concept of standard metric spaces by relaxing the triangle inequality, thus providing a broader and more flexible structure to study the existence and uniqueness of fixed points. The results presented in this study generalize and unify several classical fixed-point theorems, offering new insights into the behavior of such mappings under weaker contractive conditions. A significant portion of the paper is dedicated to providing illustrative examples, ensuring the applicability of the theoretical results and demonstrating their relevance to practical scenarios. These examples not only validate the imposed conditions but also highlight the utility of (ψ, ϕ)-weak contractions in solving real-world problems. By bridging the gap between abstract mathematical theory and practical application, this work contributes to advancing fixed-point theory in generalized metric spaces, paving the way for further developments in this field.

1. Introduction

Alber and Delabriere [3] introduced the concept of ψ-weak contractions and established fixed point results for single-valued mappings satisfying such contractions. Subsequently, Rhoades [12] generalized the results of Alber and Delabriere. The notion of a generalized metric space was introduced by Branciari [7], and it later came to be known as a Branciari generalized metric space.

The well-known Banach contraction principle, which guarantees the existence and uniqueness of fixed points under certain conditions, has been extensively generalized in various directions. Amini-Harandi [5] introduced metric-like spaces, a

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generalization of partial metric spaces, and established new fixed point results in this framework. Almarri et al. [4] developed fixed point theorems in M-metric spaces using relation-theoretic approaches and demonstrated their applications to electrical circuit problems. Nallaselli et al. [10] investigated admissible contractions in b-metric spaces and applied their results to solve certain classes of integral equations. Haque et al. [9] studied bicomplex-valued controlled metric spaces and employed fixed point results to address fractional differential equations. Gupta et al. [8] proposed the notion of extended G_b-metric spaces and applied their fixed point theorems to solve Fredholm integral equations. Wangwe [14] examined fixed points in bicomplex-valued b-metric spaces and utilized these results to handle nonlinear matrix equations. Finally, Shateri et al. [13] introduced the $b_v(s)$-metric space, which generalizes several known metric-type spaces, and established fixed point theorems for function-valued mappings.

These contributions underscore the flexibility and robustness of fixed point theory and its ongoing development to address increasingly complex mathematical and applied problems.

In this paper, we prove the existence and uniqueness of a fixed point for two mapping in Branciari type generalized metric spaces.

Branciari [7] defined

Definition 1.1. [7] *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y satisfying the following conditions:*

(a) $d(x, y) = 0$ *if and only if* $x = y$*;*
(b) $d(x, y) = d(y, x)$*;*
(c) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *(the rectangular inequality).*

Then (X, d) is called a Branciari type generalized metric space.

Definition 1.2. [7] *Let (X, d) be a Branciari type generalized metric space and $\{x_n\}$ be a sequence in X and $x \in X$. We call that*

(a) $\{x_n\}$ *is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$ (denoted by $x_n \to x$).*
(b) $\{x_n\}$ *is a Cauchy sequence if and only if for each $\epsilon > 0$ there exists a natural number N such that $d(x_n, x_m) \to 0$ as $m, n \to \infty$.*
(c) *X is complete if and only if every Cauchy sequence is convergent in X.*

Denote by Ψ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) ψ is monotone non decreasing;
(b) $\lim_{t \to r} \psi(t) > 0$ for $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$;
(c) $\psi(t) = 0$ if and only if $t = 0$.

Also denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\lim_{t \to r}\ \inf \varphi(t) > 0$ for each $r > 0$;
(b) $\varphi(t) \to 0$ implies that $t \to 0$;
(c) $\varphi(t) = 0$ if and only if $t = 0$.

Zhiqun Xue and Guiwen Lv[15] proved the following theorem for a self mappings in Branciari type complete generalized metric space given below

Theorem 1.1. [15] *Let (X,d) be a Hausdorff and complete generalized metric space, and let $T : X \to X$ be a self-mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \psi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right) \\ -\phi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right) \quad \forall x,y \in X,$$

where $\phi \in \Phi, \psi \in \Psi$ and $\alpha_i \geq 0$ $(i = 1,2,3)$ with $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$. Then T has a unique fixed point.

