Scientific Journal for Damietta Faculty of Science **15**(2) 2025, 212-225
Doi: 10.21608/sjdfs.2025.405831.1253
ISSN Print 2314-8594
ISSN Online 2314-8616

https://sjdfs.journals.ekb.eg/

On the Spectra of some Infinite Band Matrices as Operators on the Cesàro Sequence Space σ_0

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Abstract

In this paper, spectral analysis of infinite triangular double-band matrices acting as operators on the Cesàro space σ_0 is given. The study includes a detailed analysis of the spectrum, distinguishing between different types of the spectrum (e.g., point spectrum, residual spectrum, continuous spectrum, defect spectrum, compression spectrum and approximate point spectrum). Besides, a finer subdivision of the spectrum is given. A generalization of the study to symmetric and non-symmetric tridiagonal matrices is also derived. The technique used in this study is flexible enough to address the spectral problem of the underlying operators in various sequence spaces.

Keywords: Spectrum, Sequence spaces, Infinite matrices.

Introduction

Several authors have analyzed the spectra of various infinite matrix structures, such as band matrices (matrices with non-zero elements confined to diagonal band, which includes lower and upper triangular double-band matrices), Jacobi matrices (tridiagonal matrices with specific properties), and more general matrix forms. Such matrices can usually be identified with linear operators on sequence spaces. Also, several operators, like the difference operators, which are defined by difference equations, often involve infinite band matrices. It should be noted that, no general method exists for finding the spectrum of an arbitrary infinite matrix. In fact, in the case of infinite matrices, the methods used are often tailored to the specific matrix operator and the type of sequence space being considered.

In this paper, we concern ourselves with obtaining the spectra of infinite double-band matrices, in both lower and upper forms. Furthermore, tridiagonal matrices are also of our concern. Our results, in the current paper, substantially complement recent results on the difference operators and their adjoints from [Altay and Başar 2004, Altay and Başar 2005, Akhmedov and Başar 2006, Akhmedov and Başar 2007, Karakaya and Altun 2010, Dutta and Tripathy 2013, Tripathy and Das 2015, El-Shabrawy and Abu-Janah 2018, El-Shabrawy and Sawano 2021], the Jacobi operators from [El-Shabrawy and Shindy 2020], the tridiagonal non-symmetric matrices as operators from [El-Shabrawy and Shindy 2025] and other related results.

Before giving a complete description of the spectral problem we want to address, we recall some notations used in this paper.

By ℓ^∞, c and c_0, we denote the Banach spaces of bounded, convergent and null sequences of complex numbers with the supremum norm, respectively. We use ℓ^p $(1 \le p < \infty)$ to denote the Banach space of p-absolutely summable sequences with the well-known ℓ^p-norm. The symbol bv stands for the Banach space of all sequences $x = (x_k)_{k=0}^{\infty}$ for which the following norm

$$\|x\|_{\mathrm{bv}} = \left|\lim_{k\to\infty} x_k\right| + |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}|$$

is finite. Furthermore, the space $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$ is a Banach space with the bv-norm, whose dual space is norm isomorphic to the Banach space bs (cf. [Wilansky 1984, Theorems 7.2.9 and 7.3.5(ii)]), where

$$\mathrm{bs} = \left\{x = (x_k)_{k=0}^{\infty} : \|x\|_{\mathrm{bs}} = \sup_{N\ge 0}\left|\sum_{k=0}^{N} x_k\right| < \infty\right\}.$$

The space of p-bounded variation sequences, denoted by bv_p $(1 < p < \infty)$, is the Banach space of all sequences $x = (x_k)_{k=0}^{\infty}$ for which $(x_k - x_{k-1})_{k=0}^{\infty} \in \ell^p$, where $x_{-1} = 0$. The space cs is the Banach space of all sequences $x = (x_k)_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} x_k$ is convergent, with the norm

$$\|x\|_{\mathrm{cs}} = \sup_n \left|\sum_{k=0}^{n} x_k\right|.$$

The Hahn sequence space h [Rao 1990] is defined by

$$\mathrm{h} = \{x = (x_k)_{k=0}^{\infty} \in c_0 : \|x\|_h = \sum_{k=0}^{\infty} (k+1)|x_{k+1} - x_k| < \infty\},$$

which is a Banach space. Also, we consider the Cesàro sequence space σ_∞, defined by

$$\sigma_\infty = \left\{x = (x_k)_{k=0}^{\infty} : \|x\|_{\sigma_\infty} = \sup_N \frac{1}{N+1}\left|\sum_{k=0}^{N} x_k\right| < \infty\right\}.$$

In this paper, we give attention to the Cesàro-type space σ_0, which is the Banach space defined by

$$\sigma_0 = \left\{x = (x_k)_{k=0}^{\infty} : \lim_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} x_k = 0\right\}$$

with the σ_∞-norm. Moreover, it is known that $\sigma_0^* \simeq h$ (cf. [Goes and Goes 1970, Theorem 3.7 (ii)]).

Throughout the paper, we adopt the following conventions:

- Suppose $\mathbb{N} = \{1,2,3,\dots\}$ and $\mathbb{N}_0 = \{0,1,2,\dots\}$.
- The set of real numbers and the set of complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively.
- Let X be an infinite-dimensional Banach space and write $\mathcal{B}(X)$ for the space of all bounded linear operators from X into itself. For an operator $T \in \mathcal{B}(X)$, its adjoint operator $T^* \in \mathcal{B}(X^*)$, where X^* is the dual space of X.
- In a sequence space, we typically represent the zero element as $\mathbf{0} = (0,0,0,\dots)$.
- The symbol $\emptyset$ denotes the empty set.
- For a nonzero real number r, define the closed disc $\overline{\Delta}_r$, circumference $\partial\Delta_r$ and open disc Δ_r as follows:
  $\overline{\Delta}_r := \{\lambda \in \mathbb{C} : |\lambda| \le |r|\}$,
  $\partial\Delta_r := \{\lambda \in \mathbb{C} : |\lambda| = |r|\}$
  and
  $\Delta_r := \{\lambda \in \mathbb{C} : |\lambda| < |r|\}$.
  When $r = 1$, the index is omitted.

To introduce our problem, consider the infinite-dimensional lower triangular double-band matrix $B(r,s) = (b_{nk})$ [Altay and Başar 2005];

$$b_{nk} = \begin{cases} r, & if\ k = n, \\ s, & if\ k = n-1, \\ 0, & otherwise, \end{cases}$$

where $r,\ s \in \mathbb{R}$ and $s \ne 0$. On a Banach sequence space μ, this matrix can be identified with a linear operator $B(r,s)\colon \mu \to \mu$;

$(B(r,s)x)_n = r x_n + s x_{n-1}$,
where $x = (x_n)_{n=0}^{\infty} \in \mu$, $n \in \mathbb{N}_0$.

The operator $B(r,s)$ is called the generalized difference operator. In fact, if $r = 1$ and $s = -1$, the operator $B(r,s)$ is reduced to the difference operator Δ [Altay and Başar 2004]. Also, for the case $s = 1 - r$, the operator $B(r,s)$ coincides with the Zweier operator Z^r [Altay and Karakuş 2005]. The spectral problem of the operator $B(r,s)$ has been extensively studied in various sequence spaces. Notable investigations include c_0, c [Altay and

Başar 2005], ℓ^p $(1 \leq p < \infty)$, bv_p $(1 \leq p < \infty)$ [Furkan and Bilgiç 2006, Bilgiç and Furkan 2008], cs [Dutta and Tripathy 2013], bv_0, h [EL-Shabrawy and Abu-Janah 2018] and ℓ^∞, bv [El-Shabrawy and Sawano 2021].

The transpose of the matrix $B(r,s)$ is denoted by $U(r,s)$, which can be identified with a linear operator in many sequence spaces [Karakaya and Altun 2010]. If $r = 1$ and $s = -1$, the operator $U(r,s)$ coincides with the operator Δ^+ [Dündar and Başar 2013]. In many investigations [Karakaya and Altun 2010, Tripathy and Das 2015], the spectra of the operator $U(r,s)$ have been studied in the Banach spaces c_0, c and cs.

Furthermore, for $r, s, q \in \mathbb{R}$ and $n, k \in \mathbb{N}_0$, we consider the tridiagonal matrix

$$T = T(r,q,s) = (t_{nk});$$

$$t_{nk} = \begin{cases} q, & if\ k = n, \\ s, & if\ k = n-1, \\ r, & if\ k = n+1, \\ 0, & otherwise. \end{cases}$$

This infinite matrix can be identified with a linear operator on a Banach sequence space μ as $T: \mu \to \mu$;

$$(T(r,q,s)x)_n = (Tx)_n$$
$$= sx_{n-1} + qx_n + rx_{n+1}, \qquad (1)$$

where $x = (x_n)_{n=0}^{\infty} \in \mu,\ n \in \mathbb{N}_0$.

