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Acceptance Sampling Plans for the Power Inverted Topp Leone Power Series Class with Application to Reliable Renewable Energy Sources to Reduce CO2 Emissions

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Abstract

The Power Inverted Topp Leone Power Series (PITLPS) class represents a novel three-parameter iteration of the Power Inverted Topp Leone (PITL) distribution. Formed by combining a power series distribution with the PITL distribution, this compounding technique enables the creation of versatile distribution classes with significant applications in fields like biology and engineering. The PITLPS class exhibits various hazard rate shapes. Acceptance sampling plans are devised for the PITLPS class model, assuming the life test concludes at a specific time. The truncation time, representing the median lifetime of the PITLPS class distribution with the selected variables, is considered. The minimal sample size necessary to achieve the specified life test under a defined consumer's risk is determined. Ultimately, the PITL Poisson model is employed in the context of Reliable Renewable Energy Sources to Reduce CO2 Emissions, and it is fitted using the provided statistical model.

Key words: *Acceptance sampling plans, Power inverted Topp-Leone, Power series, Poisson model, Renewable energy.*

1. Introduction

Acceptance sampling, a fundamental aspect of quality assurance, involves inspecting and making decisions about batches of products and stands as one of the oldest techniques in this field. A common scenario where acceptance sampling is applied is when a company receives a shipment of products from a vendor, often components or raw materials vital to the manufacturing process. The process typically unfolds as follows:

1. Sampling: A sample is extracted from the received lot, and the pertinent quality characteristic of the units within the sample is examined.
2. Decision-Making: Based on the information obtained from the sample, a decision is reached concerning whether to accept or reject the entire lot.
    - For accepted samples, the lots proceed to production.
    - For rejected samples, the lots may be returned to the vendor or subjected to other predetermined actions.

Recently, researchers have shown a growing interest in exploring new univariate distributions due to their practical applications in modeling various types of data. Numerous strategies for generating novel distributions have been proposed and investigated in the statistical literature. One particularly valuable approach is the compounding method, which involves

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combining useful lifetimes with truncated discrete distributions. In this context, a non-negative random variable (X), determined by either the minimum or maximum values, is employed to represent the lifetime of X_i $(i = 1,2,\dots,N)$ in a system with N components, depending on whether these components are arranged in series or in parallel. The continuous random variables X_i are independent of N. The Power Series (PS) distribution emerges as a significant discrete distribution in this framework. The PS distribution encompasses various distributions such as Poisson, logarithmic, geometric, and binomial distributions (truncated at zero). The probability mass function (PMF) of the PS distribution is given by

$$P(N = n; \gamma) = \frac{a_n \gamma^n}{A(\gamma)}, n = 1,2,3, \dots \tag{1}$$

In the given context, the variable a_n is dependent solely on n, where a_n is greater than 0, representing the scale parameter. Furthermore, $A(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$.

Researchers have recently explored diverse classes and distributions by compounding well-established continuous distributions with discrete random variables. Some noteworthy examples include the Weibull-PS class by Morais and Barreto-Souza (2011), the generalized exponential PS class proposed by Mahmoudi and Jafari (2012), and the extended Weibull-PS class introduced by Silva *et al.* (2013). Other notable models include the Burr XII-geometric distribution by Korkmaz and Erisoglu (2014), the complementary Poisson Lindley-PS class by Hassan and Nassr (2019), and the Lindley-PS family by Warahena-Liyanage and Pararai (2015). Silva and Corderio (2015) presented the Burr XII-PS family, while Jafari and Tahmasebi (2016) introduced the Gompertz PS class. Further contributions include the generalized modified Weibull-PS class by Bagheri *et al.* (2016) and the generalized extended Weibull-PS class by Alkarni (2016). Elbatal *et al.* (2017) presented the exponential Pareto-PS class. Hassan and Abd-Alla (2017) introduced the exponentiated Lomax geometric distribution class, and Oluyede *et al.* (2018) contributed the Burr-Weibull PS class. The diverse range continues with models like the Marshall-Olkin generalized-G Poisson family (Korkmaz *et al.*, 2018), the power Lomax Poisson distribution (Hassan and Nassr, 2018), and the odd log-logistic class (Alizadeh *et al.*, 2018), among numerous others. This ongoing research effort showcases the richness and versatility of compounding methods in creating novel distributions for various statistical applications.

Inverted distributions have garnered significant attention and found wide applications in medical, economic, and engineering sciences, prompting various authors to delve into their study. Notable contributions include works by Calabria and Pulcini (1990), Al-Dayian (1999), Sharma *et al.* (2015), Abd AL-Fattah *et al.* (2017), Tahir *et al.* (2018), Hassan and Abd-Alla (2019), Hassen and Mohammed (2019), and Hassan and Nassr (2021). Recently, Hassan *et al.* (2020) proposed the inverted Topp-Leone (ITL) distribution, characterized by its probability density function (PDF) and cumulative distribution function (CDF) formulated as follows:

$$g(x; \varphi) = 2\varphi x^2 \, (1 + x)^{-2\varphi-1}(1 + 2x)^{\varphi-1} \, ; x, \varphi > 0,$$