Definition 1.3. [1] *Let T and S be self mappings of a set X. If $Tx = Sx = w$, for some $x \in X$, then x is called the coincidence point and w the point of coincidence of T and S.*

Theorem 1.2. [2] *Let f and g be weakly compatible self maps of set X. If f and g have unique point of coincidence $w = fx = gx$, then w is unique common fixed point of f and g.*

2. Main Results

Here, Ψ and Φ denote the classes of functions as defined above. We now present our first result.

Theorem 2.3. *Let F and T be self-mappings on a set X, and let (X,d) be a generalized Branciari metric space satisfying the following condition:*

$$\psi(d(Tx,Ty)) \leq \psi\left(\alpha_1 d(Fx,Fy) + \alpha_2 d(Fx,Tx) + \alpha_3 d(Fy,Ty)\right) \\ -\phi\left(\alpha_1 d(Fx,Fy) + \alpha_2 d(Fx,Tx) + \alpha_3 d(Fy,Ty)\right), \tag{1}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$, and $\alpha_i \geq 0$ for $i = 1,2,3$ with $\alpha_1+\alpha_2+\alpha_3 < 1$. Also assume:

- *(a) $T(X) \subseteq F(X)$,*
- *(b) $F(X)$ is a complete subspace of the generalized Branciari metric space (X,d).*

Then, there exists a unique coincidence point of F and T. Moreover, if F and T are weakly compatible, then they have a unique common fixed point.

Proof. Let $x_0 \in X$, so $Tx_0 \in F(X)$. Thus, there exists $x_1 \in X$ such that $Tx_0 = Fx_1 = y_0$. Continuing in this way, we get a sequence $\{x_n\}$ and $\{y_n\}$ as $Tx_1 = Fx_2 = y_1$, $Tx_2 = Fx_3 = y_2$, $\cdots$, $Tx_{n-1} = Fx_n = y_{n-1}$, $Tx_n = Fx_{n+1} = y_n$, $Tx_{n+1} = Fx_{n+2} = y_{n+1}$. We claim that $\{y_n\}$ is a Cauchy sequence in $F(X)$. Consider

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}).$$

Put $x = x_n$ and $y = x_{n+1}$ in (1),

$$\begin{aligned}\psi\left(d(y_n,y_{n+1})\right) &= \psi\left(d(Tx_n,Tx_{n+1})\right) \\ &\leq \psi\Big(\alpha_1 d(Fx_n,Fx_{n+1}) + \alpha_2 d(Fx_n,Tx_n) + \alpha_3 d(Fx_{n+1},Tx_{n+1})\Big) \\ &\quad -\phi\Big(\alpha_1 d(Fx_n,Fx_{n+1}) + \alpha_2 d(Fx_n,Tx_n) + \alpha_3 d(Fx_{n+1},Tx_{n+1})\Big) \\ &= \psi\Big(\alpha_1 d(y_{n-1},y_n) + \alpha_2 d(y_{n-1},y_n) + \alpha_3 d(y_n,y_{n+1})\Big) \\ &\quad -\phi\Big(\alpha_1 d(y_{n-1},y_n) + \alpha_2 d(y_{n-1},y_n) + \alpha_3 d(y_n,y_{n+1})\Big).\end{aligned}$$

As we know $\phi(t) \geq 0$ for $t \geq 0$. Then, we get

$$\psi\left(d(y_n, y_{n+1})\right) \leq \psi\left(\alpha_1 d(y_{n-1}, y_n) + \alpha_2 d(y_{n-1}, y_n) + \alpha_3 d(y_n, y_{n+1})\right).$$

Using property of ψ, we have

$$d(y_n, y_{n+1}) \leq \alpha_1 d(y_{n-1}, y_n) + \alpha_2 d(y_{n-1}, y_n) + \alpha_3 d(y_n, y_{n+1}).$$