If $s = r$, then $T(r,q,s)$ is reduced to the Jacobi matrix $J(q,r) = T(r,q,r)$ [Altun 2011, Berezanskii 1968, El-Shabrawy and Shindy 2020]. Furthermore, $B(r,s) = T(0,r,s)$ and $U(r,s) = T(s,r,0)$ are included in the class of $T(r,q,s)$; see [Altay and Başar 2005, Karakaya and Altun 2010]. So, it seems natural to firstly assume that $r,\ s \neq 0$. However, for either the case $r = 0$; or the case $s = 0$, see the conclusion in the last section. The spectra of $T(r,q,s)$ were determined in the spaces c_0, c, ℓ^1 and ℓ^∞ in [Bilgiç and Altun 2019]. More recently, this problem was studied in the Banach spaces h and bv_0 [El-Shabrawy and Shindy 2025].

To the authors' knowledge, the spectral problem has still not received enough attention in the Cesàro sequence space σ_0. So, in the current paper, we address the study of the spectra of the operators $B(r,s)$, $U(r,s)$ and $T(r,q,s)$ on σ_0. This investigation represents a natural continuation of the studies by Akhmedov and Başar (2006, 2007), Altay and Başar (2004, 2005), Karakaya and Altun 2010, Dutta and Tripathy 2013, Tripathy and Das 2015, El-Shabrawy and Abu-Janah 2018, Sawano and El-Shabrawy 2021, and El-Shabrawy and Shindy (2020, 2025).

Our work in the current paper is outlined in the following way: Section 2 provides a brief overview of basic definitions and facts related to the spectrum and various types of the spectrum. Section 3 focuses on the study of the spectra of the operators $B(r,s)$ and $U(r,s)$ acting on the sequence space σ_0. A generalization of the study to tridiagonal matrices has been obtained in Section 4. Finally, in the last section, a conclusion and future research are provided.

Preliminaries

To ensure the paper is self-contained, we briefly gather some basic definitions and preliminary facts which will be useful throughout the paper.

For any given $\lambda \in \mathbb{C}$ and $T \in \mathcal{B}(X)$, we write $T_\lambda = T - \lambda I$, where I is the identity operator on X. The *spectrum* of T, denoted by $\sigma(T,X)$, is the set of all scalars $\lambda \in \mathbb{C}$ for which T_λ is not bijective. Its complement in $\mathbb{C}$ is known as the *resolvent set* of T, denoted by $\rho(T,X)$. The spectrum $\sigma(T,X)$ can be partitioned into various subsets, classified according to the properties of $\mathcal{R}(T_\lambda)$ and the bounded invertibility of the operator T_λ. The *point spectrum* $\sigma_p(T,X)$ of T is defined by

$$\sigma_p(T,X) = \{\lambda \in \mathbb{C} :\ T_\lambda \text{ is not injective}\};$$

the *residual spectrum* $\sigma_r(T,X)$ of T is defined by

$$\sigma_r(T,X) = \{\lambda \in \mathbb{C} :\ T_\lambda \text{ is injective, but } \mathcal{R}(T_\lambda) \text{ is not dense}\};$$

the *continuous spectrum* $\sigma_c(T,X)$ of T is defined by

$$\sigma_c(T,X) = \{\lambda \in \mathbb{C} :\ T_\lambda \text{ is injective and } \mathcal{R}(T_\lambda) \text{ is dense, but } T_\lambda^{-1} \text{ is unbounded}\}.$$

Following [Appell et al. 2004] three more subsets of the spectrum can be defined as follows:

$$\sigma_\delta(T,X) = \{\lambda \in \mathbb{C} :\ T_\lambda \text{ is not surjective}\};$$

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} :\ \mathcal{R}(T_\lambda) \text{ is not dense}\};$$

$$\sigma_{ap}(T,X) = \{\lambda \in \mathbb{C} : \exists\ (x_k) \text{ in } X \text{ such that}$$

$\|x_k\| = 1 \ \forall \ k \in \mathbb{N}, \lim_{k\to\infty} \|T_\lambda x_k\| = 0\Big\}$,

which are called the *defect spectrum*, *compression spectrum* and *approximate point spectrum*, respectively. Note that these subsets of the spectrum overlap and

$$\sigma(T,X) = \sigma_{\mathrm{ap}}(T,X) \cup \sigma_\delta(T,X) \\ = \sigma_{\mathrm{ap}}(T,X) \cup \sigma_{\mathrm{co}}(T,X).$$

Another important classification of the spectrum, which is due to [Taylor and Halberg 1957], is also considered. To be more precise, let T be a linear operator on a Banach space X into itself. The operator T_λ is classified I, II or III, according as $\mathcal{R}(T_\lambda) = X$; $\overline{\mathcal{R}(T_\lambda)} = X$, but $\mathcal{R}(T_\lambda) \neq X$; or $\overline{\mathcal{R}(T_\lambda)} \neq X$. Furthermore, T_λ is classified 1, 2 or 3 according as T_λ^{-1} exists and is bounded; exists, but is not bounded; or does not exist. By combining these possibilities, we obtain different states of the operator. If $T \in \mathcal{B}(X)$, the complex plane is subdivided into parts corresponding to the states of the operator T_λ; $\mathrm{I}_1\sigma(T,X)$, $\mathrm{I}_2\sigma(T,X)$, $\mathrm{I}_3\sigma(T,X)$, $\mathrm{II}_1\sigma(T,X)$, $\mathrm{II}_2\sigma(T,X)$, $\mathrm{II}_3\sigma(T,X)$, $\mathrm{III}_1\sigma(T,X)$, $\mathrm{III}_2\sigma(T,X)$ and $\mathrm{III}_3\sigma(T,X)$. Consequently, we obtain a complete disjoint subdivision of the spectrum. Precisely, the following relations hold:

$$\sigma(T,X) = \mathrm{I}_3\sigma(T,X) \cup \mathrm{II}_2\sigma(T,X) \cup \mathrm{II}_3\sigma(T,X) \\ \cup \mathrm{III}_1\sigma(T,X) \cup \mathrm{III}_2\sigma(T,X) \\ \cup \mathrm{III}_3\sigma(T,X);$$

$$\sigma_p(T,X) = \mathrm{I}_3\sigma(T,X) \cup \mathrm{II}_3\sigma(T,X) \\ \cup \mathrm{III}_3\sigma(T,X);$$

$$\sigma_r(T,X) = \mathrm{III}_1\sigma(T,X) \cup \mathrm{III}_2\sigma(T,X);$$

$$\sigma_c(T,X) = \mathrm{II}_2\sigma(T,X).$$

It should be noted that $\mathrm{II}_1\sigma(T,X) = \emptyset$ since any boundedly invertible operator on a Banach space into itself should have a closed range (cf. [Taylor and Halberg 1957, Theorem 10]). Furthermore, $\mathrm{I}_2\sigma(T,X) = \emptyset$ as a consequence of the closed graph theorem. We observe that $\lambda \in \rho(T,X)$ if and only if $T_\lambda \in \mathrm{I}_1\sigma(T,X)$; otherwise $\lambda \in \sigma(T,X)$.

From the definition, we notice that

$$\sigma_\delta(T,X) = \sigma(T,X) \backslash \mathrm{I}_3\sigma(T,X).$$

Also, we have

$$\sigma_{\mathrm{ap}}(T,X) = \sigma(T,X) \backslash \mathrm{III}_1\sigma(T,X)$$

(cf. [Taylo and Lay 1986, p. 282]).

It is worthwhile to assert that, if $T \in \mathcal{B}(\sigma_0)$ is represented by a matrix A, then its adjoint T^* $\in \mathcal{B}(\sigma_0^*)$ is represented by the transpose matrix A^t; see [Taylo and Lay 1986, Problem 7, P.233].

For the sake of simplicity for the reader, we recall the following theorems which are concerned with the spectra of the operators $U(r,s)$ and $B(r,s)$ on the Hahn sequence space h. In fact, these results are crucial in the sequel.

Theorem 2.1. *[El-Shabrawy and Shindy 2025, Theorem 3.4] We have $U(r,s) \in \mathcal{B}(h)$. Moreover, the following results are satisfied:*

(1): $\sigma(U(r,s),h) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(2): $\sigma_p(U(r,s),h) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(3): $\sigma_p(U(r,s)^*,h^*) = \emptyset$.
(4): $\sigma_r(U(r,s),h) = \emptyset$.
(5): $\sigma_c(U(r,s),h) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.

Theorem 2.2. *[EL-Shabrawy and Abu-Janah 2018] We have $B(r,s) \in \mathcal{B}(h)$. Moreover, the following results are satisfied:*

(1): $\sigma(B(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(2): $\sigma_{\mathrm{p}}(B(r,s),\mathrm{h}) = \emptyset$.
(3): $\sigma_{\mathrm{p}}(B(r,s)^*,\mathrm{h}^*) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(4): $\sigma_{\mathrm{r}}(B(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(5): $\sigma_{\mathrm{c}}(B(r,s),\mathrm{h}) = \emptyset$.
(6): $\sigma_{\mathrm{ap}}(B(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(7): $\sigma_\delta(B(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(8): $\sigma_{\mathrm{co}}(B(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

Furthermore, we report on some recent results concerning the spectra of the operator T acting on the Hahn space h.