and

$$G(x; \varphi) = 1 - \left\{\frac{(1 + 2x)^{\varphi}}{(1 + x)^{2\varphi}}\right\}.$$

where φ is the shape parameter. In the context of estimating parameters from a censored sample, particularly focusing on the shape parameter, researchers have explored various approaches to enhance the goodness of fit. One such approach involves the utilization of the power transformation, as proposed by Box and Cox (1964). This method provides an additional option that enhances the flexibility of hazard functions, making them more adept at describing different types of real data. Addressing this, Abushal *et al.* (2021) introduced the power inverted Topp-Leone (PITL) distribution. The probability density function (PDF) of the PITL distribution, as presented by Abushal *et al.* (2021), is given by

$$g(x;\delta,\varphi) = 2\delta\varphi x^{2\delta-1}\,(1+x^{\delta})^{-2\varphi-1}(1+2x^{\delta})^{\varphi-1}\,;\, x,\delta,\varphi > 0 \tag{2}$$

where δ and φ are shape parameters. The CDF associated with (2) is given by:

$$G(x;\delta,\varphi) = 1 - \left\{\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\} \tag{3}$$

In comparison to the inverted Topp-Leone (ITL) distribution, the power inverted Topp-Leone (PITL) distribution boasts several distinctive characteristics. Notably, it introduces added flexibility to both density and hazard rate functions, along with an impact on kurtosis. Building upon this, a novel compound class named the PITLPS distribution is introduced, considering a system with parallel components and amalgamating the PITL and Power Series (PS) distributions. The introduction of this class is motivated by its ability to (i) manifest a spectrum of hazard rate shapes, (ii) encompass a new compound class and several new compound distributions, and (iii) yield superior fits compared to certain other distributions.

Before delving into a specific distribution within the class, we compile the key attributes of the PITLPS distribution. The estimation methodology utilized is Maximum Likelihood (ML), a frequentist approach. Additionally, for the PITL Poisson (PITLP) distribution, an acceptance sampling plan is conducted. The overarching objective of this research is to formulate a sampling plan, define its operating characteristic function, and articulate the corresponding decision rule. This entails establishing a systematic approach to guide decisions regarding lot acceptance or rejection based on inspection outcomes, ultimately contributing to the enhancement of efficiency and reliability in quality control processes within manufacturing and supply chain management. In the final phase, the study evaluates the efficacy of the PITLP distribution as a sub-model using real data, aiming to improve energy reliability and significantly reduce CO2 emissions. The PITLP distribution emerges as a promising sub-model, illustrating its potential impact on both energy systems and environmental sustainability.

The remainder of this paper is structured as follows. Section 2 establishes the definitions for the density and limiting behavior. Moving on to Section 3, we derive the ML estimator for the class. Section 4 is dedicated to exploring the acceptance sampling plans of the PITLP distribution. Section 5 offers real-world data examples to illustrate the versatility of the PITLP distribution, applied specifically to energy reliability and the substantial reduction of CO2 emissions. The conclusions of the paper are presented in the final section.

2. The Power Inverted Topp-Leone Power Series Class

El-Saeed *et al.* (2023) proposed the power inverted Topp Leone power series (PITLPS) class. Considering independent and identically distributed (iid) random variables $X_1,\ldots,X_N$ with PDF (2) and CDF (3), and assuming that N is a discrete random variable following a PS distribution (truncated at zero) with PMF (1), the density function of the PITLPS distribution is computed given $X = \max\{X_i\}_{i=1}^{N}$, the conditional CDF of $X|N$ is given by:

$$F_{X|N=n}(x) = [G(x;\delta,\varphi)]^n = \left\{1 - \frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}^n$$

The PITLPS distribution is characterized by the marginal CDF of X, which is expressed as:

$$F(x;\delta,\varphi,\gamma)=\sum_{n=1}^{\infty}\frac{a_n\gamma^n}{A(\gamma)}\left(1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right)^n=\frac{1}{A(\gamma)}A\left(\gamma\left(1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right)\right). \quad (4)$$

Moreover, the CDF of the PITLP distribution is derived by selecting $A(\gamma)=e^{\gamma}-1$, where $\gamma\in(0,\infty)$, and is expressed as follows:

$$F(x;\delta,\varphi,\gamma)=\frac{1}{(e^{\gamma}-1)}\left(e^{\gamma\left\{1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}}-1\right) \quad (5)$$

A random variable with CDF (5), adhering to the PITLPS distribution with parameters $\delta,\varphi,$ and γ, is represented as $X\sim$ PITLPS (δ,φ,γ). The PDF of the PITLPS family associated with (5) is expressed as follows:

$$f(x;\delta,\varphi,\gamma)=\frac{2\delta\varphi\gamma x^{2\delta-1}(1+x^{\delta})^{-2\varphi-1}(1+2x^{\delta})^{\varphi-1}}{(e^{\gamma}-1)}e^{\gamma\left\{1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}} \quad (6)$$