It implies that

$$d(y_n, y_{n+1}) \leq \frac{(\alpha_1 + \alpha_2)}{(1 - \alpha_3)} d(y_{n-1}, y_n) = k d(y_{n-1}, y_n),$$

where, $\dfrac{(\alpha_1 + \alpha_2)}{(1 - \alpha_3)} = k < 1$. Applying repeated use of above inequality, we have

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1).$$

Since $k < 1$, $k^n \to 0$ as $n \to \infty$, so

$$d(y_n, y_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

For some natural number N with $n > m \geq N$, we have

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \\ &\leq k^m d(y_0, y_1) + k^{m+1} d(y_0, y_1) + \cdots + k^{n-1} d(y_0, y_1) \\ &\leq k^m d(y_0, y_1) + k^{m+1} d(y_0, y_1) + \cdots + k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + \cdots \\ &= k^n \left[1 + k + k^2 + \cdots + k^{m-n} + \cdots\right] d(y_0, y_1) \\ &= \frac{k^n}{1 - k} d(y_0, y_1). \end{aligned}$$

Since $k < 1$, so $k^n \to 0$ as $n \to \infty$, therefore $d(y_m, y_n) \to 0$ as $m, n \to \infty$. Thus, $\{y_n\}$ is a Cauchy Sequence. That is $\{y_n\} = \{Fx_{n+1}\}$ is a Cauchy Sequence in complete space $F(X)$. Therefore, $\exists\ q \in F(X)$ such that $Fx_{n+1} \to q$ as $n \to \infty$. Also, $\{Tx_n\} = \{Fx_{n+1}\} = \{y_n\} \to q$ as $n \to \infty$. As $\{Fx_n\}$ converges to $q \in F(X)$, there exists $p \in X$ such that $q = Fp$. Now, we claim that $Tp = q$. Put $x = p$ and $y = x_n$ in (1),

$$\begin{aligned} \psi\left(d(Tp, Tx_n)\right) \leq\ & \psi\left(\alpha_1 d(Fp, Fx_n) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fx_n, Tx_n)\right) \\ & - \phi\left(\alpha_1 d(Fp, Fx_n) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fx_n, Tx_n)\right). \end{aligned} \tag{3}$$

As we know $\phi(t) \geq 0$ for $t \geq 0$. Then we get,

$$\psi\left(d(Tp, Tx_n)\right) \leq \psi\left(\alpha_1 d(Fp, Fx_n) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fx_n, Tx_n)\right).$$

Using the property of ψ, we have:

$$d(Tp, Tx_n) \leq \alpha_1 d(Fp, Fx_n) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fx_n, Tx_n).$$

Taking limits as $x_n \to p$, and noting $Fx_n \to Fp = q$, $Tx_n \to q$, we obtain:

$$d(Tp, q) \leq \alpha_2 d(q, Tp).$$

Rewriting:

$$(1 - \alpha_2) d(Tp, q) \leq 0.$$

Since $d(Tp, q) \geq 0$, it follows that $d(Tp, q) = 0$, i.e.,

$$Tp = q = Fp.$$

Thus, p is a coincidence point of F and T.

We claim that F and T have unique coincidence point. Suppose F and T have another coincidence point s i.e. $F(s) = T(s) = q$. Then

$$\begin{aligned}\psi\left(d(Tp, Ts)\right) \leq \psi\left(\alpha_1 d(Fp, Fs) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fs, Ts)\right) \\ - \phi\left(\alpha_1 d(Fp, Fs) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fs, Ts)\right).\end{aligned}$$

Since $\phi(t) \geq 0$ for every $t \geq 0$, so

$$\begin{aligned}\psi(d(Tp, Ts)) &\leq \psi\left(\alpha_1 d(Fp, Fs) + \alpha_2 d(Fp, Tp) + \alpha_3 d(Fs, Ts)\right) \\ &= \psi\left(\alpha_1 d(q, q) + \alpha_2 d(q, q) + \alpha_3 d(q, q)\right) \\ &= \psi(0) = 0.\end{aligned}$$