Theorem 2.3. *[El-Shabrawy and Shindy 2025] For $|r| < |s|$, we have $T \in \mathcal{B}(h)$. Moreover, the following results are satisfied:*

(1): $\sigma(T,\mathrm{h}) = Q\left(\overline{\Delta} \backslash \Delta_{\frac{r}{s}}\right)$.
(2): $\sigma_{\mathrm{p}}(T,\mathrm{h}) = \emptyset$.
(3): $\sigma_{\mathrm{p}}(T^*,\mathrm{h}^*) = Q\left(\overline{\Delta} \backslash \Delta_{\frac{r}{s}}\right)$.
(4): $\sigma_{\mathrm{r}}(T,\mathrm{h}) = Q\left(\overline{\Delta} \backslash \Delta_{\frac{r}{s}}\right)$.
(5): $\sigma_{\mathrm{c}}(T,\mathrm{h}) = \emptyset$.

Theorem 2.4. *[El-Shabrawy and Shindy 2025] For $|r| > |s|$, we have $T \in \mathcal{B}(h)$. Moreover, the following results are satisfied:*

(1): $\sigma(T,\mathrm{h}) = Q\left(\overline{\Delta}_{\frac{r}{s}} \backslash \Delta\right)$.
(2): $\sigma_{\mathrm{p}}(T,\mathrm{h}) = Q\left(\Delta_{\frac{r}{s}} \backslash \overline{\Delta}\right)$.
(3): $\sigma_{\mathrm{p}}(T^*,\mathrm{h}^*) = \emptyset$.

(4): $\sigma_{\mathrm{r}}(T,\mathrm{h}) = \emptyset$.
(5): $\sigma_{\mathrm{c}}(T,\mathrm{h}) = Q\left(\partial\Delta \cup \partial\Delta_{\frac{r}{s}}\right)$.

Theorem 2.5. *[El-Shabrawy and Shindy 2025] For $|r| = |s|$, we have $T \in \mathcal{B}(h)$. Moreover, the following results are satisfied:*
(1): $\sigma(T,\mathrm{h}) = Q(\partial\Delta)$.
(2): $\sigma_{\mathrm{p}}(T,\mathrm{h}) = \emptyset$.
(3): $\sigma_{\mathrm{p}}(T^*,\mathrm{h}^*) = \begin{cases} Q(\partial\Delta\backslash\{1\}), & \text{if } r = s, \\ Q(\partial\Delta), & \text{if } r = -s. \end{cases}$
(4): $\sigma_{\mathrm{r}}(T,\mathrm{h}) = \begin{cases} Q(\partial\Delta\backslash\{1\}), & \text{if } r = s, \\ Q(\partial\Delta), & \text{if } r = -s. \end{cases}$
(5): $\sigma_{\mathrm{c}}(T,\mathrm{h}) = \begin{cases} Q(\{1\}), & \text{if } r = s, \\ \emptyset, & \text{if } r = -s. \end{cases}$

Spectra of the operators $B(r,s)$ and $U(r,s)$ on σ_0

In this section, we completely determine the spectrum and various parts of the spectrum of the operators $B(r,s)$ and $U(r,s)$ on the Cesàro space σ_0. Firstly, the following theorem, which completes the results in Theorem 2.1 is given. It is necessary for our proofs in the current section.

Theorem 3.1. *We have the following results:*
(1): $\sigma_{\mathrm{ap}}(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(2): $\sigma_{\delta}(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(3): $\sigma_{\mathrm{co}}(U(r,s),\mathrm{h}) = \emptyset$.
Proof.
(1): By utilizing Theorem 2.1(4), we obtain that $\mathrm{III}_1\sigma(U(r,s),\mathrm{h}) = \emptyset$. Combined this with the fact that
$$\sigma_{\mathrm{ap}}(U(r,s),\mathrm{h}) = \sigma(U(r,s),\mathrm{h})\backslash\mathrm{III}_1\sigma(U(r,s),\mathrm{h}),$$
implies
$\sigma_{\mathrm{ap}}(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$, where we have used the result in Theorem 2.1(1).
(2): In fact, we have
$$\mathrm{I}_3\sigma(U(r,s),\mathrm{h}) \subseteq \sigma_p(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}.$$
Conversely, let $\lambda \in \mathbb{C}$ such that $|\lambda - r| < |s|$. Then, $U(r,s) - \lambda I$ is not injective. Furthermore, from [El-Shabrawy and Shindy 2025, Proposition 3.2(3)], $U(r,s) - \lambda I$ is surjective. Then $\lambda \in \mathrm{I}_3\sigma(U(r,s),\mathrm{h})$. This concludes that, $\mathrm{I}_3\sigma(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$. Thus,
$$\sigma_{\delta}(U(r,s),\mathrm{h}) = \sigma(U(r,s),\mathrm{h})\backslash\mathrm{I}_3\sigma(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\},$$
where we have used the result in Theorem 2.1(1).
(3): Follows immediately from the relation $\sigma_{\mathrm{co}}(U(r,s),\mathrm{h}) = \sigma_p(U(r,s)^*,\mathrm{h}^*)$ and using Theorem 2.1(3).

The next is our first main theorem.

Theorem 3.2. *We have $B(r,s) \in \mathcal{B}(\sigma_0)$. Furthermore, the following statements are satisfied:*
(1): $\sigma(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(2): $\sigma_{\mathrm{p}}(B(r,s),\sigma_0) = \emptyset$.
(3): $\sigma_{\mathrm{p}}(B(r,s)^*,\sigma_0^*) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(4): $\sigma_{\mathrm{r}}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(5): $\sigma_{\mathrm{c}}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(6): $\sigma_{\mathrm{ap}}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(7): $\sigma_{\delta}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(8): $\sigma_{\mathrm{co}}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(9): $\mathrm{I}_3\sigma(B(r,s),\sigma_0) = \mathrm{II}_3\sigma(B(r,s),\sigma_0) = \mathrm{III}_3\sigma(B(r,s),\sigma_0) = \emptyset$.
(10): $\mathrm{II}_2\sigma(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(11): $\mathrm{III}_1\sigma(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(12): $\mathrm{III}_2\sigma(B(r,s),\sigma_0) = \emptyset$.
Proof.
(1): The result follows from [Appell et al. 2004, Proposition 1.3] and Theorem 2.1(1). In fact, we have
$$\sigma(B(r,s),\sigma_0) = \sigma(B(r,s)^*,\sigma_0^*) = \sigma(U(r,s),\mathrm{h}) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}.$$
(2): It can be shown that, for all $\lambda \in \mathbb{C}$, $(B(r,s) - \lambda)x = \mathbf{0}$ has only the zero solution for x.
(3): Since
$\sigma_{\mathrm{p}}(B(r,s)^*,\sigma_0^*) = \sigma_{\mathrm{p}}(U(r,s),\mathrm{h})$,
then applying Theorem 2.1(2) yields the desired result.
(4): Follows immediately from the relation
$$\sigma_{\mathrm{r}}(B(r,s),\sigma_0) = \sigma_{\mathrm{p}}(B(r,s)^*,\sigma_0^*)\backslash\sigma_{\mathrm{p}}(B(r,s),\sigma_0)$$
and then applying Statements (2) and (3).
(5): Since $\sigma_{\mathrm{p}}(B(r,s),\sigma_0)$, $\sigma_{\mathrm{r}}(B(r,s),\sigma_0)$ and $\sigma_{\mathrm{c}}(B(r,s),\sigma_0)$ form a disjoint subdivision of $\sigma(B(r,s),\sigma_0)$, then, by applying Statements (1), (2) and (4), we obtain that $\sigma_{\mathrm{c}}(B(r,s),\sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(6) − (7): Follow immediately from [Appell et al. 2004, Proposition 1.3] and using Theorem 3.1(1)-(2). Indeed, we have
$$\sigma_{\mathrm{ap}}(B(r,s),\sigma_0) = \sigma_{\delta}(B(r,s)^*,\sigma_0^*) = \sigma_{\delta}(U(r,s),\mathrm{h})$$

$= \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$
and
$\sigma_\delta(B(r,s), \sigma_0) = \sigma_{ap}(B(r,s)^*, \sigma_0^*)$
$= \sigma_{ap}(U(r,s), \mathrm{h})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(8): Follows immediately from [Appell et al. 2004, Proposition 1.3] and Statement (3). Indeed, we have
$\sigma_{co}(B(r,s), \sigma_0) = \sigma_p(B(r,s)^*, \sigma_0^*)$
$= \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

(9): The result follows from Statement (2) and the fact that
$\sigma_p(B(r,s), \sigma_0) = \mathrm{I}_3\sigma(B(r,s), \sigma_0)$
$\cup \mathrm{II}_3\sigma(B(r,s), \sigma_0) \cup \mathrm{III}_3\sigma(B(r,s), \sigma_0)$.

(10): Simply observe that
$\mathrm{II}_2\sigma(B(r,s), \sigma_0) = \sigma_c(B(r,s), \sigma_0)$
and then apply Statement (5).