The survival function and hazard rate function (HRF) of the class are given by

$$S(x;\delta,\varphi,\gamma)=1-\frac{1}{(e^{\gamma}-1)}\left(e^{\gamma\left\{1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}}-1\right).$$

and

$$h(x;\delta,\varphi,\gamma)=2\delta\varphi y^{2\delta-1}\gamma\,(1+x^{\delta})^{-2\varphi-1}(1+2x^{\delta})^{\varphi-1}e^{\gamma\left\{1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}}\left(e^{\gamma}-e^{\gamma\left\{1-\frac{(1+2x^{\delta})^{\varphi}}{(1+x^{\delta})^{2\varphi}}\right\}}\right)^{-1}$$

Fig 1 illustrates the PDF and HRF plots of the PITLP distribution. The PDF plots exhibit right skewness, unimodality, and variations in the increasing and decreasing patterns for different parameter values, as depicted below. Additionally, the HRF forms for the PITLP distribution, corresponding to the specified parameter values, demonstrate an initial increase followed by a decrease (increasing-decreasing) and an upside-down shape.

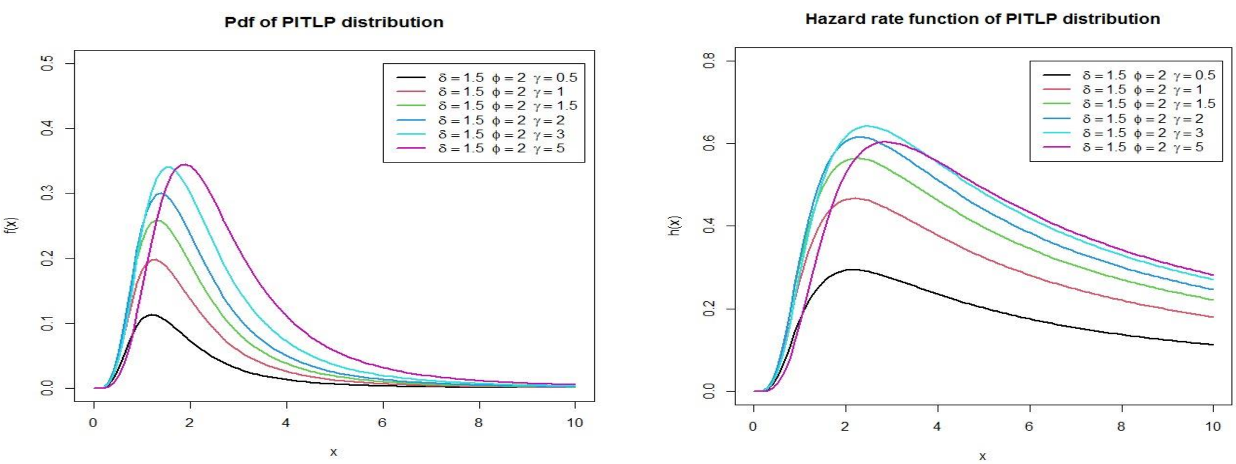

Figure 1: The PDF and HRF plots of the PITLP distribution

3. Maximum Likelihood Estimation

Consider a random sample $X_1, X_2, \ldots, X_n$from the PITLP distribution. The log-likelihood function for the parameters, denoted as lnℓ, is expressed as (El-Saeed *et al.* (2023)):

$$\ln\ell = n\ln(2\delta\varphi\gamma) + (2\delta-1)\sum_{i=1}^{n}\ln x_i - (2\varphi+1)\sum_{i=1}^{n}\ln K_i + (\varphi-1)\sum_{i=1}^{n}\ln D_i - n\ln(e^{\gamma}-1) + \gamma\sum_{i=1}^{n}\left(1 - D_i^{\varphi}K_i^{-2\varphi}\right),$$

where, $K_i = \left(1 + x_i^{\delta}\right)$ and $D_i = \left(1 + 2x_i^{\delta}\right)$. The partial derivatives of lnℓ with respect to γ, δ and φ are expressed as follows:

$$\frac{\partial\, ln\, \ell}{\partial\gamma} = \frac{n}{\gamma} - \frac{ne^{\gamma}}{(e^{\gamma}-1)} + \sum_{i=1}^{n}\left(1 - D_i^{\varphi}K_i^{-2\varphi}\right),$$

$$\frac{\partial\, ln\, \ell}{\partial\delta} = \frac{n}{\delta} + 2\sum_{i=1}^{n}\ln y_i - (2\varphi+1)\sum_{i=1}^{n}x_i^{\delta}\ln x_i\, K_i^{-1} + 2(\varphi-1)\sum_{i=1}^{n}x_i^{\delta}\ln x_i\, D_i^{-1} - 2\varphi\gamma\sum_{i=1}^{n}x_i^{\delta}\ln x_i\, D_i^{\varphi}K_i^{-2\varphi}\left(D_i^{-1} - K_i^{-1}\right),$$

and,

$$\frac{\partial\, ln\, \ell}{\partial\varphi} = \frac{n}{\varphi} - 2\sum_{i=1}^{n}\ln K_i + \sum_{i=1}^{n}\ln D_i - \gamma\sum_{i=1}^{n}D_i^{\varphi}K_i^{-2\varphi}\left(\ln D_i - 2\ln K_i\right).$$

The ML estimators for γ, δ and φ, denoted as $\hat{\gamma}, \hat{\delta}$ and $\hat{\varphi}$ respectively, are obtained by solving the nonlinear equations $\frac{\partial\, ln\, \ell}{\partial\gamma} = 0$, $\frac{\partial\, ln\, \ell}{\partial\delta} = 0$ and $\frac{\partial\, ln\, \ell}{\partial\varphi} = 0$ through an iterative approach.