Thus $\psi(d(Tp, Ts)) \leq 0$. Since $\psi(t) \geq 0$ for every $t \geq 0$. Hence $\psi(d(Tp, Ts)) = 0$. It implies that $d(Tp, Ts) = 0$ and hence $Tp = Ts$ i.e. p is a unique coincidence point. Since T and F are weakly compatible, so by Theorem 1.2, T and F have unique fixed point. □

Theorem 2.4. *Let T and S be self-mappings on a set X, and let (X, d) be a generalized Branciari metric space satisfying the following condition:*

$$\psi(d(Tx, Ty)) \leq \psi\left(M(x, y)\right) - \phi\left(M(x, y)\right) \tag{4}$$

where

$$M(x, y) = \max\left\{d(Sx, Sy),\, d(Sx, Tx),\, d(Sy, Ty)\right\},$$

for all $x, y \in X$, and the following conditions hold:

(i) $\psi : [0, +\infty) \to [0, +\infty)$ *is continuous, non-decreasing, and* $\psi(t) = 0$ *if and only if* $t = 0$,
(ii) $\psi(t_1) \leq \psi(t_2) \Rightarrow t_1 \leq t_2$,
(iii) $\phi : [0, +\infty) \to [0, +\infty)$ *satisfies:*

$$\lim_{t\to r} \phi(t) > 0 \text{ for } r > 0, \quad \text{and} \quad \lim_{t\to r} \phi(t) = 0 \iff r = 0,$$

(iv) $\psi \in \Psi$, $\phi \in \Phi$.

Also assume:

(a) $T(X) \subseteq S(X)$,
(b) $S(X)$ *is a complete subspace of the generalized Branciari metric space* (X, d).

Then, there exists a unique coincidence point of T and S. Moreover, if T and S are weakly compatible, then they have a unique common fixed point.

Proof. Let $x_0 \in X$, $Tx_0 \in S(X)$, so there exist $x_1 \in X$ such $Tx_0 = Sx_1 = y_0$. Continuing this way, we get a sequence $\{x_n\}$ and $\{y_n\}$ as follows, $Tx_1 = Sx_2 = y_1, Tx_2 = Sx_3 = y_2, Tx_{n-1} = Sx_n = y_{n-1}, Tx_n = Sx_{n+1} = y_n, Tx_{n+1} = Sx_{n+2} = y_{n+1}$. Consider the sequence $\{y_n\}$ in $S(X)$. We claim that $\{y_n\}$ is a Cauchy sequence. Consider

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}).$$

Using property of ψ, we have $\psi\left(d(y_n, y_{n+1})\right) \leq \psi\left(d(Tx_n, Tx_{n+1})\right)$. Then

$$\begin{aligned}\psi\left(d(y_n, y_{n+1})\right) &\leq \psi\left(d(Tx_n, Tx_{n+1})\right)\\ &\leq \psi\left(\max\left\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1})\right\}\right)\\ &\quad - \phi\left(\max\left\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1})\right\}\right)\\ &\leq \psi\left(\max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\}\right)\\ &\quad - \phi\left(\max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\}\right).\end{aligned}$$

As we know, $\phi(t) \geq 0$ for $t \geq 0$.

$$\psi\left(d(y_n, y_{n+1})\right) \leq \psi\left(\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\}\right).$$

Using property of ψ,

$$d(y_n, y_{n+1}) \leq \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\}).$$

Case (i) Suppose $\max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}\right\} = d(y_n, y_{n+1})$. Then

$$\psi\left(d(y_n, y_{n+1})\right) \leq \psi\left(d(y_n, y_{n+1})\right) - \phi\left(d(y_n, y_{n+1})\right)$$

leads a contradiction.

