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Original Article

Cohomology theory through infinity algebras.

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Abstract:
This research explores the foundational structures and theoretical framework of A_∞-algebras and $\mathbb{L}_\infty$-algebras within the wider setting of homological algebra and higher category theory. These "infinity" algebras extend classical associative and Lie algebras by encoding operations that satisfy generalized coherence relations up to homotopy. Their flexible and homotopically rich structure provides a unifying language for dealing with complex algebraic phenomena that are not accessible through traditional means.
The study begins by reviewing some definitions and algebraic properties of A_∞- and $\mathbb{L}_\infty$-algebras, emphasizing their realization as differential graded structures governed by an infinite sequence of multilinear operations. This hierarchy of operations is structured via higher associativity or higher Jacobi-type identities, which hold up to coherent homotopies. Operads are introduced as essential organizing tools that capture and formalize these intricate patterns of relations.
The homological dimensions of these algebras are developed through constructions such as the bar complexes, as well as the theory of Maurer–Cartan elements, which serve as central objects in encoding deformations. A special emphasis is placed on Hochschild and Chevalley–Eilenberg cohomology, which classify extensions and control deformation theory in these settings.
Furthermore, the notion of homotopy equivalence between infinity morphisms is investigated to understand equivalence classes of algebraic structures. This provides a natural framework for studying moduli problems and organizing higher algebraic invariants in a coherent way.

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1. Introduction

Homology theory is a fundamental branch of mathematics that studies algebraic structures associated with topological spaces, allowing for the classification and analysis of their properties. The origins of homology trace back to the 19th century when Bernhard Riemann investigated the connectivity of surfaces, laying the groundwork for what would later become algebraic topology. His work, particularly in relation to Green's theorem, established that homologous curves produce the same integral values, thus introducing a concept that would be extensively developed in the following decades (Hatcher, 2002).

The formalization of homology as an algebraic tool began in the 20th century with an important work of Gerhard Hochschild, who introduced homology theory in the context of associative algebras over fields in 1945. Hochschild's contributions were essentialin establishing homological techniques to study algebraic structures, leading to the modern framework of homological algebra (Hochschild, 1945). Subsequently, Henri Cartan and Samuel Eilenberg extended homology theory to more general algebraic contexts, particularly in noncommutative rings and module categories. Their work introduced derived functors, such as Ext and Tor, which became indispensable tools in algebra and topology (Cartan, Eilenberg, 1956).

A major advancement in homotopy theory came in the 1960s with Jim Stasheff's introduction of A_∞-spaces and A_∞-algebras. Stasheff originally developed these concepts to study higher homotopy associativity in topological spaces, leading to the formulation of A_∞-algebras as structures that generalize associative algebras by incorporating higher homotopy relations (Stasheff, 1963). These algebras play an essential role in areas such as derived algebraic geometry, category theory, and mathematical physics. The concept of A_∞-operads further extends this framework, offering an approach to describing deformations in homotopy theory and enabling deeper connections between homological algebra and geometry (Loday & Vallette, 2012).

Parallel to the development of A_∞-algebras, the study of $\mathbb{L}_\infty$-algebras (or strong homotopy Lie algebras) gained prominence due to their applications in deformation theory and mathematical physics. L_∞-algebras, introduced in the 1990 and generalized Lie algebras by incorporating higher-order brackets that satisfy homotopy-invariant versions of the Jacobi identity (Lada &Stasheff, 1993). These structures arise in string field theory, where they govern the interactions of fields in a way that respects higher homotopy structures. Moreover, $\mathbb{L}_\infty$-algebras are essential in Poisson geometry, derived brackets, and the study of deformation quantization, linking them to modern research in theoretical physics and symplectic geometry (Kontsevich, 2003).

One of the most significant applications of A_∞- and $\mathbb{L}_\infty$-algebras lies in their role within homological mirror symmetry. Which introduced by Maxim Kontsevich in 1994, homological mirror symmetry conjectures a deep duality between the derived Fukaya category of a symplectic manifold and the derived category of coherent sheaves on a mirror complex variety. In this setting, A_∞-categories and A_∞-modules provide the necessary algebraic structures to describe the deformation theory of holomorphic objects, while L_∞-algebras govern the deformation spaces of Poisson structures (Kontsevich, 1994). These ideas had major effects in both mathematics and theoretical physics, influencing the study of Calabi-Yau manifolds, Floer homology, and topological field theories.

In addition to their theoretical importance, A_∞- and $\mathbb{L}_\infty$-algebras have found an applications in computational algebra, category theory, and even data analysis. The flexibility of these structures allows for the encoding of complex homotopy information in algebraic terms, enabling more accurate methods in homological computations and derived categories. The connection between these algebras and operadic structures has led to new insights into higher-category theory, making them fundamental tools in modern mathematical research (Keller, 2001).

This paper aims to further explore $\mathbb{L}_\infty$-algebras and their homological properties, particularly their applications in deformation theory and mathematical physics. Through this study, we seek to contribute to the broader understanding of homotopical and categorical algebra, emphasizing the deep connections between these advanced algebraic structures and their applications in topology, geometry, and physics.

2. Hochschild of an infinity algebras

This part provides definitions of an infinity algebras and the Hochschild homology theory for A_∞-algebras.

A graded algebra is a module $A = \bigoplus_{i\in\mathbb{Z}} A_i$, where the unit 1 is in$A_0$, and the grading kept intact by multiplication. A map $\partial: A \to A$ with degree $+1$, where $\partial^2 = 0$, defines a differential graded algebra. The degree of an element x is written as either $|x|$ or $deg\ x$.