4. Acceptance Sampling Plans

In our analysis, we assume the product's lifetime conforms to the PITLP distribution characterized by parameters $(\delta, \varphi, \gamma)$, as described by the CDF in (5), with a specified median lifetime denoted as M_0. Our objective is to draw conclusions regarding the acceptance or rejection of the proposed lot. This determination hinges on the criterion that the actual median lifetime (M) of the units surpasses the claimed lifetime M_0. In life testing, a standard practice involves concluding the test at a predetermined time T_0 and recording the number of failures. To observe the median lifetime, the experiment runs for a duration of T_0, which is a multiple of the claimed median lifetime with any positive constant a. The decision to accept the proposed lot is based on the evidence that $M \geq M_0$, with a specified probability of at least α (consumer's risk). This approach follows a single acceptance sampling plan (ASP), as outlined by Singh and Tripathi (2017).

Randomly select n units from the proposed lot and conduct a T_0-unit experiment. If, during the experiment, the number of units failing (acceptance number, denoted as n_0 or fewer) is observed, then accept the entire lot; otherwise, reject the lot. Note that the probability of accepting a lot, assuming sufficiently large-sized lots to apply the binomial distribution, under the proposed sampling plan is expressed as:

$$L(p) = \sum_{i=0}^{n_0} \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 1,2,\dots,n$$

where $p = F_{\{PITLP\}}(T_0; \delta, \varphi, \gamma)$, as defined by equation (5). The function $L(p)$ represents the operating characteristic function of the sampling plan, specifically the acceptance probability of the lot as a function of the failure probability. Additionally, by employing $T_0 = aM_0$, the expression for p can be formulated as:

$$p = \frac{1}{(e^{\gamma}-1)}\left(e^{\gamma\left\{1-\frac{(1+2T_0^{\delta})^{\varphi}}{(1+T_0^{\delta})^{2\varphi}}\right\}} - 1\right) \tag{7}$$

Now, the problem is to determine for given values of α $(0 < \alpha < 1)$, T_0 and n_0, the smallest positive integer n such that

$$L(p) = \sum_{i=0}^{n_0} \binom{n}{i} p^i (1-p)^{n-i} \leq 1-\alpha \tag{8}$$

where p is given by (7). The minimum values of n satisfying the inequality in (8) and its corresponding operating characteristic probability are obtained and displayed from Table (1) to Table (9) for the following assumed parameters:

1. The parameters of the PITLP distribution are assumed as follows: $\delta = \phi = (1.5, 2.5)$ and $\gamma = (0.5, 1.5, 2.5)$.
2. The parameters of the acceptance sampling plan are assumed as follows: $p = 0.05, 0.25, 0.50, 0.75, 0.99$, $n_0 = 0, 5, 10, 15, 20, 30$, and $a = 0.25, 0.50, 0.75, 1$. Note that when $a = 1$, thus $T_0 = M_0 = 0.5$ for all δ, ϕ, γ.

From the results obtained in Table (1) to Table (9), we observe that:

- With an increase in p and n_0, the required sample size n and $L(p)$ are increasing.
- With an increase in a, the required sample size n and $L(p)$ are decreasing.

Finally, for all the obtained results, we verified that $L(p) \leq 1-\alpha$. Also, when $a = 1$, we have $p = 0.5$ as $T_0 = M_0$, and hence all results $(n, L(p))$ for any vector of parameters (δ, ϕ, γ) are the same.

Table (1): ASP for PITLP distribution at $\delta = 1.5, \varphi = 1.5, \gamma = 0.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	2	0.95864	1	1.00000	1	1.00000	1	1.00000
	5	65	0.95123	16	0.95633	9	0.97306	7	0.98437
	10	152	0.95020	36	0.95374	20	0.96208	15	0.97131
	15	246	0.95026	57	0.95624	32	0.95280	24	0.95342
	20	344	0.95030	80	0.95132	44	0.95138	32	0.96461
	30	547	0.95048	126	0.95153	68	0.95624	50	0.95727