(11): Let $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$. Since
$\sigma_r(B(r,s), \sigma_0)$
$= \mathrm{III}_1\sigma(B(r,s), \sigma_0) \cup \mathrm{III}_2\sigma(B(r,s), \sigma_0)$,
then, to show $\lambda \in \mathrm{III}_1\sigma(B(r,s), \sigma_0)$, it suffices to show that $B(r,s)^* - \lambda I$ is surjective [Taylor and Halberg 1957, Theorem 4]. This follows from the fact that
$\sigma_\delta(B(r,s)^*, \sigma_0^*) = \sigma_\delta(U(r,s), \mathrm{h})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$,
where we have used Theorem 3.1(2). Thus, we conclude that
$\{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$
$\subseteq \mathrm{III}_1\sigma(B(r,s), \sigma_0)$.
The second inclusion follows by using [Gindler and Taylor 1962, Theorem 3.3]. In fact, we have
$\mathrm{III}_1\sigma(B(r,s), \sigma_0)$
$\subseteq \mathrm{int}(\{\lambda \in \mathbb{C} : |\lambda - r| < |s|\})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

(12): Follows immediately.

Next, we give our second main theorem, which is concerned with the spectra of the operator $U(r,s)$ on the Cesàro space σ_0.

Theorem 3.3. *We have* $U(r,s) \in \mathcal{B}(\sigma_0)$. *Furthermore, the following statements are satisfied:*

(1): $\sigma(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(2): $\sigma_p(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \backslash \{r+s\}$.
(3): $\sigma_p(U(r,s)^*, \sigma_0^*) = \emptyset$.
(4): $\sigma_r(U(r,s), \sigma_0) = \emptyset$.
(5): $\sigma_c(U(r,s), \sigma_0) = \{r+s\}$.
(6): $\sigma_{ap}(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
(7): $\sigma_\delta(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.
(8): $\sigma_{co}(U(r,s), \sigma_0) = \emptyset$.
(9): $\mathrm{I}_3\sigma(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.
(10): $\mathrm{III}_3\sigma(U(r,s), \sigma_0) = \emptyset$.
(11): $\mathrm{II}_3\sigma(U(r,s), \sigma_0) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\} \backslash \{r+s\}$.
(12): $\mathrm{II}_2\sigma(U(r,s), \sigma_0) = \{r+s\}$.
(13): $\mathrm{III}_1\sigma(U(r,s), \sigma_0) = \mathrm{III}_2\sigma(U(r,s), \sigma_0) = \emptyset$.

Proof.

(1): The required result follows from [Appell et al. 2004, Proposition 1.3] and Theorem 2.2(1). In fact, we have
$\sigma(U(r,s), \sigma_0) = \sigma(U(r,s)^*, \sigma_0^*)$
$= \sigma(B(r,s), \mathrm{h})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(2): Firstly, we recall that,
$\sigma_p(U(r,s), \sigma_0) \subseteq \sigma(U(r,s), \sigma_0)$
$= \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.
Furthermore, suppose that
$(U(r,s) - \lambda I)x = \mathbf{0}$ for $x \neq \mathbf{0}$. Thus,
$$x_n = \left(\frac{\lambda - r}{s}\right)^n x_0, \qquad n \in \mathbb{N}.$$
Then, we should assume that $x_0 \neq 0$ and $\frac{\lambda - r}{s} \neq 1$ since otherwise we would obtain either $x = \mathbf{0}$ or $x \notin \sigma_0$. With this, if $|\lambda - r| \leq |s|$,
$$\sum_{k=0}^{n} x_k = x_0 \sum_{k=0}^{n} \left(\frac{\lambda - r}{s}\right)^k = x_0 \frac{1 - \left(\frac{\lambda - r}{s}\right)^{n+1}}{1 - \left(\frac{\lambda - r}{s}\right)}.$$
Then, $\frac{1}{n+1}\sum_{k=0}^{n} x_k \to 0$ as $n \to \infty$. That is, $x = (x_k) \in \sigma_0$ and so,
$\lambda \in \sigma_p(U(r,s), \sigma_0)$.

(3): The result follows immediately from the fact that
$\sigma_p(U(r,s)^*, \sigma_0^*) = \sigma_p(B(r,s), \mathrm{h})$
and then applying Theorem 2.2(2).

(4): Using the relation
$\sigma_r(U(r,s), \sigma_0)$
$= \sigma_p(U(r,s)^*, \sigma_0^*) \backslash \sigma_p(U(r,s), \sigma_0)$
along with Statements (2) and (3), the required result follows.

(5): The result follows based on the fact that $\sigma_p(U(r,s), \sigma_0)$, $\sigma_r(U(r,s), \sigma_0)$ and $\sigma_c(U(r,s), \sigma_0)$ form a disjoint partition of $\sigma(U(r,s), \sigma_0)$, and then applying the results in Statements (1), (2) and (4).

(6) - (7): Follow immediately from [Appell et al. 2004, Proposition 1.3] and then using Theorem 2.2 (6)-(7). Indeed, we have
$\sigma_{ap}(U(r,s), \sigma_0) = \sigma_\delta(U(r,s)^*, \sigma_0^*)$
$= \sigma_\delta(B(r,s), \mathrm{h})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$

and
$\sigma_\delta(U(r,s),\sigma_0) = \sigma_{\rm ap}(U(r,s)^*,\sigma_0^*)$
$= \sigma_{\rm ap}(B(r,s),{\rm h})$
$= \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$.

(8): It follows from [Appell et al. 2004, Proposition 1.3] and Statement (3) that $\sigma_{\rm co}(U(r,s),\sigma_0) = \sigma_p(U(r,s)^*,\sigma_0^*) = \emptyset$.

(9): It is known that $I_3\sigma(U(r,s),\sigma_0) \subseteq \sigma_{\rm p}(U(r,s),\sigma_0)$. Then, applying [Gindler and Taylor 1962, Theorem 4.2], we obtain $I_3\sigma(U(r,s),\sigma_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$. Conversely, let $\lambda \in \mathbb{C}$ such that $|\lambda - r| < |s|$. Then $\lambda \notin \sigma_\delta(U(r,s),\sigma_0)$ and $\lambda \in \sigma_{\rm p}(U(r,s),\sigma_0)$. This implies that $U(r,s) - \lambda I$ is surjective and not injective. Consequently $\lambda \in I_3\sigma(U(r,s),\sigma_0)$. This completes the proof of the statement.

(10): Clearly, $III_3\sigma(U(r,s),\sigma_0) \subseteq \sigma_{\rm p}(U(r,s),\sigma_0)$ $= \{\lambda \in \mathbb{C} : |\lambda - r| \le |s|\} \backslash \{r+s\}$. Conversely, for all $\lambda \in \mathbb{C}$ such that $|\lambda - r| \le |s|$ and $\lambda \ne r+s$, we have $\lambda \in \sigma_{\rm p}(U(r,s),\sigma_0)$ and $\lambda \notin \sigma_{\rm p}(U(r,s)^*,\sigma_0^*)$. This implies that $U(r,s) - \lambda I$ is not injective and $U(r,s) - \lambda I$ has a dense range (cf. [Taylor and Halberg 1957, Theorem 1]). Consequently, $\lambda \notin III_3\sigma(U(r,s),\sigma_0)$. Thus, $III_3\sigma(U(r,s),\sigma_0) = \emptyset$.

(11): From the definition of $II_3\sigma(U(r,s),\sigma_0)$ and using Statements (2), (9) and (10), we obtain
$II_3\sigma(U(r,s),\sigma_0)$
$= \sigma_{\rm p}(U(r,s),\sigma_0) \backslash$
$[I_3\sigma(U(r,s),\sigma_0) \cup III_3\sigma(U(r,s),\sigma_0)]$
$= \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\} \backslash \{r+s\}$.

(12): Follows from the fact that $II_2\sigma(U(r,s),\sigma_0) = \sigma_{\rm c}(U(r,s),\sigma_0)$ and use Statement (5).

(13): It is known that $\sigma_{\rm r}(U(r,s),\sigma_0) = III_1\sigma(U(r,s),\sigma_0) \cup III_2\sigma(U(r,s),\sigma_0)$. With this and the result in Statement (4), we obtain the required result.

Spectra of the operator $T(r,q,s)$ on σ_0

For the sake of brevity, if there is no confusion, we sometimes use T instead of $T(r,q,s)$, especially when combined with another symbol. Now, the method on which we proceed in order to find the spectra of T depends on examining the injectivity and surjectivity of the complex functions;

$Q(z) = sz + q + rz^{-1}$ and

$P(z) = rz + q + sz^{-1}$,

where r, q and s are fixed real numbers with $r, s \ne 0$. Observe that, if α_1 and α_2 are the roots of Q, they are nonzero, and α_1^{-1} and α_2^{-1} are the roots of P. Furthermore, the following relations are satisfied:

$$\alpha_1 + \alpha_2 = \frac{-q}{s} \quad \text{and} \quad \alpha_1\alpha_2 = \frac{r}{s}. \tag{2}$$

The right-shift operator and the left-shift operator are defined by $(Rx)_n = x_{n-1}$ and $(Lx)_n = x_{n+1}$, respectively: they yield a factorization of the operator T;

$$T = s(I - \alpha_1 L) \circ (R - \alpha_2 I), \tag{3}$$

where α_1 and α_2 are the roots of the function Q and interchangeably. As an immediate consequence, the boundedness of T on σ_0 follows from the boundedness of R and L.