The classical operad in a multi-category $\mathcal{C}$ includes:

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- $\mathcal{C}_0$, which is the set of objects $a, a_1, a_2, \cdots$.
- For each $n \in \mathbb{N}$, the set of morphisms $Hom(a_1, \cdots, a_n; a)$,
- A composition map $\gamma_{k_1,\dots,k_n}$ that combines morphisms as follows:

$$(\theta; \theta_1, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \cdots, \theta_n).$$

$$Hom(a_1, \cdots, a_n; a) \times Hom(a_{11}, \cdots, a_1 k_1; a) \times \cdots \times Hom(a_{n1}, \cdots, a_n k_n; a) \\ \to Hom(a_{n1}, \cdots, a_n k_n; a).$$

The operad ensures that the composition is associative and has an identity, where each component a has identity morphism $1_a \in Hom(a, a)$, this known as an operad (Mahmoud et al., 2024).

The non-symmetric operad L includes:

- A sequence $(\xi(n))_{n\in\mathbb{N}}$ of unique n-ary operations on ξ.
- A composition map $\gamma_{k_1,\dots,k_n}$for integers $n,\ k_1, \cdots, k_n$which combines operations as:

$$\xi(n) \times \xi(k_1) \times \cdots \times \xi(k_n) \to \xi(k_1 + \cdots + k_n),$$

with composition $(\theta; \theta_1, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \cdots, \theta_n)$.

For each identity $1 \in \xi(1)$, the operad satisfies:

- Associativity: $\theta \circ \left(\theta_1 \circ \left(\theta_{1,1}, \cdots, \theta_{1,k_1}\right), \cdots, \theta_n \circ \left(\theta_{n,1}, \cdots, \theta_{n,k_n}\right)\right)$

$$= \left(\theta \circ (\theta_1, \cdots, \theta_n)\right) \circ \left(\theta_{1,1}, \cdots, \theta_{1,k_1}, \cdots, \theta_{n,1}, \cdots, \theta_{n,k_n}\right),$$

- Unity: $\theta \circ (1, \cdots, 1) = \theta = 1 \circ \theta$.

2.1 Definition

In terms of the monoidal structure $\circ$, the operad is both a monad in the category $\mathcal{C}$ and a monoid in the category $E(\mathcal{C})$. Specifically, the operad ξ is an element of $E(\mathcal{C})$, which is represented by the functor $\xi : \mathcal{C} \to \mathcal{C}$. This functor includes the natural transformation maps $\gamma: \xi \circ \xi \to \xi$ and $i: I \to \xi$. Both of these maps satisfy the conditions for associativity and unity, as shown by the following commutative diagrams:

$$\begin{array}{ccc} \xi \circ (\xi \circ \xi) \simeq (\xi \circ \xi) \circ \xi & \xrightarrow{\gamma \otimes id} & \xi \circ \xi \\ \gamma \otimes id \downarrow & & \downarrow \gamma \\ \xi \circ \xi & \xrightarrow{\gamma} & \xi \end{array} \quad , \quad \begin{array}{ccccc} I \circ \xi & \xrightarrow{i \otimes id} & \xi \circ \xi & \xrightarrow{id \otimes i} & \xi \circ I \\ & \searrow^{\simeq} & \downarrow \gamma & \swarrow^{\simeq} & \\ & & \xi & & \end{array}$$

In (Abo-Quota et al., 2023), has given that ℓ is a linear map and $\mathcal{C}$ is a vector space, the composition $\circ$ of two endo-functors $\xi,\ \tau \in E(\mathcal{C})$ can be written as:

$$(\xi \circ \tau)(\ell) = \xi\left(\tau(\ell)\right), \quad (\xi \circ \tau)(\mathcal{C}) = \xi\left(\tau(\mathcal{C})\right).$$

Additional operations on endo-functors within V, such as direct sums and tensor products, can be described as:

$$(\xi \otimes \tau)(\ell) = \xi(\ell) \otimes \tau(\ell), \qquad (\xi \otimes \tau)(\mathcal{C}) = \xi(\mathcal{C}) \otimes \tau(\mathcal{C})$$

and

$$(\xi \oplus \tau)(\ell) = \xi(\ell) \oplus \tau(\ell), \qquad (\xi \oplus \tau)(\mathcal{C}) = \xi(\mathcal{C}) \oplus \tau(\mathcal{C}).$$

2.2 Definition

Just as associative algebras are algebras over the naturally generated non-symmetric operad, A_∞-algebras are algebras over a specific non-symmetric operad A_∞ (Karar et al., 2024).

Let $(A, m_1, m_2, \cdots)$, where $m_k \in Hom(A^{\otimes k}, A)$and $degm_k = k - 2$ define the A_∞-algebras. The relation for $n = 1$ can be written as:

$$\sum_{\substack{k=p+1+r \\ n=p+q+r \\ k,q\geq 1}} (-1)^{qr+p} m_k \left(\underbrace{id, \cdots, id}_{(p)}, m_q, \underbrace{id, \cdots, id}_{(r)} \right) = 0 \qquad , n \geq 1$$

For a graded vector space A, the chain complex structure implies $m_1 \circ m_1 = 0$, where $d := -m_1 \in End_{-1}(A)$. Therefore, $A^{\otimes n}$ represents the chain complex with the differential: $d_{A^{\otimes n}} = \sum_{n=p+1+r}(id^{\otimes p} \otimes d \otimes id^{\otimes r})$.

Thus, the differential $\partial = [d, -] = [-m_1, -]$ has a chain complex described by $Hom(A^{\otimes n}, A)$. The relation for ∂m_ngiven by:

$$\partial m_n = -m_1(m_n) + (-1)^{n-2} m_n \left(\sum_{n=p+1+r} (id^{\otimes p} \otimes m_1 \otimes id^{\otimes r}) \right)$$

and for $n \geq 2$, we have:

$$\partial m_n = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k,q\geq 2}} (-1)^{p+qr} m_k \left(id^{\otimes p} \otimes m_q \otimes id^{\otimes r}\right).$$

Since the A_∞-algebra is a chain complex (A, d) equipped with operations $m_n \in Hom(A^{\otimes n}, A)$ for $n \geq 2$ with degree $(n - 2)$, it shows that the graded operad A_∞ is a non-symmetric differential operad, which properly handles the relations described.