0.25	0	7	0.77613	2	0.81499	1	1.00000	1	1.00000
	5	103	0.75285	24	0.75792	13	0.77781	9	0.85546
	10	210	0.75055	48	0.75774	26	0.75711	19	0.75964
	15	319	0.75356	73	0.75147	39	0.75781	28	0.77894
	20	431	0.75139	98	0.75264	52	0.76387	37	0.79745
	30	657	0.75095	148	0.76203	79	0.75326	56	0.79059
0.50	0	17	0.50874	4	0.54131	2	0.64574	1	1.00000
	5	137	0.50542	31	0.51076	16	0.55036	11	0.62303
	10	258	0.50318	58	0.50825	30	0.54247	21	0.58807
	15	379	0.50210	85	0.50726	44	0.54000	31	0.57219
	20	500	0.50140	112	0.50674	59	0.50022	41	0.56264
	30	742	0.50047	166	0.50625	87	0.50742	61	0.55123
0.75	0	33	0.25881	7	0.29302	4	0.26926	2	0.49999
	5	179	0.25118	39	0.27044	20	0.28347	14	0.29051
	10	314	0.25098	69	0.26433	36	0.25446	25	0.27060
	15	446	0.25066	99	0.25157	51	0.25936	35	0.30376
	20	576	0.25134	128	0.25087	66	0.25865	46	0.27570
	30	834	0.25027	185	0.25417	96	0.25121	67	0.26924
0.99	0	110	0.01001	23	0.01110	11	0.01261	7	0.01562
	5	313	0.01024	67	0.01089	33	0.01190	22	0.01330
	10	482	0.01022	104	0.01069	52	0.01086	35	0.01215
	15	641	0.01014	139	0.01042	70	0.01019	47	0.01294
	20	794	0.01014	173	0.01008	87	0.01057	59	0.01237
	30	1090	0.01015	238	0.01027	120	0.01127	82	0.01282

Table (2): ASP for PITLP distribution at $\delta = 1.5, \varphi = 1.5, \gamma = 1.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	2	0.95837	1	1.00000	1	1.00000	1	1.00000
	5	64	0.95307	16	0.95968	9	0.97430	7	0.98437
	10	151	0.95027	37	0.95013	20	0.96455	15	0.97131
	15	244	0.95090	58	0.95607	32	0.95651	24	0.95343
	20	342	0.95006	81	0.95354	44	0.95581	32	0.96462
	30	543	0.95097	128	0.95249	69	0.95217	50	0.95728

0.25	0	7	0.77483	2	0.81832	1	1.00000	1	1.00000
	5	102	0.75524	24	0.77129	13	0.78502	9	0.85547
	10	208	0.75368	49	0.75430	26	0.76783	19	0.75966
	15	317	0.75338	74	0.75692	39	0.77089	28	0.77897
	20	428	0.75215	99	0.76425	52	0.77873	37	0.79748
	30	653	0.75035	151	0.75790	79	0.77195	56	0.79062
0.50	0	17	0.50646	4	0.54798	2	0.64898	1	1.00000
	5	136	0.50628	31	0.52981	16	0.56094	11	0.62305
	10	256	0.50495	59	0.50952	31	0.50304	21	0.58810
	15	377	0.50020	86	0.51840	45	0.51324	31	0.57223
	20	497	0.50056	114	0.50749	59	0.52090	41	0.56268
	30	737	0.50113	169	0.50672	88	0.50099	61	0.55129
0.75	0	33	0.25650	7	0.30029	4	0.27333	2	0.50000
	5	178	0.25043	40	0.26381	20	0.29354	14	0.29053
	10	312	0.25087	71	0.25164	36	0.26743	25	0.27063
	15	443	0.25102	100	0.26386	51	0.27505	35	0.30380
	20	572	0.25209	130	0.25562	66	0.27655	46	0.27574
	30	828	0.25164	188	0.25872	96	0.27262	67	0.26929
0.99	0	109	0.01013	23	0.01214	11	0.01325	7	0.01562
	5	311	0.01023	68	0.01131	33	0.01309	22	0.01330
	10	479	0.01019	106	0.01068	52	0.01227	35	0.01215
	15	637	0.01010	142	0.01008	70	0.01177	47	0.01295
	20	789	0.01011	176	0.01033	88	0.01031	59	0.01237
	30	1083	0.01014	242	0.01066	121	0.01160	82	0.01283

Table (3): ASP for PITLP distribution at $\delta = 1.5, \varphi = 1.5,\ \gamma = 2.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	2	0.96329	1	1.00000	1	1.00000	1	1.00000
	5	73	0.95108	17	0.95933	10	0.95102	7	0.98437
	10	170	0.95168	39	0.95374	21	0.95342	15	0.97131
	15	276	0.95117	62	0.95541	33	0.95088	24	0.95343
	20	387	0.95035	87	0.95094	45	0.95417	32	0.96462
	30	616	0.95017	137	0.95179	70	0.95578	50	0.95728