We need the following proposition.

Proposition 4.1. *$T \in (\sigma_0 : \sigma_0)$ is injective if and only if one of the following conditions holds:*
(i): *the function Q has a root inside Δ,*
(ii): *either 1 or -1 is a double root of Q.*

Proof.
The operator T is not injective if and only if there exists $x = (x_n) \ne \mathbf{0}$ in σ_0 with $Tx = \mathbf{0}$. From [Bilgiç and Altun 2019, Lemma 1.1], the solution of $Tx = \mathbf{0}$ is given by

$$x_n = \begin{cases} c\left(\frac{\alpha_2}{\alpha_1^n} - \frac{\alpha_1}{\alpha_2^n}\right), & if\ \alpha_1 \ne \alpha_2, \\ c\,\frac{1+n}{\alpha^n}, & if\ \alpha_1 = \alpha_2 = \alpha, \end{cases}$$

where α_1 and α_2 are the roots of the function Q. So, in the forthcoming, we will validate the result by considering three cases for the possibilities of α_1 and α_2.

(i): Suppose $\alpha_1 \ne \alpha_2$ and $|\alpha_1| = |\alpha_2| = |\alpha|$. We may assume that $\alpha_1 = |\alpha|(\cos\theta + i\sin\theta)$ and $\alpha_2 = |\alpha|(\cos\theta - i\sin\theta)$ for some $0 < \theta < \pi$. Then
$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} x_k = \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} c\left(\frac{|\alpha|(\cos\theta - i\sin\theta)}{|\alpha|^k(\cos\theta + i\sin\theta)^k} - \frac{|\alpha|(\cos\theta + i\sin\theta)}{|\alpha|^k(\cos\theta - i\sin\theta)^k}\right)$$

$= -2ic \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{|\alpha|^{k-1}} \sin\big((k+1)\theta\big) = 0$

for $|\alpha_1| = |\alpha_2| = |\alpha| \geq 1$. However, for $|\alpha_1| = |\alpha_2| = |\alpha| < 1$, we have $x = (x_n) \notin \sigma_0$ [El-Shabrawy and Shindy 2025, Proposition 2.2(1)]. Thus, T is not injective in σ_0 if and only if $|\alpha_1| = |\alpha_2| = |\alpha| \geq 1$.

(ii): Suppose $\alpha_1 \neq \alpha_2$ and $|\alpha_1| \neq |\alpha_2|$. We have

$$\sum_{k=0}^{n} x_k = c \sum_{k=0}^{n} \left(\frac{\alpha_2}{\alpha_1^k} - \frac{\alpha_1}{\alpha_2^k}\right) = c\left[\sum_{k=0}^{n} \frac{\alpha_2}{\alpha_1^k} - \sum_{k=0}^{n} \frac{\alpha_1}{\alpha_2^k}\right].$$

Hence, we have the following two cases:

(a): If $|\alpha_1| > |\alpha_2| \geq 1$ or $|\alpha_2| > |\alpha_1| \geq 1$, then $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k = 0$.

(b): If $|\alpha_1| < |\alpha_2| \leq 1$, $|\alpha_2| < |\alpha_1| \leq 1$, $|\alpha_1| < 1 < |\alpha_2|$ or $|\alpha_2| < 1 < |\alpha_1|$, then $x = (x_n) \notin \sigma_\infty$. So, $x = (x_n) \notin \sigma_0$.

Thus, in Case (ii), T is not injective in σ_0 if and only if $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$.

(iii): Suppose $\alpha_1 = \alpha_2 = \alpha$. Then, α_1 and α_2 are real. So, we study three cases:

(a): If $|\alpha| > 1$, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k = c \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \frac{1+k}{\alpha^k} = 0.$$

(b): If $|\alpha| < 1$, then $x = (x_n) \notin \sigma_\infty$. So, $x = (x_n) \notin \sigma_0$.

(c): If $|\alpha| = 1$, then $\alpha = -1$ or $\alpha = 1$. For $\alpha = -1$, we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k = c \lim_{n\to\infty} \frac{1}{n+1} \begin{cases} \frac{n+2}{2}, & \text{if } n \text{ is even}, \\ -\frac{n+1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

So,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k \neq 0.$$

However, for $\alpha = 1$, we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k = \frac{c}{2} \lim_{n\to\infty} (n+2).$$

From Cases (a), (b) and (c), we conclude that, in the case where $\alpha_1 = \alpha_2$, T is not injective in σ_0 if and only if $|\alpha_1| = |\alpha_2| > 1$.

As an immediate result of Proposition 4.1, we have the following:

Corollary 4.1. *$T - \lambda I \in (\sigma_0 : \sigma_0)$ is injective if and only if one of the following conditions holds:*

(i): *the function $Q - \lambda$ has a root inside Δ,*

(ii): *either 1 or -1 is a double root of $Q - \lambda$.*

Now, we consider the following lemma for the right-shift operator R:

Lemma 4.2. *Let $\alpha \in \mathbb{C}$. Then, $R - \alpha I \in (\sigma_0 : \sigma_0)$ is surjective if and only if $\alpha \notin \overline{\Delta}$.*

Proof. The result follows directly from Theorem 3.2(7).

As a consequence of Theorem 3.3(7), we establish the following lemma.

Lemma 4.3. *Let $\alpha \in \mathbb{C}$. Then, $I - \alpha L \in (\sigma_0 : \sigma_0)$ is surjective if and only if $\alpha \notin \partial\Delta$.*

With the help of Lemmas 4.2 and 4.3, we can introduce the following analogy to [El-Shabrawy and Shindy 2025, Proposition 2.3], whose proof can be derived in a similar manner.

Proposition 4.2. *$T \in (\sigma_0 : \sigma_0)$ is surjective if and only if the roots of the function Q do not lie on $\partial\Delta$ and at least one root of Q is outside $\overline{\Delta}$.*

Consequently, from Proposition 4.2, we have the following corollary.

Corollary 4.2. *$T - \lambda I \in (\sigma_0 : \sigma_0)$ is surjective if and only if the roots of $Q - \lambda$ do not lie on $\partial\Delta$ and at least one root of $Q - \lambda$ is outside $\overline{\Delta}$.*

Next, we give our first main theorem on the spectra of the operator T.

Theorem 4.1 *For $|r| < |s|$, we have $T \in \mathcal{B}(\sigma_0)$. Furthermore, the following statements hold:*

(1): $\sigma(T, \sigma_0) = Q\left(\overline{\Delta} \backslash \Delta_{\frac{r}{s}}\right)$.

(2): $\sigma_p(T, \sigma_0) = \emptyset$.

(3): $\sigma_p(T^*, \sigma_0^*) = Q\left(\Delta \backslash \overline{\Delta}_{\frac{r}{s}}\right)$.

(4): $\sigma_r(T, \sigma_0) = Q\left(\Delta \backslash \overline{\Delta}_{\frac{r}{s}}\right)$.

(5): $\sigma_c(T, \sigma_0) = Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$.

(6): $\mathrm{I}_3\sigma(T, \sigma_0) = \mathrm{II}_3\sigma(T, \sigma_0) = \mathrm{III}_3\sigma(T, \sigma_0) = \emptyset$.

(7): $\mathrm{II}_2\sigma(T, \sigma_0) = Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$.

(8): $\mathrm{III}_1\sigma(T, \sigma_0) = Q\left(\Delta \backslash \overline{\Delta}_{\frac{r}{s}}\right)$.

(9): $\mathrm{III}_2\sigma(T, \sigma_0) = \emptyset$.

(10): $\sigma_{ap}(T, \sigma_0) = Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$.

(11):$\sigma_\delta(T,\sigma_0)=Q\left(\overline{\Delta}\backslash\Delta_{\frac{r}{s}}\right)$.

(12):$\sigma_{co}(T,\sigma_0)=Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right)$.

Proof.

(1): From [Appell et al. 2004, Proposition 1.3], we have $\sigma(T,\sigma_0)=\sigma(T^*,\sigma_0^*)=\sigma(T^*,h)$, where T^* is the transpose of T. Now, by applying Theorem 2.4(1), with swapping r and s, we thereby obtain $\sigma(T,\sigma_0)=P\left(\overline{\Delta}_{\frac{s}{r}}\backslash\Delta\right)=Q\left(\overline{\Delta}\backslash\Delta_{\frac{r}{s}}\right)$, where swapping r and s implies replacing Q by P.

(2): Since the product of the two roots of $Q-\lambda$ equals $\frac{r}{s}$, then, $Q-\lambda$ has a root inside Δ. By Corollary 4.1, $T-\lambda I$ is injective. Therefore, T has no eigenvalues in σ_0, so that $\sigma_p(T,\sigma_0)=\emptyset$.

(3): Follows immediately from Theorem 2.4(2). Indeed, we have

$$\sigma_p(T^*,\sigma_0^*)=\sigma_p(T^*,h)=Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right).$$

(4): By using the relation $\sigma_r(T,\sigma_0)=\sigma_p(T^*,\sigma_0^*)\backslash\sigma_p(T,\sigma_0)$ and applying Statements (2) and (3), the result follows directly.