2.3 Definition

A graded vector space $\mathbb{L}$, together with a degree-preserving anti-symmetric linear map $[\cdot,\cdot]: \mathbb{L} \otimes \mathbb{L} \longrightarrow \mathbb{L}$, defines a graded Lie algebra. The Lie bracket satisfies the Jacobi identity, given by:

$$\big[a,[b,c]\big] = \big[[a,b],c\big] \pm (-1)^{|a||b|}\big[b,[a,c]\big] = 0\,, \quad \forall\, a,b,c \in \mathbb{L} \text{ are homogeneous.}$$

If $\mathbb{L}$ is ungraded, this leads to the standard concept of Lie algebras. Note that the derivation of graded algebras $(\mathbb{L},[\cdot,\cdot])$simply given by $[x,\cdot]$.

If the elementary algebras are Lie algebras that are graded, then the differential graded algebras referred to as differential graded Lie algebras.

2.4 Definition

In a graded Lie algebra $(\mathbb{L},[\cdot,\cdot],d)$, the element of degree 1 called the Maurer-Cartan element. The Maurer-Cartan equation given by:

$$(a) + \frac{1}{2}[a,a] = 0.$$

2.5 Definition

$\mathbb{L}_\infty$-algebras are a combination of a graded vector space $\mathbb{L}$ and anti-symmetric linear maps $l_k: \mathbb{L}^{\otimes k} \to \mathbb{L}$ for higher degree brackets, where $|l_k| = 2-k$. For $1 \le k < \infty$, the generalized Jacobi identity given by:

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh_{2,1}^{-1}} (-1)^{i(j-1)} a(\sigma)\,, \qquad l_j(l_i(a_1,\cdots,a_i),a_{i+1},\cdots,a_n) = 0,$$

which holds for all $n \ge 1$ and homogeneous elements $a_1,\cdots,a_n \in \mathbb{L}$.

2.6 Theorem (Noreldeen et al., 2024)

For each $p \in \mathbb{Z}$, the anti-symmetric linear maps $l: \mathcal{S}^{\otimes n} \to \mathcal{S}$ with degree $(p-n+1)$ and the symmetric linear maps $\delta: (\mathcal{S}\,[1])^{\otimes n} \to \mathcal{S}\,[1]$ with degree p have a one-to-one correspondence, given by the formulas:

$$l = \uparrow \circ\, \delta\, \circ \downarrow^{\otimes n}, \qquad \delta = (-1)^{\frac{k(k-1)}{2}} \downarrow \circ\, l \circ \uparrow^{\otimes n}.$$

Since $\mathcal{S}$ is a graded vector space, let the differential $d: \mathcal{S} \to \mathcal{S}$ be a degree one linear map such that $d^2 = 0$. The pair $(\mathcal{S}, d)$ called a differential graded vector space.

The linear map $f: \mathcal{S} \to H$ preserves degree, and the differential graded vector spaces $(\mathcal{S}, d)$ and (H, d') are homomorphisms if $d' \circ f = f \circ d$. Chain complexes refer to differential graded vector spaces, and if $(\mathcal{S}, d)$ is a chain complex, we define:

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- when $d(s) = 0$, $s \in \mathcal{S}_n$ is an n-cycle element.
- when $s = d(h)$, $s \in \mathcal{S}_{n-1}$ is an n-boundary for some $h \in \mathcal{S}_{n-1}$.

The graded vector space:

$$H(\mathcal{S}) = \frac{ker(d)}{im(d)}$$

checks if the sequence is non-exact:

$$\dots \xleftarrow{d} \mathcal{S}_{n-1} \xleftarrow{d} \mathcal{S}_n \xleftarrow{d} \mathcal{S}_{n+1} \xleftarrow{d} \dots,$$

which is called the homology of $(\mathcal{S}, d)$ and is denoted by:

$$H_n(\mathcal{S}) = \frac{n - cycles}{n - boundaries}$$

the n^{th} homology group (Kozae et al., 2022).

2.7 Definition

Recall that the degree one symmetric map $\delta_k: S^k(\mathbb{L}[1]) \to L[1]$ is equivalent to the degree $(2-k)$ anti-symmetric map $l_k: \mathbb{L}^{\otimes k} \to \mathbb{L}$, where $l_k = \uparrow\circ\, \delta_k \circ\downarrow^{\otimes k}$.

Next, the generalized Jacobi identity for the $\mathbb{L}_\infty$-structure expressed as:

$$\sum_{i+j=n+1} \sum_{\sigma\in Sh_{2,1}^{-1}} (-1)^i (j-1) a(\sigma)\, l_j(l_i(a_1, \cdots, a_i), a_{i+1}, \cdots, a_n) = 0$$

In terms of the δ_k-sections, the $\mathbb{L}_\infty$-structure of $\mathbb{L}$ in the symmetric bracket parts described by:

$$\sum_{j+i=n+1} \sum_{\Omega\in Sh_{i,n-i}^{-1}} (-1)^{(j-1)i} l_j(l_i \otimes id_{\mathbb{L}}^{\otimes(j-1)})\hat{a}(\Omega) = 0,$$

This is equivalent to the following expression with isomorphisms $\downarrow$ and $\uparrow^{\otimes n}$:

$$0 = (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega\in Sh_{i,n-i}^{-1}} (-1)^{(j-1)i} \downarrow\circ\, l_j \left(l_i \otimes id_{\mathbb{L}}^{\otimes(j-1)}\right) \hat{a}(\Omega) \circ\uparrow^{\otimes n}$$

By substituting δ_j for l_i, this becomes:

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega\in Sh_{i,n-i}^{-1}} (-1)^{(j-1)i} \delta_j \circ\downarrow^{\otimes j} \left(l_i \otimes id_{\mathbb{L}}^{\otimes(j-1)}\right) \uparrow^{\otimes n}\circ\, \hat{\varepsilon}(\Omega)$$

Finally, this leads to:

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega\in Sh_{i,n-i}^{-1}} \delta_j\left((\downarrow\circ\, l_i) \otimes \downarrow^{\otimes(j-1)}\right) \uparrow^{\otimes n}\circ\, \hat{\varepsilon}(\Omega)$$

$$= \sum_{j+i=n+1} \sum_{\Omega\in Sh_{i,n-i}^{-1}} \delta_j \left(\delta_i \otimes id_{\mathbb{L}[1]}^{\otimes(j-1)}\right) \hat{\varepsilon}(\Omega).$$

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2.8 Corollary

Assumed the graded vector space $\mathbb{L}$, the $\mathbb{L}_\infty$-structure is related with the linear map $\delta: \bar{S}(\mathbb{L}[1]) \to \mathbb{L}[1]$of degree one. This map satisfies the condition:

$$\delta \circ \mu_S(\delta \otimes id_S)\Delta_S = 0.$$

2.9 Definition (Reinhold, 2019)

Let $\mathbb{L}$ be $\mathbb{Z}_{\leq 0}$-graded finite $\mathbb{L}_\infty$-algebra, and let E be a trivial, finite-dimensional space in negative degrees. The differential graded algebras $S(\mathbb{L}[1])^*$ and $(S(\mathbb{L}[1]) \otimes \mathcal{S}^*)^*$ defined as follows:

$$S(\mathbb{L}[1])^* \cong S(\mathbb{L}[1]^*)\,, \qquad \mathcal{S} \cong \mathcal{S}^{**}, \qquad (S(\mathbb{L}[1]) \otimes \mathcal{S}^*)^* \cong S(\mathbb{L}[1]^*) \otimes \mathcal{S}.$$

Assume $d_{\mathcal{CS}} = -d^*$, defining a differential on $S(\mathbb{L}[1]^*)$. The left $S(\mathbb{L}[1]^*)$-module given by the term $S(\mathbb{L}[1]^*) \otimes \mathcal{S}$ from the maps:

$$S(\mathbb{L}[1]^*) \otimes (S(\mathbb{L}[1]^*) \otimes \mathcal{S}) \to S(\mathbb{L}[1]^*) \otimes \mathcal{S}$$

where

$$\big(\mu \otimes (\Omega \otimes \mathscr{s})\big) \mapsto (\mu \vee \Omega) \otimes \mathscr{s}.$$

The linear map $D_{\mathcal{CS}}: S(\mathbb{L}[1]^*) \otimes \mathcal{S} \to S(\mathbb{L}[1]^*) \otimes \mathcal{S}$ is a degree one derivation. Expanding $D_{\mathcal{CS}}$, for any $\mu, \Omega \in S(L[1]^*)$, $e \in \mathcal{S}$ homogeneous, we get:

$$D_{\mathcal{CS}}\big(\mu \vee (\Omega \otimes \mathscr{s})\big) = D_{\mathcal{CS}}\, \mu \vee (\Omega \otimes \mathscr{s}) + (-1)^{|\mu|}\mu \vee D_{\mathcal{CS}}(\Omega \otimes \mathscr{s}).$$

2.10 Proposition

The $\mathbb{L}_\infty$-algebras $\mathbb{L}$ on $\mathcal{S}$ has a representation ρ given by the derivation:

$$D_{\mathcal{CS}}: S(\mathbb{L}[1]^*) \otimes \mathcal{S} \to: S(\mathbb{L}[1]^*) \otimes \mathcal{S},$$

which extends $d_{\mathcal{CS}}$ and satisfies $D_{\mathcal{CS}}^2 = 0$.

For example, setting $\mathrm{D}_{\mathcal{CS}} = -\mathrm{D}^*$, where D a co-derivation is extending d, we see that D generated by the two-way representation $\rho^{\mathcal{S}}$. Given a representation ρ of the $\mathbb{L}_\infty$-algebra $\mathbb{L}$ on $\mathcal{S}$, we obtain the complex $S(\mathbb{L}[1]^*) \otimes \mathcal{S}$, which closely resembles the generalized Chevalley–Eilenberg complex, with $D_{\mathcal{CS}}$ acting as a co-boundary operator.

2.11 Definition (Noreldeen, 2020)

Let $\mathcal{S}$ be Lie algebra over a field $\mathcal{F}$ with bracket operation $[\cdot,\cdot]$. This bracket defines a linear map:

$$\Lambda^2 \mathcal{S} \to \mathcal{S}$$

due to its anti-symmetry. Now, let $\mathcal{M}$ be an $\mathcal{S}$-module, and define the space of n-cochains as:

$$C^n(\mathcal{S}, \mathcal{M}) = Hom(\Lambda^n \mathcal{S}, \mathcal{M})$$

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which consists of n-multilinear anti-symmetric functions on $\mathcal{S}$ with values in $\mathcal{M}$. This structure provides a degree n-cochain module on $\mathcal{S}$ with values in $\mathcal{M}$.