0.25	0	8	0.76965	2	0.83052	1	1.00000	1	1.00000
	5	116	0.75201	26	0.75979	13	0.80141	9	0.85547
	10	236	0.75211	52	0.76249	26	0.79195	19	0.75966
	15	360	0.75065	79	0.75982	40	0.76367	28	0.77897
	20	485	0.75224	106	0.76360	53	0.78142	37	0.79748
	30	740	0.75064	162	0.75413	81	0.76355	56	0.79062
0.50	0	19	0.51005	4	0.57286	2	0.65655	1	1.00000
	5	155	0.50074	34	0.50374	17	0.51228	11	0.62305
	10	291	0.50125	63	0.51400	31	0.53807	21	0.58810
	15	427	0.50172	93	0.50209	46	0.51219	31	0.57223
	20	563	0.50212	122	0.51000	60	0.53151	41	0.56269
	30	836	0.50009	181	0.50822	89	0.52938	61	0.55129
0.75	0	38	0.25061	8	0.27255	4	0.28301	2	0.50000
	5	201	0.25410	43	0.26156	21	0.26507	14	0.29053
	10	354	0.25041	76	0.25411	37	0.26024	25	0.27063
	15	502	0.25219	108	0.25362	52	0.27972	35	0.30380
	20	649	0.25146	139	0.26083	68	0.26185	46	0.27574
	30	939	0.25178	202	0.25501	99	0.25291	67	0.26929
0.99	0	124	0.01005	25	0.01160	11	0.01488	7	0.01562
	5	353	0.01024	74	0.01018	34	0.01241	22	0.01330
	10	544	0.01013	114	0.01062	54	0.01036	35	0.01215
	15	723	0.01010	152	0.01063	72	0.01102	47	0.01295
	20	896	0.01002	189	0.01042	90	0.01048	59	0.01237
	30	1229	0.01013	260	0.01062	124	0.01133	82	0.01283

Table (4): ASP for PITLP distribution at $\delta = 1.5, \varphi = 2.5,\ \gamma = 0.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	2	0.97060	1	1.00000	1	1.00000	1	1.00000
	5	91	0.95022	19	0.95232	10	0.95894	7	0.98437
	10	212	0.95108	42	0.95693	21	0.96416	15	0.97131
	15	344	0.95095	68	0.95315	34	0.95192	24	0.95343
	20	482	0.95060	95	0.95073	46	0.95954	32	0.96462
	30	768	0.95022	150	0.95059	73	0.95032	50	0.95728

0.25	0	10	0.76450	2	0.84540	1	1.00000	1	1.00000
	5	144	0.75451	28	0.76952	14	0.76424	9	0.85546
	10	294	0.75310	57	0.75928	27	0.78721	19	0.75965
	15	449	0.75091	87	0.75080	41	0.77811	28	0.77895
	20	606	0.75027	117	0.75043	55	0.77777	37	0.79747
	30	923	0.75159	177	0.75766	84	0.76139	56	0.79061
0.50	0	24	0.50346	5	0.51080	2	0.66892	1	1.00000
	5	193	0.50277	37	0.50967	17	0.55465	11	0.62304
	10	363	0.50213	69	0.51391	32	0.54443	21	0.58809
	15	533	0.50196	102	0.50020	47	0.54069	31	0.57222
	20	703	0.50191	134	0.50520	63	0.50319	41	0.56267
	30	1043	0.50193	199	0.50041	93	0.50857	61	0.55127
0.75	0	47	0.25347	9	0.26092	4	0.29931	2	0.50000
	5	252	0.25096	47	0.26474	22	0.25559	14	0.29052
	10	442	0.25097	83	0.25958	38	0.27499	25	0.27062
	15	628	0.25024	118	0.25953	54	0.27887	35	0.30378
	20	811	0.25094	153	0.25514	71	0.25100	46	0.27573
	30	1174	0.25015	222	0.25126	102	0.26967	67	0.26927
0.99	0	155	0.01010	28	0.01073	12	0.01200	7	0.01562
	5	442	0.01017	81	0.01069	36	0.01062	22	0.01330
	10	681	0.01003	126	0.01006	56	0.01083	35	0.01215
	15	904	0.01012	167	0.01075	75	0.01083	47	0.01295
	20	1120	0.01006	208	0.01025	93	0.01165	59	0.01237
	30	1537	0.01008	286	0.01044	129	0.01121	82	0.01283

Table (5): ASP for PITLP distribution at $\delta = 1.5, \varphi = 2.5,\ \gamma = 1.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	2	0.97242	1	1.00000	1	1.00000	1	1.00000
	5	96	0.95196	20	0.95241	10	0.96372	7	0.98438
	10	226	0.95078	45	0.95253	22	0.95539	15	0.97131
	15	367	0.95038	72	0.95259	34	0.96158	24	0.95343
	20	514	0.95017	100	0.95302	47	0.96009	32	0.96462
	30	818	0.95038	158	0.95337	74	0.95662	50	0.95728