(5): It is known that

$$\begin{aligned}\sigma_c(T,\sigma_0)&=\sigma(T,\sigma_0)\backslash\left[\sigma_p(T,\sigma_0)\cup\sigma_r(T,\sigma_0)\right]\\&=Q\left(\overline{\Delta}\backslash\Delta_{\frac{r}{s}}\right)\backslash Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right)\\&\subseteq Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right).\end{aligned}$$

Conversely, let $\lambda\in Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right)$. Then, $\lambda\in P\left(\partial\Delta_{\frac{s}{r}}\cup\partial\Delta\right)$. Therefore, there exists a root γ of $P-\lambda$ such that $\gamma\in\partial\Delta_{\frac{s}{r}}\cup\partial\Delta$. That is $|\gamma|=\left|\frac{s}{r}\right|$ or $|\gamma|=1$. Therefore, there exists a root of $P-\lambda$, which lies on $\partial\Delta$. Hence, by [El-Shabrawy and Shindy 2025, Proposition 2.1], $T^*-\lambda I$ is injective, and then, $\lambda\notin\sigma_p(T^*,\sigma_0^*)$. So, $\lambda\notin\sigma_r(T,\sigma_0)$. This implies that $\lambda\in\sigma_c(T,\sigma_0)$. Consequently,

$$Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right)\subseteq\sigma_c(T,\sigma_0).$$

Thus, we conclude that

$$\sigma_c(T,\sigma_0)=Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right).$$

(6): The result follows immediately from Statement (2) and the fact that $\sigma_p(T,\sigma_0)=\mathrm{I}_3\sigma(T,\sigma_0)\cup\mathrm{II}_3\sigma(T,\sigma_0)\cup\mathrm{III}_3\sigma(T,\sigma_0)$.

(7): Simply observe that $\mathrm{II}_2\sigma(T,\sigma_0)=\sigma_c(T,\sigma_0)$. It remains to apply Statement (5).

(8): We have

$$\mathrm{III}_1\sigma(T,\sigma_0)\subseteq\sigma_r(T,\sigma_0)=Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right).$$

Conversely, let $\lambda\in Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right)=P\left(\Delta_{\frac{s}{r}}\backslash\overline{\Delta}\right)$. In this case, there exists a root γ of $P-\lambda$ such that $\gamma\in\Delta_{\frac{s}{r}}\backslash\overline{\Delta}$. From [El-Shabrawy and Shindy 2025, Proposition 2.3], $T^*-\lambda I$ is surjective. That is $T-\lambda I$ has a bounded inverse (cf. [Taylor and Halberg 1957, Theorem 4]). Additionally, we have $\lambda\in\sigma_p(T^*,\sigma_0^*)$, which implies that $T^*-\lambda I$ is not injective. Then, $T-\lambda I$ does not have a dense range (cf. [Taylor and Halberg 1957, Theorem 1]). That is we have $\lambda\in\mathrm{III}_1\sigma(T,\sigma_0)$. Thus,

$$\mathrm{III}_1\sigma(T,\sigma_0)=Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right).$$

(9): Based on the fact that $\mathrm{III}_2\sigma(T,\sigma_0)=\sigma_r(T,\sigma_0)\backslash\mathrm{III}_1\sigma(T,\sigma_0)$ and then applying Statements (4) and (8), the result follows immediately.

(10): We have

$$\begin{aligned}\sigma_\delta(T^*,\sigma_0^*)=\sigma_{ap}(T,\sigma_0)&=\sigma(T,\sigma_0)\backslash\mathrm{III}_1\sigma(T,\sigma_0)\\&=Q\left(\overline{\Delta}\backslash\Delta_{\frac{r}{s}}\right)\backslash Q\left(\Delta\backslash\overline{\Delta}_{\frac{r}{s}}\right)\\&\subseteq Q\left(\partial\Delta\cup\partial\Delta_{\frac{r}{s}}\right).\end{aligned}$$

Conversely, let

$$\lambda\in Q\left(\partial\Delta\cup\partial\Delta_{\frac{r}{s}}\right)=P\left(\partial\Delta\cup\partial\Delta_{\frac{s}{r}}\right).$$

Then, there exists a root γ of $P-\lambda$ such that $\gamma\in\partial\Delta\cup\partial\Delta_{\frac{s}{r}}$. Therefore, $P-\lambda$ should have a root on $\partial\Delta$. By [El-Shabrawy and Shindy 2025, Proposition 2.3], $T^*-\lambda I$ is not surjective. So, $\lambda\in\sigma_\delta(T^*,\sigma_0^*)$. Therefore,

$$\begin{aligned}Q\left(\partial\Delta\cup\partial\Delta_{\frac{r}{s}}\right)&\subseteq\sigma_\delta(T^*,\sigma_0^*)\\&=\sigma_{ap}(T,\sigma_0).\end{aligned}$$

As a result

$$\sigma_{ap}(T,\sigma_0)=\sigma_\delta(T^*,\sigma_0^*)$$

$$= Q\left(\partial\Delta \cup \partial\Delta_{\frac{r}{s}}\right).$$

(11): The result follows directly from the fact that
$$\sigma_\delta(T,\sigma_0) = \sigma(T,\sigma_0)\backslash \mathrm{I}_3\sigma(T,\sigma_0) = Q\left(\overline{\Delta}\backslash\Delta_{\frac{r}{s}}\right).$$

(12): It is a direct consequence of Statement (3) and the fact that
$$\sigma_{\mathrm{co}}(T,\sigma_0) = \sigma_{\mathrm{p}}(T^*,\sigma_0^*).$$

Theorem 4.2. *For $|r| > |s|$, we have $T \in \mathcal{B}(\sigma_0)$. Furthermore, the following statements hold:*

(1): $\sigma(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)$.

(2): $\sigma_{\mathrm{p}}(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)$.

(3): $\sigma_{\mathrm{p}}(T^*,\sigma_0^*) = \emptyset$.

(4): $\sigma_{\mathrm{r}}(T,\sigma_0) = \emptyset$.

(5): $\sigma_{\mathrm{c}}(T,\sigma_0) = \emptyset$.

(6): $\mathrm{I}_3\sigma(T,\sigma_0) = Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right)$.

(7): $\mathrm{II}_3\sigma(T,\sigma_0) = Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$.

(8): $\mathrm{III}_3\sigma(T,\sigma_0) = \emptyset$.

(9): $\mathrm{II}_2\sigma(T,\sigma_0) = \emptyset$.

(10): $\mathrm{III}_1\sigma(T,\sigma_0) = \mathrm{III}_2\sigma(T,\sigma_0) = \emptyset$.

(11): $\sigma_{\mathrm{ap}}(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)$.

(12): $\sigma_\delta(T,\sigma_0) = Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$.

(13): $\sigma_{\mathrm{co}}(T,\sigma_0) = \emptyset$.

Proof.

(1): From Theorem 2.3(1), we have
$$\sigma(T,\sigma_0) = \sigma(T^*,\sigma_0^*) = \sigma(T^*,\mathrm{h}) = P\left(\overline{\Delta}\backslash\Delta_{\frac{s}{r}}\right) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right).$$

(2): Firstly, we have
$$\sigma_{\mathrm{p}}(T,\sigma_0) \subseteq \sigma(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right).$$
Conversely, suppose that $\lambda \in Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)$. Then, there exists a root β of $Q-\lambda$ such that $\beta \in \overline{\Delta}_{\frac{r}{s}}\backslash\Delta$. So, $1 \le |\beta| \le \left|\frac{r}{s}\right|$. Therefore, the two roots of $Q-\lambda$ lie outside Δ. Since it can never happen that 1 or -1 is a double root of $Q-\lambda$, then, by Corollary 4.1, $T-\lambda I$ is not injective, and then, $\lambda \in \sigma_{\mathrm{p}}(T,\sigma_0)$. Thus, we conclude that
$$\sigma_{\mathrm{p}}(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right).$$

(3): Follows immediately from Theorem 2.3(2). Indeed, we have
$$\sigma_{\mathrm{p}}(T^*,\sigma_0^*) = \sigma_{\mathrm{p}}(T^*,\mathrm{h}) = \emptyset.$$

(4): By using the relation
$$\sigma_{\mathrm{r}}(T,\sigma_0) = \sigma_{\mathrm{p}}(T^*,\sigma_0^*)\backslash\sigma_{\mathrm{p}}(T,\sigma_0)$$
and applying Statement (3), the result follows immediately.

(5): Follows immediately.

(6): Let $\lambda \in \mathrm{I}_3\sigma(T,\sigma_0)$. Then, $T-\lambda I$ is surjective and not injective. By, Corollaries 4.1 and 4.2, it follows that both roots of Q must be outside $\overline{\Delta}$. This implies that $Q-\lambda$ has a root β satisfying $1 < |\beta| < \left|\frac{r}{s}\right|$. Hence, $\beta \in \Delta_{\frac{r}{s}}\backslash\overline{\Delta}$, and so, $\lambda = Q(\beta) \in Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right)$. Thus, we conclude that
$$\mathrm{I}_3\sigma(T,\sigma_0) \subseteq Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right).$$
Conversely, let $\lambda \in Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right)$. Then, there exists a root β of $Q-\lambda$ such that $1 < |\beta| < \left|\frac{r}{s}\right|$. So, by Corollaries 4.1 and 4.2, $T-\lambda I$ is surjective and not injective. Thus $\lambda \in \mathrm{I}_3\sigma(T,\sigma_0)$. So, $Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right) \subseteq \mathrm{I}_3\sigma(T,\sigma_0)$.
This ends the proof of Statement (6).