The co-boundary operator for Lie algebra cohomology given by:

$$d\varphi(s_1,\cdots,s_{n+1}) = \sum_{1\le i<j\le n+1} (-1)^{i+j-1}\phi\left([s_i,s_j],s_1,\cdots,\hat{s}_i,\cdots,\hat{s}_j,\cdots,s_{n+1}\right) + \sum_{1\le i\le n+1} (-1)^i s_i\cdot\phi(s_1,\cdots,\hat{s}_i,\cdots,s_{n+1}).$$

The cohomology of $\mathcal{S}$ with coefficients in $\mathcal{M}$ then defined as

$$H^n(\mathcal{S},\mathcal{M}) = \frac{\ker\left(d:\mathcal{C}^n(\mathcal{S},\mathcal{M})\to\mathcal{C}^{n+1}(\mathcal{S},\mathcal{M})\right)}{im\left(d:\mathcal{C}^{n-1}(\mathcal{S},\mathcal{M})\to\mathcal{C}^n(\mathcal{S},\mathcal{M})\right)}.$$

A special case occurs when $\mathcal{M}=\mathcal{S}$, in which case we write $\mathcal{C}^n(\mathcal{S},\mathcal{S})$ as $\mathcal{C}^n(\mathcal{S})$ and denote the cohomology as $H^n(\mathcal{S})$. Here, the adjoint action givethe action of $\mathcal{S}$ on itself.

3. Main results

This work delves into key aspects of algebraic topology and homology in L_∞-algebras, emphasizing how homology isomorphisms hold under certain conditions. It introduces simplicial and bar homology with module coefficients, explores H-unitality, and examines the connections between homological structures and quasi-isomorphisms.

The first theorem focuses on excision in $\mathbb{L}_\infty$-algebras, proving that homology isomorphisms remain intact through inclusion maps. The proof relies on chain complexes and homotopy equivalence arguments.

3.1. Theorem

Let $\mathcal{E}$ be a subset of an $\mathbb{L}_\infty$-algebra such that $\mathcal{E}\subset\mathcal{A}\subset\mathcal{X}$. Then, for all n, the inclusion $(\mathcal{X}\backslash\mathcal{E},\mathcal{A}\backslash\mathcal{E})\hookrightarrow(\mathcal{X},\mathcal{A})$ induces the isomorphism:

$$\mathcal{H}_n(\mathcal{X}\backslash\mathcal{E},\mathcal{A}\backslash\mathcal{E})\xrightarrow{\cong}\mathcal{H}_n(\mathcal{X},\mathcal{A}).$$

If $\mathcal{X}$ is covered by the interiors of the subspaces $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A},\mathcal{B}\subset\mathcal{X}$, then the inclusion $(\mathcal{B},\mathcal{A}\cap\mathcal{B})\hookrightarrow(\mathcal{X},\mathcal{A})$ similarly induces the isomorphism:

$$\mathcal{H}_n(\mathcal{B},\mathcal{A}\cap\mathcal{B})\xrightarrow{\cong}\mathcal{H}_n(\mathcal{X},\mathcal{A}),$$

for all n, where $\mathcal{B}$is defined as $\mathcal{B}=\mathcal{X}\backslash\mathcal{E}$.

Proof: Using (Noreldeenet al., 2021), we consider $\mathcal{X}$ as the union of $\mathcal{A}$ and $\mathcal{B}$ with their interiors covering $\mathcal{X}$. This setup provides natural inclusion maps:

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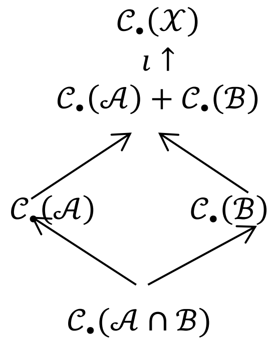

From this, we obtain the equivalence:

$$\mathcal{C}_\bullet(\mathcal{X})/\mathcal{C}_\bullet(\mathcal{A}) = \mathcal{C}_\bullet(\mathcal{B})/\mathcal{C}_\bullet(\mathcal{A} \cap \mathcal{B}),$$

whenever ι is an isomorphism. However, some simplices may intersect nontrivially with both $(\mathcal{A} - \mathcal{A} \cap \mathcal{B})$ and $(\mathcal{B} - \mathcal{A} \cap \mathcal{B})$, which prevents $\iota$from being an isomorphism.

To resolve this, we use the chain map $\xi: \mathcal{C}_\bullet(\mathcal{X}) \to \mathcal{C}_\bullet(\mathcal{A}) + \mathcal{C}_\bullet(\mathcal{B})$ to decompose problematic simplices into smaller, well-behaved ones without altering homology. We show that $\mathcal{C}_\bullet(\mathcal{A}) + \mathcal{C}_\bullet(\mathcal{B})$ is a homotopy retract of $\mathcal{C}_\bullet(\mathcal{X})$, satisfying

$\xi \circ \iota = Id$ and $\iota \circ \xi = d\mathfrak{D} + \mathfrak{D}d$ for some chain homotopy $\mathfrak{D}$.

Choosing $\mathfrak{D}$ appropriately preserves the subcomplexes $\mathcal{C}_\bullet(\mathcal{A})$ and$\mathcal{C}_\bullet(\mathcal{B})$, leading to the chain homotopy equivalence:

$$\mathcal{C}_\bullet(\mathcal{X})/\mathcal{C}_\bullet(\mathcal{A}) \to \mathcal{C}_\bullet(\mathcal{B})/\mathcal{C}_\bullet(\mathcal{A} \cap \mathcal{B}).$$

Finally, we establish simplicial homology isomorphisms within $\mathbb{L}_\infty$-algebras, focusing on their behavior under subspace inclusions.

3.2. Definition

For a space $\mathcal{X}$ and a subset $\mathcal{E}$ in an $\mathbb{L}_\infty$-algebra, where $\mathcal{E} \subset \mathcal{A} \subset \mathcal{X}$, the inclusion $(\mathcal{X}\backslash\mathcal{E}, \mathcal{A}\backslash\mathcal{E}) \hookrightarrow (\mathcal{X}, \mathcal{A})$ induces simplicial homology isomorphisms for all n:

$$\mathcal{HH}_n(\mathcal{X}\backslash\mathcal{E}, \mathcal{A}\backslash\mathcal{E}) \xrightarrow{\cong} \mathcal{HH}_n(\mathcal{X}, \mathcal{A}).$$

Setting $\mathcal{B} = \mathcal{X}\backslash\mathcal{E}$ and assuming $\mathcal{X}$ covered by the interiors of $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$, the equivalent statement follows from the inclusion $(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \hookrightarrow (\mathcal{X}, \mathcal{A})$:

$$\mathcal{HH}_n(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \to \mathcal{HH}_n(\mathcal{X}, \mathcal{A}) \quad \forall n. \tag{7}$$

Next, we define the bar homology of $\mathbb{L}_\infty$-algebras with module coefficients and describe the corresponding boundary maps.