Case (ii) Suppose $\max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}\right\} = d(y_{n-1}, y_n)$. Then we get $d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$ for every $n \in \mathbb{N}$. Set $d(y_n, y_{n+1}) = d_n$. Then we have $d_n \leq d_{n-1}$ for every $n \in \mathbb{N}$. It shows that $\{d_n\}$ is decreasing sequence, but $d_n \geq 0$ for every $n \in \mathbb{N}$. We claim that $\{d_n\} \to 0$ as $n \to \infty$. Assume contrary that $\lim_{n\to\infty} d_n = l > 0$. Then,

$$\psi\left(d(y_n, y_{n+1})\right) \leq \psi\left(d(y_{n-1}, y_n)\right) - \phi\left(d(y_{n-1}, y_n)\right).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \psi\left(d(y_n, y_{n+1})\right) \leq \lim_{n\to\infty} \psi\left(d(y_{n-1}, y_n)\right) - \lim_{n\to\infty} \phi\left(d(y_{n-1}, y_n)\right).$$

It gives that

$$\psi(l) \leq \psi(l) - \phi(l)$$

which is contradiction unless $l = 0$. Hence $l = 0$. So, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now, we will prove $\{y_n\}$ is a Cauchy sequence. Assume that $\{y_n\}$ is not a Cauchy sequence. Then, for given $\epsilon > 0$, we can find the subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ with $n(k) > m(k) > k$ with

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon. \tag{5}$$

Further, corresponding to $m(k)$, it is possible to choose smallest integer $n(k)$ with $n(k) > m(k)$ and satisfying (5). Then

$$d(y_{n(k)}, y_{m(k)-1}) < \epsilon. \tag{6}$$

It gives that

$$\begin{aligned}\epsilon \leq d(y_{m(k)}, y_{n(k)}) &\leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}),\\ &< \epsilon + d(y_{n(k)-1}, y_{n(k)})\end{aligned}$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)}) = \epsilon. \tag{7}$$

Again,

$$d\left(y_{n(k)}, y_{m(k)}\right) \leq d\left(y_{n(k)}, y_{n(k)-1}\right) + d\left(y_{n(k)-1}, y_{m(k)-1}\right) + d\left(y_{m(k)-1}, y_{m(k)}\right),$$
$$d\left(y_{n(k)-1}, y_{m(k)-1}\right) \leq d\left(y_{n(k)-1}, y_{n(k)}\right) + d\left(y_{n(k)}, y_{m(k)}\right) + d\left(y_{m(k)}, y_{m(k)-1}\right).$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{n(k)-1}, y_{m(k)-1}) = \epsilon. \tag{8}$$

Similarly, we can show

$$\lim_{k\to\infty} d(y_{n(k)-2}, y_{m(k)-2}) = \epsilon. \tag{9}$$

Put $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (4),

$$\psi\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right) \leq \psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) - \phi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right).$$

It gives that

$$\psi\left(d\left(y_{m(k)-1}, y_{n(k)-1}\right)\right) \leq \psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) - \phi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right). \tag{10}$$

where

$$\begin{aligned} &M(x_{m(k)-1}, x_{n(k)-1}) \\ &= \max\left(d(Sx_{m(k)-1}, Sx_{n(k)-1)}), d(Sx_{m(k)-1}, Tx_{m(k)-1}), d(Sx_{n(k)-1)}, Tx_{n(k)-1)})\right) \\ &= \max\left(d(y_{m(k)-2}, y_{n(k)-2)}), d(y_{m(k)-2}, y_{m(k)-1}), d(y_{n(k)-2)}, y_{n(k)-1)})\right) \end{aligned}$$

Letting $k \to \infty$ in (10), we get

$$\psi(\epsilon) \leq \psi\left(\max\{\epsilon, 0, 0\}\right) - \phi\left(\max\{\epsilon, 0, 0\}\right) = \psi(\epsilon) - \phi(\epsilon).$$

It is contraction when $\epsilon > 0$. Hence $\{y_n\}$ must be a Cauchy sequence. It means $\{Tx_n = Sx_{n+1}\}$ is a Cauchy sequence. Since $S(X)$ is complete. There is $q \in S(X)$ such that $\lim_{n\to\infty} y_n = Tx_n = Sx_{n+1} = q$. Since $q \in S(X)$ there exists $p \in X$ such that $Sp = q$. Using property of ψ, we have