0.25	0	11	0.75603	2	0.85425	1	1.00000	1	1.00000
	5	154	0.75172	30	0.75953	14	0.78426	9	0.85547
	10	314	0.75073	60	0.76537	28	0.77201	19	0.75966
	15	478	0.75202	92	0.75304	42	0.77792	28	0.77897
	20	645	0.75192	124	0.75000	57	0.75852	37	0.79748
	30	984	0.75092	188	0.75358	86	0.76350	56	0.79062
0.50	0	25	0.51108	5	0.53252	2	0.67724	2	0.50000
	5	206	0.50119	39	0.51513	18	0.51525	12	0.50000
	10	387	0.50165	73	0.51671	33	0.53476	22	0.50000
	15	568	0.50212	108	0.50260	49	0.50825	32	0.50000
	20	750	0.50007	142	0.50647	64	0.52418	42	0.50000
	30	1112	0.50120	211	0.50085	95	0.51930	62	0.50000
0.75	0	50	0.25400	9	0.28358	4	0.31061	3	0.25000
	5	268	0.25288	50	0.26228	22	0.28225	14	0.29053
	10	471	0.25131	88	0.26057	39	0.27480	25	0.27063
	15	669	0.25106	126	0.25044	56	0.26145	35	0.30380
	20	864	0.25178	162	0.25871	72	0.27313	46	0.27574
	30	1251	0.25074	235	0.25600	105	0.26248	67	0.26929
0.99	0	165	0.01019	30	0.01037	12	0.01374	7	0.01563
	5	472	0.01004	86	0.01081	37	0.01068	22	0.01330
	10	726	0.01005	133	0.01087	57	0.01220	35	0.01215
	15	964	0.01010	178	0.01032	77	0.01100	47	0.01295
	20	1194	0.01008	221	0.01024	96	0.01083	59	0.01237
	30	1639	0.01004	304	0.01032	133	0.01048	82	0.01283

Table (6): ASP for PITLP distribution at $\delta = 1.5, \varphi = 2.5,\ \gamma = 2.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	3	0.95553	1	1.00000	1	1.00000	1	1.00000
	5	118	0.95067	22	0.95574	10	0.97054	7	0.98437
	10	277	0.95025	50	0.95587	23	0.95205	15	0.97131
	15	449	0.95060	81	0.95288	36	0.95305	24	0.95342
	20	629	0.95047	113	0.95189	49	0.95836	32	0.96461
	30	1002	0.95042	179	0.95122	77	0.95621	50	0.95727

0.25	0	13	0.76113	3	0.75924	1	1.00000	1	1.00000
	5	189	0.75002	34	0.75478	15	0.75631	9	0.85546
	10	384	0.75251	68	0.76118	29	0.77702	19	0.75963
	15	586	0.75153	104	0.75242	44	0.76866	28	0.77893
	20	791	0.75108	140	0.75188	59	0.76868	37	0.79745
	30	1206	0.75119	213	0.75038	90	0.75422	56	0.79058
0.50	0	31	0.50541	6	0.50227	2	0.69063	1	1.00000
	5	252	0.50303	44	0.51716	18	0.56329	11	0.62302
	10	475	0.50013	83	0.50992	35	0.50682	21	0.58806
	15	697	0.50080	122	0.50620	51	0.51169	31	0.57219
	20	919	0.50132	161	0.50370	67	0.51547	41	0.56264
	30	1364	0.50050	239	0.50028	99	0.52140	61	0.55123
0.75	0	61	0.25544	11	0.25228	4	0.32941	2	0.49999
	5	329	0.25214	57	0.25706	23	0.28055	14	0.29050
	10	578	0.25079	100	0.25769	41	0.26523	25	0.27060
	15	821	0.25043	142	0.25900	58	0.27449	35	0.30375
	20	1061	0.25004	184	0.25562	76	0.25272	46	0.27570
	30	1535	0.25011	267	0.25158	110	0.25469	67	0.26924
0.99	0	203	0.01011	34	0.01062	13	0.01178	7	0.01562
	5	579	0.01012	98	0.01067	39	0.01008	22	0.01330
	10	891	0.01008	152	0.01027	60	0.01143	35	0.01215
	15	1184	0.01003	202	0.01055	81	0.01023	47	0.01294
	20	1466	0.01004	251	0.01033	100	0.01162	59	0.01237
	30	2012	0.01000	345	0.01050	139	0.01063	82	0.01282

Table (7): ASP for PITLP distribution at $\delta = 2.5, \varphi = 2.5,\ \gamma = 0.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	23	0.95194	1	1.00000	1	1.00000	1	1.00000
	5	1170	0.95011	51	0.95172	13	0.95749	7	0.98437
	10	2761	0.95005	119	0.95030	29	0.95224	15	0.97131
	15	4491	0.95002	192	0.95097	46	0.95122	24	0.95343
	20	6296	0.95004	268	0.95159	63	0.95538	32	0.96462
	30	10041	0.95001	427	0.95060	99	0.95570	50	0.95728