3.3.Definition

Let $\mathcal{J}$ be an $\mathbb{L}_\infty$-algebra, possibly non-unital, and let $\mathcal{R}$ be a right $\mathcal{J}$-module. The homology $\mathcal{H}B'_*(\mathcal{J}, \mathcal{R})$ corresponds to the bar homology of $\mathcal{J}$ with $\mathcal{R}$ as its coefficients:

$$(\mathcal{R}\otimes\mathcal{J}^{\otimes *},\rho'_*):=\mathcal{R}\overset{\rho'_1}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\overset{\rho'_2}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\otimes\mathcal{J}\overset{\rho'_3}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\otimes\mathcal{J}\otimes\mathcal{J}\overset{\rho'_4}{\leftarrow}\dots$$

A tensor product defined over $\mathbb{L}_\infty$-algebras, with the boundary map expressed as:

$$\rho'_n(a_0\otimes\dots\otimes a_n)=\sum_{i=0}^{n-1}(-1)^i a_0\otimes\dots\otimes a_i a_{i+1}\otimes\dots\otimes a_n.$$

The following definition introduces the simplicial homology of complexes and examines the associated boundary maps, highlighting their relationship with bar homology.

3.4. Definition

The homology of complexes $\mathcal{HH}_*(\mathcal{J},\mathcal{R})$ describes the simplicial homology of $\mathcal{J}$ with coefficients in $\mathcal{R}$, represented by the sequence:

$$(\mathcal{R}\otimes\mathcal{J}^{\otimes *},\rho_*):=\mathcal{R}\overset{\rho_1}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\overset{\rho_2}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\otimes\mathcal{J}\overset{\rho_3}{\leftarrow}\mathcal{R}\otimes\mathcal{J}\otimes\mathcal{J}\otimes\mathcal{J}\overset{\rho_4}{\leftarrow}\dots,$$

where the boundary map is defined as:

$$\rho_n(a_0\otimes\dots\otimes a_n)=\rho'_n(a_0\otimes\dots\otimes a_n)+(-1)^n a_n a_0\otimes a_1\otimes\dots\otimes a_{n-1}.$$

Additionally, we derive a corollary linking H-homology and bar homology, exploring their connections in $\mathbb{L}_\infty$-algebras and the exact sequences that appear.

3.5. Corollary

The $\mathbb{L}_\infty$-algebras derived from $\mathcal{J}$ by adding unity denoted as $\tilde{\mathcal{J}}=k\times\mathcal{J}$. The H-homology $H_*(\mathcal{J})$is given by $\mathcal{H}_*(\mathcal{J}):=\mathcal{H}_*(\mathcal{J},\mathcal{J})$, while the bar homology $\mathcal{H}B_*(\mathcal{J})$ is defined by $\mathcal{H}B_*(\mathcal{J}):=\mathcal{H}B'_*(\mathcal{J},\mathcal{J})$. Additionally, the simplicial homology of $\mathcal{J}$ is represented by $\mathcal{HH}_*(\mathcal{J}):=\bar{\mathcal{H}}_*(\mathcal{J},\tilde{\mathcal{J}})$, where $\bar{\mathcal{H}}_n(\mathcal{J},\tilde{\mathcal{J}})=\mathcal{H}_n(\mathcal{J},\tilde{\mathcal{J}})$for $n>0$and$\bar{\mathcal{H}}_0(\mathcal{J},\tilde{\mathcal{J}})=\mathcal{H}_0(\mathcal{J},\tilde{\mathcal{J}})/k$. Furthermore, the homology $\mathcal{HH}_*(\mathcal{J})$ corresponds to the homology of the associated double complex:

$$\mathcal{CC}(\mathcal{J})^{|2|}\coloneqq(\mathcal{J}\otimes\mathcal{J}^{\otimes *},\rho_*)\xleftarrow{1-t}(\mathcal{J}\otimes\mathcal{J}^{\otimes *},-\rho'_*). \tag{8}$$

As a result, an exact sequence arises:

$$\dots\leftarrow H_{n-1}(\mathcal{J})\leftarrow\mathcal{H}B_{n-1}(\mathcal{J})\leftarrow\mathcal{HH}_n(\mathcal{J})\leftarrow H_n(\mathcal{J})\leftarrow\mathcal{H}B_n(\mathcal{J})\leftarrow\mathcal{HH}_{n+1}(\mathcal{J})\leftarrow\dots.$$

The following definition explores the notion of $\mathcal{H}$-unitarity in $\mathbb{L}_\infty$-algebras, highlighting the criteria that determine when a module $\mathcal{R}$ that classified as $\mathcal{H}$-unitary.

3.6. Definition

Let $\mathcal{R}$ be an $\mathcal{J}$-bimodule, where $\mathcal{J}$ is an $\mathbb{L}_\infty$-algebra. If every $\mathbb{L}_\infty$-module $\mathcal{G}$ admits an exact complex $(\mathcal{R}\otimes\mathcal{J}^{\otimes *},\rho_*)\otimes\mathcal{G}$, then $\mathcal{R}$ is considered $\mathcal{H}$-unitary.

An algebra $\mathcal{J}$ is $\mathcal{H}$-unital when $\mathcal{R}=\mathcal{J}$, making $\mathcal{R}$ a left $\mathcal{J}$-module. Consequently, if $\mathcal{J}$ is $\mathcal{H}$-unital, then $\mathcal{R}\otimes\mathcal{J}$ is also $\mathcal{H}$-unitary.