(7)- (8): Simply observe
$$\mathrm{II}_3\sigma(T,\sigma_0) \cup \mathrm{III}_3\sigma(T,\sigma_0) = \sigma_{\mathrm{p}}(T,\sigma_0)\backslash \mathrm{I}_3\sigma(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)\backslash Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right) \subseteq Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right).$$
Conversely, let $\lambda \in Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$. Then, $\lambda \in P\left(\partial\Delta_{\frac{s}{r}} \cup \partial\Delta\right)$. Therefore, there exists a root γ of $P-\lambda$ such that $\gamma \in \partial\Delta_{\frac{s}{r}} \cup \partial\Delta$. This implies that, $P-\lambda$ has the two roots inside $\overline{\Delta}$. So, by [El-Shabrawy and Shindy 2025, Proposition 2.1], $T^*-\lambda I$ is injective. From [Taylor and Halberg 1957, Theorem 1], $T-\lambda I$ has a dense range. On the other hand, $\lambda \in Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right)$ implies that there exists a root β of $Q-\lambda$ such that $|\beta| = 1$ or $|\beta| = \left|\frac{r}{s}\right|$. Therefore, one of the roots of $Q-\lambda$

lies on $\partial\Delta$ and the other one is not inside Δ. So, by Corollaries 4.1 and 4.2, $T-\lambda I$ is neither surjective nor injective.
Consequently, $\lambda \in \mathrm{II}_3\sigma(T,\sigma_0)$. Thus,
$Q\left(\partial\Delta_{\frac{r}{s}} \cup \partial\Delta\right) = \mathrm{II}_3\sigma(T,\sigma_0)$
and $\mathrm{III}_3\sigma(T,\sigma_0)=\emptyset$.

(9): Simply observe that
$\mathrm{II}_2\sigma(T,\sigma_0)=\sigma_c(T,\sigma_0)$.
It remains to apply Statement (5).

(10): Since
$\sigma_{\mathrm{r}}(T,\sigma_0)=\mathrm{III}_1\sigma(T,\sigma_0)\cup \mathrm{III}_2\sigma(T,\sigma_0)$, then the desired result follows by applying Statement (4).

(11): Indeed, we have
$$\sigma_{\mathrm{ap}}(T,\sigma_0)=\sigma(T,\sigma_0)\backslash \mathrm{III}_1(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right).$$

(12): We have
$$\sigma_\delta(T,\sigma_0)=\sigma(T,\sigma_0)\backslash \mathrm{I}_3(T,\sigma_0) = Q\left(\overline{\Delta}_{\frac{r}{s}}\backslash\Delta\right)\backslash Q\left(\Delta_{\frac{r}{s}}\backslash\overline{\Delta}\right) \subseteq Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right).$$
Conversely, let $\lambda\in Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right)$. One can show that $T-\lambda I$ is not surjective, and then, $\lambda\in\sigma_\delta(T,\sigma_0)$. Thus, we conclude that
$$\sigma_\delta(T,\sigma_0)=Q\left(\partial\Delta_{\frac{r}{s}}\cup\partial\Delta\right).$$

(13): It is a direct consequence of Statement (3) and the fact that
$\sigma_{\mathrm{co}}(T,\sigma_0)=\sigma_{\mathrm{p}}(T^*,\sigma_0^*)$.

Theorem 4.3. *For* $|r|=|s|$, *we have* $T\in\mathcal{B}(\sigma_0)$. *Furthermore, the following statements hold:*

(1): $\sigma(T,\sigma_0)=Q(\partial\Delta)$.

(2): $\sigma_{\mathrm{p}}(T,\sigma_0)=\begin{cases}Q(\partial\Delta\backslash\{-1,1\}), & \text{if } r=s,\\ Q(\partial\Delta), & \text{if } r=-s.\end{cases}$

(3): $\sigma_{\mathrm{p}}(T^*,\sigma_0^*)=\emptyset$.

(4): $\sigma_{\mathrm{r}}(T,\sigma_0)=\emptyset$.

(5): $\sigma_{\mathrm{c}}(T,\sigma_0)=\begin{cases}Q(\{-1,1\}), & \text{if } r=s,\\ \emptyset, & \text{if } r=-s.\end{cases}$

(6): $\mathrm{I}_3\sigma(T,\sigma_0)=\emptyset$.

(7): $\mathrm{II}_3\sigma(T,\sigma_0)=\begin{cases}Q(\partial\Delta\backslash\{-1,1\}), & \text{if } r=s,\\ Q(\partial\Delta), & \text{if } r=-s.\end{cases}$

(8): $\mathrm{III}_3\sigma(T,\sigma_0)=\emptyset$.

(9): $\mathrm{II}_2\sigma(T,\sigma_0)=\begin{cases}Q(\{-1,1\}), & \text{if } r=s,\\ \emptyset, & \text{if } r=-s.\end{cases}$

(10): $\mathrm{III}_1\sigma(T,\sigma_0)=\mathrm{III}_2\sigma(T,\sigma_0)=\emptyset$.

(11): $\sigma_{\mathrm{ap}}(T,\sigma_0)=Q(\partial\Delta)$.

(12): $\sigma_\delta(T,\sigma_0)=Q(\partial\Delta)$.

(13): $\sigma_{\mathrm{co}}(T,\sigma_0)=\emptyset$.

Proof.

(1): From Theorem 2.5(1), we have
$$\sigma(T,\sigma_0)=\sigma(T^*,\sigma_0^*) = \sigma(T^*,\mathrm{h}) = P(\partial\Delta) = Q(\partial\Delta).$$

(2): Let $r=s$. Suppose that $\lambda\in\sigma_{\mathrm{p}}(T,\sigma_0)$. Then, $T-\lambda I$ is not injective. Since the product of the two roots of $Q-\lambda$ equals 1, therefore, by Corollary 4.1, the two roots lie on $\partial\Delta$ and that neither -1 nor 1 is a double root of $Q-\lambda$. Let β be a root of $Q-\lambda$. Then, $\beta\in\partial\Delta\backslash\{-1,1\}$. That is, we have
$\lambda=Q(\beta)\in Q(\partial\Delta\backslash\{-1,1\})$. So,
$\sigma_{\mathrm{p}}(T,\sigma_0)\subseteq Q(\partial\Delta\backslash\{-1,1\})$.
Conversely, let $\lambda\in Q(\partial\Delta\backslash\{-1,1\})$. Then, there exists $\beta\in\partial\Delta\backslash\{-1,1\}$ such that $\lambda=Q(\beta)$. Therefore, β is a root of $Q-\lambda$ that satisfies $|\beta|=1$ and $\beta\notin\{-1,1\}$. By Corollary 4.1, $T-\lambda I$ is not injective, and so, $\lambda\in\sigma_{\mathrm{p}}(T,\sigma_0)$. Hence,
$Q(\partial\Delta\backslash\{-1,1\})\subseteq\sigma_{\mathrm{p}}(T,\sigma_0)$.
This concludes that
$\sigma_{\mathrm{p}}(T,\sigma_0)=Q(\partial\Delta\backslash\{-1,1\})$.
Let $r=-s$. The proof is similar to case $r=s$.

(3): This result follows directly from Theorem 2.5(2). Indeed, we have $\sigma_{\mathrm{p}}(T^*,\sigma_0^*)=\sigma_{\mathrm{p}}(T^*,\mathrm{h})=\emptyset$.

(4): Based on the relation
$\sigma_{\mathrm{r}}(T,\sigma_0)=\sigma_{\mathrm{p}}(T^*,\sigma_0^*)\backslash\sigma_{\mathrm{p}}(T,\sigma_0)$
and using Statement (3), the result follows immediately.

(5): If $r=-s$, the proof follows immediately from the relation
$\sigma_{\mathrm{c}}(T,\sigma_0)$
$=\sigma(T,\sigma_0)\backslash[\sigma_{\mathrm{p}}(T,\sigma_0)\cup\sigma_{\mathrm{r}}(T,\sigma_0)]$,
and then applying Statements (1), (2) and (4). Now, let $r=s$. In fact, we have
$$\sigma_{\mathrm{c}}(T,\sigma_0)=Q(\partial\Delta)\backslash Q(\partial\Delta\backslash\{-1,1\}) \subseteq Q(\{-1,1\}).$$
On the other hand, let $\lambda\in Q(\{-1,1\})$. Then, -1 and 1 are the double roots of $Q-\lambda$. This implies that $T-\lambda I$ is injective. Therefore, $\lambda\notin\sigma_{\mathrm{p}}(T,\sigma_0)$. Consequently, $\lambda\in\sigma_{\mathrm{c}}(T,\sigma_0)$.
We conclude that
$Q(\{-1,1\})\subseteq\sigma_{\mathrm{c}}(T,\sigma_0)$.
This concludes the result.