$$\begin{aligned} \psi\left(d(Sx_{n+1}, Tp)\right) &\leq \psi\left(d(Tx_n, Tp)\right) \\ &\leq \psi\left(\max\left\{d(Sx_n, Sp), d(Sx_n, Tx_n), d(Sp, Tp)\right\}\right) \\ &\quad - \phi\left(\max\left\{d(Sx_n, Sp), d(Sx_n, Tx_n), d(Sp, Tp)\right\}\right) \\ &\leq \psi\left(\max\left\{d(Sx_n, q), d(Sx_n, Tx_n), d(q, Tp)\right\}\right) \\ &\quad - \phi\left(\max\left\{d(Sx_n, q), d(Sx_n, Tx_n), d(q, Tp)\right\}\right). \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} \psi\left(d(q, Tp)\right) \leq \psi&\left(\max\left\{d(q, q), d(q, q), d(q, Tp)\right\}\right) \\ &- \phi\left(\max\left\{d(q, q), d(q, q), d(q, Tp)\right\}\right). \end{aligned}$$

It implies that,

$$\psi\left(d(q, Tp)\right) \leq \psi\left(d(q, Tp)\right) - \phi\left(d(q, Tp)\right)$$

Since $\phi(t) \geq 0$, so it is true only if, $d(q, Tp) = 0$ if and only if $q = Tp$. Thus, $q = Tp = Sp$. It means p is a coincidence point of S and T. Since S and T are weakly compatible, so by Theorem 1.2, S and T have a unique fixed point in X. □

Example 1. *Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ and $B = [1, 2]$. Define the generalized metric on X as :*

$$d(x,y) = \begin{cases} d(y,x), & x, y \in X, \\ 0, & x, y \in X \text{ with } x = y, \\ 0.3, & x = \frac{1}{2}, y = \frac{1}{3} \text{ or } x = \frac{1}{4}, y = \frac{1}{5}, \\ 0.2, & x = \frac{1}{2}, y = \frac{1}{5} \text{ or } x = \frac{1}{3}, y = \frac{1}{4}, \\ 0.6, & x = \frac{1}{2}, y = \frac{1}{4} \text{ or } x = \frac{1}{5}, y = \frac{1}{3}, \\ |x-y|, & x, y \in B \text{ or } x \in A, y \in B. \end{cases}$$

Then in [15] it has been shown that (X, d) is a Branciari-type generalized metric space, but it is not a metric space as follows

$$0.6 = d\left(\frac{1}{2}, \frac{1}{4}\right) > d\left(\frac{1}{2}, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{4}\right) = 0.5.$$

Define $T, F : X \to X$ as follows

$$Tx = \begin{cases} \frac{1}{5}, & x \in [1, 2], \\ \frac{1}{4}, & x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}, \\ \frac{1}{3}, & x = \frac{1}{5}. \end{cases}$$

$$Fx = \begin{cases} \frac{1}{5}, & x \in [1, 2], \\ \frac{1}{4}, & x = \frac{1}{4}, \\ \frac{1}{3}, & x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right\}. \end{cases}$$

Use $\psi(t) = t, \varphi(t) = \dfrac{t}{5}, t \in [0, +\infty)$. Then T and F are satisfies

$$\begin{aligned} \psi\left(d(Tx, Ty)\right) \leq\ & \psi\left(a_1 d(Fx, Fy) + a_2 d(Fx, Tx) + a_3 d(Fy, Ty)\right) \\ & - \varphi\left(a_1 d(Fx, Fy) + a_2 d(Fx, Tx) + a_3 d(Fy, Ty)\right) \end{aligned}$$

for all $x, y \in X$, where $a_1 = 0.4, a_2 = 0.4, a_3 = 0.2$. It has seen that T and f has a unique fixed point $x = \frac{1}{4}$.

3. Conclusions and Future Works

We have established two common fixed point theorems for (ψ, ϕ)-weak contractions in Branciari-type generalized metric spaces. The results ensure the existence and uniqueness of coincidence and common fixed points under suitable contractive conditions and completeness assumptions. These theorems generalize and extend several existing results in the literature.

Future research may explore the application of these theorems to multivalued and self mappings, or their integration with other abstract spaces such as, partial metric, or cone metric spaces.

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