Next, we present a theorem on quasi-isomorphisms between complexes in the framework of $\mathbb{L}_\infty$-algebras and bimodules, establishing results based on specific conditions related to $\mathcal{H}$-unitarity.

3.7. Theorem

Consider the extension of $\mathbb{L}_\infty$-algebras given by the exact sequence:

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0,$$

Let $\mathcal{G}$ be an $\mathbb{L}_\infty$-module and $\mathcal{R}$ an $\mathcal{A}$-bimodule. The following canonical inclusions:

$$i: (\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho_*) \otimes \mathcal{G} \hookrightarrow (\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G}, \quad (9)$$

$$i': (\mathcal{R} \otimes \mathcal{J}^{\otimes *}, \rho'_*) \otimes \mathcal{G} \hookrightarrow (\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho'_*) \otimes \mathcal{G} \quad (10)$$

are quasi-isomorphisms provided that $\mathcal{R}$, as an $\mathcal{J}$-bimodule, is $\mathcal{H}$-unitary.

Proof: To prove that the inclusions i and i' are quasi-isomorphisms, we consider the filtration F^ℓ of the complex$(\mathcal{R} \otimes \mathcal{A}^{\otimes *}, \rho_*)$ and analyze its graded components. Define the filtration as follows:

$$F^\ell = \{\mathcal{R} \overset{\rho_1}{\leftarrow} \mathcal{R} \otimes \mathcal{A} \overset{\rho_2}{\leftarrow} \mathcal{R} \otimes \mathcal{A}^{\otimes 2} \overset{\rho_3}{\leftarrow} \dots \overset{\rho_\ell}{\leftarrow} \mathcal{R} \otimes \mathcal{A}^{\otimes p} \overset{\rho_{\ell+1}}{\longleftarrow} \mathcal{R} \otimes \mathcal{J} \otimes \mathcal{A}^{\otimes \ell} \overset{\rho_{\ell+2}}{\longleftarrow} \mathcal{R} \otimes \mathcal{J}^{\otimes 2} \otimes \mathcal{A}^{\otimes \ell} \overset{\rho_{\ell+3}}{\longleftarrow} \dots\}$$

For all $\ell \geq 0$.The associated graded terms satisfy:

$$\left(F^{\ell+1} \otimes \frac{\mathcal{G}}{F^\ell} \otimes \mathcal{G}\right)_* = \left(\mathcal{R} \otimes \mathcal{J}^{\otimes\, *-\ell-1}, \rho'_*\right) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes \ell} \otimes \mathcal{G}, \quad (11)$$

Since $\mathcal{R}$ is $\mathcal{H}$-unitary, this sequence is exact. Using the long exact sequence

$$0 \to F^n \otimes \mathcal{G} \to F^{n+1} \otimes \mathcal{G} \to \frac{F^{n+1} \otimes \mathcal{G}}{F^n \otimes \mathcal{G}} \to 0 \quad (12)$$

we see that $F^0 \to F^\ell$ is a quasi-isomorphism for every ℓ, which implies that ℓ is a quasi-isomorphism. By a similar argument, ℓ' also follows as a quasi-isomorphism, thus proving the theorem.

Remark: It is important to note that Theorem (3.7) above can also be proven in the case where $\mathcal{J}$ is a right ideal of $\mathcal{A}$ rather than a two-sided ideal.

The following corollary offers additional understanding of quasi-isomorphisms within the context of extensions of $\mathbb{L}_\infty$-algebras, particularly regarding modules and the criteria for $\mathcal{H}$-unitarity.

3.8. Corollary

Consider the exact sequence $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$ of the $\mathbb{L}_\infty$-algebras where $\mathcal{G}$ is a k-moduleand $\mathcal{J} \subset \mathcal{A} \subset \mathcal{B}$. The canonical arrows:

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$$\pi: \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*\right) \otimes \mathcal{G} \to \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*\right) \otimes \mathcal{G},$$

$$\pi': \left(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, {\rho'}_*\right) \otimes \mathcal{G} \to \left(\mathcal{B} \otimes \mathcal{B}^{\otimes *}, {\rho'}_*\right) \otimes \mathcal{G},$$

remain quasi-isomorphisms, when $\mathcal{J}$ is $\mathcal{H}$-unital.

Proof: To prove that the canonical maps $\pi$and π' are quasi-isomorphisms, we consider the quotient complex $\tilde{F}^{\ell}$ for $(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*)$, defined as follows:

$$\tilde{F}^{\ell} \coloneqq \mathcal{B} \overset{\rho_1}{\leftarrow} \mathcal{B} \otimes \mathcal{B} \overset{\rho_2}{\leftarrow} \mathcal{B} \otimes \mathcal{B}^{\otimes 2} \overset{\rho_3}{\leftarrow} \dots \overset{\rho_\ell}{\leftarrow} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \overset{\rho_{\ell+1}}{\longleftarrow} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A} \overset{\rho_{\ell+2}}{\longleftarrow} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A}^{\otimes 2} \overset{\rho_{\ell+3}}{\longleftarrow} \dots .$$

Next, check if π is a quasi-isomorphism, we analyze the canonical projections π^{ℓ}:

$$\pi^{\ell}: \tilde{F}^{\ell} \otimes \mathcal{G} \to \tilde{F}^{\ell+1} \otimes \mathcal{G}.$$

Since $\mathcal{B}(\ell) = \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{J}$, a simple calculation shows that:

$$Ker(\pi^{\ell}) = (\mathcal{B}(\ell) \otimes \mathcal{A}^{\otimes * - \ell - 1}, \rho_*) \otimes \mathcal{G}.$$

By applying Theorem (3.7), we deduce that $Ker(\pi^{\ell})$ is quasi-isomorphic to:

$$\left(\mathcal{B}(\ell) \otimes \mathcal{J}^{\otimes * - \ell - 1}, \rho_*\right) \otimes \mathcal{G} = (\mathcal{B}(\ell) \otimes \mathcal{J}^{\otimes * - \ell - 1}, \rho'_*) \otimes \mathcal{G}, \quad (13)$$

this is exact by assumption. A similar proof holds for π', confirming that both π and π' are quasi-isomorphisms.