(6): Let $r=s$. From the fact that

$\mathrm{I}_3\sigma(T,\sigma_0) \subseteq \sigma_{\mathrm{p}}(T,\sigma_0)$.
We have
$\mathrm{I}_3\sigma(T,\sigma_0) \subseteq Q(\partial\Delta\backslash\{-1,1\})$.
Conversely, let $\lambda \in Q(\partial\Delta\backslash\{-1,1\})$. Therefore, there exists a root β of $Q-\lambda$ such that $\beta \in \partial\Delta\backslash\{-1,1\}$. So, by Corollaries 4.1 and 4.2, $T-\lambda I$ is neither injective nor surjective. So, $\lambda \notin \mathrm{I}_3\sigma(T,\sigma_0)$. This concludes that $\mathrm{I}_3\sigma(T,\sigma_0)=\emptyset$.
Let $r=-s$. The proof is similar to case $r=s$.

(7)-(8): Let $r=s$. Simply observe that
$$\mathrm{II}_3\sigma(T,\sigma_0) \cup \mathrm{III}_3\sigma(T,\sigma_0) \subseteq \sigma_{\mathrm{p}}(T,\sigma_0)\backslash \mathrm{I}_3\sigma(T,\sigma_0) = Q(\partial\Delta\backslash\{-1,1\}).$$
Now, let $\lambda \in Q(\partial\Delta\backslash\{-1,1\})$. Then, $\lambda \in P(\partial\Delta\backslash\{-1,1\})$. Therefore, there exists a root γ of $P-\lambda$ such that $\gamma \in \partial\Delta\backslash\{-1,1\}$. This implies that the two roots of $P-\lambda$ are inside $\overline{\Delta}$. So, by [El-Shabrawy and Shindy 2025, Proposition 2.1], $T^*-\lambda I$ is injective. Also, from [Taylor and Halberg 1957, Theorem 1], $T-\lambda I$ has a dense range, and consequently, $\lambda \in \mathrm{II}_3\sigma(T,\sigma_0)$. Therefore,
$\mathrm{II}_3\sigma(T,\sigma_0)=Q(\partial\Delta\backslash\{-1,1\})$.
and
$\mathrm{III}_3\sigma(T,\sigma_0)=\emptyset$.
The case $r=-s$ follows by a similar argument.

(9): Simply observe that
$\mathrm{II}_2\sigma(T,\sigma_0)=\sigma_{\mathrm{c}}(T,\sigma_0)$.
It remains to use Statement (5).

(10): Since
$$\sigma_{\mathrm{r}}(T,\sigma_0) = \mathrm{III}_1\sigma(T,\sigma_0) \cup \mathrm{III}_2\sigma(T,\sigma_0),$$
then the desired result follows by applying Statement (4).

(11): Indeed, from Statements (1) and (10), we have
$$\sigma_{\mathrm{ap}}(T,\sigma_0)=\sigma(T,\sigma_0)\backslash \mathrm{III}_1(T,\sigma_0) = Q(\partial\Delta).$$

(12): Follows directly from Statements (1) and (6) that
$$\sigma_\delta(T,\sigma_0)=\sigma(T,\sigma_0)\backslash \mathrm{I}_3(T,\sigma_0) = Q(\partial\Delta).$$

(13): It is a direct consequence of Statement (3) and the fact that
$$\sigma_{\mathrm{co}}(T,\sigma_0)=\sigma_{\mathrm{p}}(T^*,\sigma_0^*).$$

An important result is in order before ending this section. Consider the special case where $r=s>0$. Then, the operator $T(r,q,s)$ is reduced to the Jacobi operator $J(q,r)$. If this is the case, we observe that
$$Q(\partial\Delta)=[q-2r,q+2r],$$
$$Q(\partial\Delta\backslash\{-1,1\})=(q-2r,q+2r),$$
$$Q(\{-1,1\})=\{q-2r,q+2r\}.$$
Consequently, we have the following important corollary:

Corollary 4.5. *Let r and q be fixed real numbers with $r>0$. We have $J(q,r)\in\mathcal{B}(\sigma_0)$. Furthermore, the following statements hold:*

(1): $\sigma(J(q,r),\sigma_0)=[q-2r,q+2r]$.
(2): $\sigma_{\mathrm{p}}(J(q,r),\sigma_0)=(q-2r,q+2r)$.
(3): $\sigma_{\mathrm{p}}(J(q,r)^*,\sigma_0^*)=\emptyset$.
(4): $\sigma_{\mathrm{r}}(J(q,r),\sigma_0)=\emptyset$.
(5): $\sigma_{\mathrm{c}}(J(q,r),\sigma_0)=\{q-2r,q+2r\}$.
(6): $\mathrm{I}_3\sigma(J(q,r),\sigma_0)=\emptyset$.
(7): $\mathrm{II}_3\sigma(J(q,r),\sigma_0)=(q-2r,q+2r)$.
(8): $\mathrm{III}_3\sigma(J(q,r),\sigma_0)=\emptyset$.
(9): $\mathrm{II}_2\sigma(J(q,r),\sigma_0)=\{q-2r,q+2r\}$.
(10): $\mathrm{III}_1\sigma(J(q,r),\sigma_0)=\mathrm{III}_2\sigma(J(q,r),\sigma_0)=\emptyset$.
(11): $\sigma_{\mathrm{ap}}(J(q,r),\sigma_0)=[q-2r,q+2r]$.
(12): $\sigma_\delta(J(q,r),\sigma_0)=[q-2r,q+2r]$.
(13): $\sigma_{\mathrm{co}}(J(q,r),\sigma_0)=\emptyset$.

In Corollary 4.5, the results established in Statements (1)-(5) agree with the results in [El-Shabrawy and Shindy 2020, Theorem 4.1]. While, the remaining results are new contributions.

Conclusion and work in progress

In this paper, a detailed study on the spectra of the infinite matrices $B(r,s)$, $U(r,s)$ and $T(r,q,s)$ as operators on the Cesàro sequence space σ_0 has been given. In fact, the class of the operators $T(r,q,s)$ includes $B(r,s)=T(0,r,s)$ and $U(r,s)=T(s,r,0)$. We assert that the technique presented in Section 4 is flexible to be adapted to study the spectral problem of the operators $B(r,s)$ and $U(r,s)$. In fact, analogous to Corollaries 4.1, 4.2, 4.3 and 4.4, the reader can easily verify the following proposition.

Proposition 5.1. *The following statements are satisfied:*

(1): $B(r,s)-\lambda I\in(\sigma_0:\sigma_0)$ *is injective.*
(2): $B(r,s)^*-\lambda I\in(\mathrm{h}:\mathrm{h})$ *is injective if and only if* $|\lambda-r|\geq|s|$.
(3): $B(r,s)-\lambda I\in(\sigma_0:\sigma_0)$ *is surjective if*

and only if $|\lambda - r| > |s|$.

(4): $B(r,s)^* - \lambda I \in (\mathrm{h}:\mathrm{h})$ *is surjective if and only if* $|\lambda - r| \neq |s|$.

(5): $U(r,s) - \lambda I \in (\sigma_0:\sigma_0)$ *is injective if either* $|\lambda - r| > |s|$ *or* $\lambda = r + s$.

(6): $U(r,s)^* - \lambda I \in (\mathrm{h}:\mathrm{h})$ *is injective*.

(7): $U(r,s) - \lambda I \in (\sigma_0:\sigma_0)$ *is surjective if and only if* $|\lambda - r| \neq |s|$.

(8): $U(r,s)^* - \lambda I \in (\mathrm{h}:\mathrm{h})$ *is surjective if and only if* $|\lambda - r| > |s|$.

Then, by similar calculations like in the preceding section, we can obtain results that agree with those in Section 3. This gives a new technique for determining the spectra of the operators $B(r,s)$ and $U(r,s)$.

Further study on the spectra of the operator $T(r,q,s)$ on sequence spaces, like σ_∞ and bs, is required. However, some difficulties caused some problems in characterizing the residual and the continuous spectra of $T(r,q,s)$ on σ_∞ and bs. The problem is in progress and we hope to have advances regarding these results in future publication.

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الملخص العربي

عنوان البحث: حول أطياف بعض المصفوفات ذات النطاق لانهائية البعد كمؤثرات على فراغ المتتابعات لشيزارو σ_0

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قدَّم هذا العمل دراسةً طيفيةً لبعض المصفوفات المعروفة، وهي المصفوفات المثلثية السفلية والعلوية ثنائية النطاق، كمؤثرات على فراغ المتتابعات لشيزارو σ_0 . تتضمن الدراسة تحليلاً دقيقاً لطيف هذه المؤثرات، مع التمييز بين أنواع الطيف المختلفة مثل: الطيف النقطي، الطيف المتبقي، الطيف المتصل، طيف العيب، طيف الضغط، والطيف التقريبي النقطي. كما يُقدِّم تقسيمًا أدق للطيف. وتمتد الدراسة لتشمل تعميمًا على المصفوفات ثلاثية الأقطار المتماثلة وغير المتماثلة .ويتميز الأسلوب المُتَّبَع في البحث بمرونة تسمح بتطبيقه لدراسة المسألة الطيفية للمؤثرات موضع الدراسة في فراغات متتابعات أخرى.