Now, we complete with a theorem that connects H-unitality, excision, and homology conditions for $\mathbb{L}_\infty$-algebras, highlighting their equivalence.

3.9. Theorem

Let $\mathcal{J}$ be an $\mathbb{L}_\infty$-algebra. Then, the following statements are equivalent:

1. $\mathcal{J}$ is H-unital.
2. $\mathcal{J}$ satisfies the H-homology excision property.
3. $\mathcal{J}$ satisfies the excision property for bar homology.
4. $\mathcal{J}$satisfies the excision property for simplicial homology.

Proof: To establish the equivalence of the given propositions, we analyze the homological properties of the $\mathbb{L}_\infty$-algebra $\mathcal{J}$. Consider the short exact sequence of $\mathbb{L}_\infty$-algebras $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$,where $\mathcal{A}$is a pure extension, and let $\mathcal{G}$ be a k-module. The canonical projection defined as:

$$\pi: (\mathcal{A} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} \to (\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*) \otimes \mathcal{G}$$

This projection induces the following commutative diagram of short exact sequences:

$$0 \to (\mathcal{J} \otimes \mathcal{A}^{\otimes\,*}, \rho_*) \otimes \mathcal{G} \to (\mathcal{A} \otimes \mathcal{A}^{\otimes\,*}, \rho_*) \otimes \mathcal{G} \to (\mathcal{B} \otimes \mathcal{A}^{\otimes\,*}, \rho_*) \otimes \mathcal{G} \to 0$$
$$|j \qquad\qquad | = \qquad\qquad |\pi_1$$
$$0 \to \quad ker\,(\pi) \quad \to (\mathcal{A} \otimes \mathcal{A}^{\otimes\,*}, \rho_*) \otimes \mathcal{G} \xrightarrow{\pi} (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} \to 0$$

Using Corollary (3.8), we know that π_1 is a quasi-isomorphism. As a result, j must also be a quasi-isomorphism, proving the equivalence of (1) and (2). A similar argument applies to show that (1) and (3) are equivalent.

Furthermore, considering the long exact sequence of homology:

$$\cdots \leftarrow H_{n-1}(\mathcal{J}) \leftarrow \mathcal{H}B_{n-1}(\mathcal{J}) \leftarrow \mathcal{H}\mathcal{H}_n(\mathcal{J}) \leftarrow H_n(\mathcal{J}) \leftarrow \mathcal{H}B_n(\mathcal{J}) \leftarrow \mathcal{H}\mathcal{H}_{n+1}(\mathcal{J}) \leftarrow \cdots$$

we conclude that (2) and (4) are equivalent.

To establish the equivalence of (2) and (1), we assume that $\mathcal{A} = \mathcal{J} \oplus \mathcal{G}$ is a k-algebra with a k-module $\mathcal{G}$, and define the projection:

$$\pi: (\mathcal{A} \otimes \mathcal{A}^{\otimes\,*}, \rho_*) \to (\mathcal{G} \otimes \mathcal{G}^{\otimes *}, \rho_*),$$

where the multiplication is given by$(u, v)(u', v') = (uu', 0)$.

Since the kernel of π satisfies:

$$\ker(\pi) = \mathcal{G} \otimes \left(\mathcal{J} \otimes \mathcal{J}^{\otimes\,*-1}, \rho'_*\right) \oplus (\mathcal{J} \otimes \mathcal{J}^{\otimes *}, \rho'_*),$$

it follows that $\mathcal{J}$ satisfies the excision property for$H$-homology, ensuring exactness in the associated complexes. By similar reasoning, we establish the equivalence of (3) and (1).

For (4) and (1), we introduce the canonical projection:

$$\bar{\pi}: \mathcal{C}_{**}(\mathcal{A}) \to \mathcal{C}_{**}(\mathcal{G}),$$

and define a sub-complex β in $ker\,(\bar{\pi})$ consisting of elements $(a_0 \otimes \cdots \otimes a_n, a'_0 \otimes \cdots \otimes a'_{n-1})$ that include some a_i and a'_n in $\mathcal{G}$. The exactness of β ensures that:

$$ker(\bar{\pi}) = \mathcal{C}_{**}(\mathcal{J}) \oplus \beta,$$

Proving that $\mathcal{J}$ satisfies the simplicial homology excision property.

Finally, assuming that $\mathcal{J}$ is not H-unital leads to a contradiction. Suppose there exists $x \in \mathcal{G} \otimes \mathcal{J}^{\otimes n}$representing a cycle that does not represent a boundary under ρ'_n. Then, the element$\big(0, N(x)\big)$forms a cycle of degree $n+1$ that is not a boundary, contradicting the exactness of β. This contradiction confirms that all four propositions are equivalent, completing the proof.

Conclusion

We have thoroughly examined the homological behavior of ∞-algebras through various frameworks, including excision, simplicial, and bar homologies. Our findings highlight the preservation of homological structures under certain algebraic mappings and inclusions. The relationship between different types of homology has been made clear, with particular emphasis on their preservation under quasi-isomorphisms. Additionally, we have discussed the conditions required for H-unit and the implications of these conditions for the broader theory of ∞-algebras. The results of this paper contribute to a deeper understanding of how these algebraic structures interact with homological properties, with potential applications in the study of algebraic topology and homotopy theory.

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All parts contained in the research were carried out by the authors through hard work and a review of the various references and contributions in the field of mathematics. Authors have read and approved the final manuscript.

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