0.25	0	129	0.75084	6	0.76121	2	0.76340	1	1.00000
	5	1888	0.75003	80	0.75757	19	0.76226	9	0.85546
	10	3856	0.75005	164	0.75012	38	0.75599	19	0.75965
	15	5883	0.75002	249	0.75295	57	0.76460	28	0.77895
	20	7941	0.75010	336	0.75201	77	0.75561	37	0.79747
	30	12114	0.75003	512	0.75174	117	0.75103	56	0.79060
0.50	0	310	0.50068	13	0.51954	3	0.58278	1	1.00000
	5	2536	0.50007	107	0.50401	24	0.52891	11	0.62304
	10	4771	0.50008	201	0.50373	45	0.52520	21	0.58808
	15	7006	0.50020	295	0.50385	66	0.52434	31	0.57221
	20	9242	0.50011	389	0.50405	88	0.50042	41	0.56266
	30	13714	0.50002	578	0.50055	130	0.50551	61	0.55126
0.75	0	620	0.25012	26	0.25558	6	0.25927	2	0.50000
	5	3318	0.25025	139	0.25338	31	0.25314	14	0.29052
	10	5821	0.25009	244	0.25299	54	0.26034	25	0.27061
	15	8265	0.25013	347	0.25136	77	0.25586	35	0.30378
	20	10678	0.25012	448	0.25266	99	0.26572	46	0.27572
	30	15449	0.25004	649	0.25094	144	0.25918	67	0.26927
0.99	0	2057	0.01002	85	0.01022	18	0.01016	7	0.01562
	5	5858	0.01001	243	0.01027	52	0.01025	22	0.01330
	10	9003	0.01001	375	0.01006	80	0.01123	35	0.01215
	15	11953	0.01001	498	0.01017	107	0.01112	47	0.01295
	20	14797	0.01000	617	0.01017	133	0.01122	59	0.01237
	30	20295	0.01000	848	0.01002	184	0.01077	82	0.01282

Table (8): ASP for PITLP distribution at $\delta = 2.5, \varphi = 2.5,\ \gamma = 1.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	24	0.95143	2	0.95042	1	1.00000	1	1.00000
	5	1210	0.95008	54	0.95343	13	0.96512	7	0.98438
	10	2855	0.95006	127	0.95088	30	0.95380	15	0.97132
	15	4644	0.95002	205	0.95168	47	0.95853	24	0.95344
	20	6511	0.95001	287	0.95125	66	0.95240	32	0.96463
	30	10383	0.95002	457	0.95062	104	0.95082	50	0.95730

0.25	0	133	0.75144	6	0.77549	2	0.77343	1	1.00000
	5	1952	0.75015	86	0.75416	20	0.75273	9	0.85548
	10	3987	0.75015	175	0.75322	39	0.77304	19	0.75969
	15	6083	0.75012	267	0.75098	60	0.75138	28	0.77900
	20	8212	0.75005	360	0.75100	80	0.76161	37	0.79752
	30	12527	0.75002	548	0.75249	122	0.75200	56	0.79067
0.50	0	321	0.50019	14	0.51630	3	0.59820	2	0.50001
	5	2622	0.50023	115	0.50025	25	0.53020	12	0.50003
	10	4933	0.50025	215	0.50527	47	0.52387	22	0.50004
	15	7245	0.50016	316	0.50351	69	0.52138	32	0.50005
	20	9557	0.50013	417	0.50234	91	0.52012	42	0.50006
	30	14181	0.50011	619	0.50079	136	0.50040	62	0.50007
0.75	0	641	0.25019	28	0.25335	6	0.27676	3	0.25001
	5	3432	0.25004	149	0.25281	32	0.26460	14	0.29056
	10	6019	0.25018	262	0.25033	56	0.26991	25	0.27067
	15	8547	0.25011	372	0.25038	80	0.26441	35	0.30384
	20	11042	0.25013	480	0.25244	104	0.25612	46	0.27579
	30	15976	0.25002	695	0.25163	151	0.25144	67	0.26935
0.99	0	2128	0.01000	91	0.01029	18	0.01268	7	0.01563
	5	6058	0.01001	261	0.01010	54	0.01107	22	0.01331
	10	9311	0.01000	402	0.01005	84	0.01084	35	0.01216
	15	12361	0.01000	534	0.01011	112	0.01110	47	0.01295
	20	15301	0.01001	662	0.01001	140	0.01023	59	0.01238
	30	20987	0.01000	908	0.01018	193	0.01025	82	0.01283

Table (9): ASP for PITLP distribution at $\delta = 2.5, \varphi = 2.5, \ \gamma = 2.5$

p	n_0	$a = 0.25$		$a = 0.50$		$a = 0.75$		$a = 1$	
		n	$L(p)$	n	$L(p)$	n	$L(p)$	n	$L(p)$
0.05	0	29	0.95153	2	0.95902	1	1.00000	1	1.00000
	5	1475	0.95012	65	0.95300	14	0.96331	7	0.98437
	10	3482	0.95002	153	0.95093	32	0.95628	15	0.97131
	15	5663	0.95005	248	0.95062	51	0.95565	24	0.95343
	20	7940	0.95003	347	0.95049	71	0.95359	32	0.96462
	30	12663	0.95002	552	0.95051	112	0.95243	50	0.95728

0.25	0	163	0.75015	7	0.77798	2	0.79103	1	1.00000
	5	2381	0.75001	104	0.75246	21	0.77391	9	0.85547
	10	4863	0.75006	212	0.75023	43	0.75013	19	0.75966
	15	7419	0.75012	322	0.75334	64	0.76971	28	0.77896
	20	10015	0.75014	435	0.75132	87	0.75324	37	0.79748
	30	15278	0.75008	663	0.75114	132	0.75268	56	0.79062
0.50	0	391	0.50052	17	0.51198	3	0.62573	1	1.00000
	5	3198	0.50020	139	0.50022	27	0.53188	11	0.62304
	10	6017	0.50016	261	0.50003	51	0.52077	21	0.58809
	15	8837	0.50008	383	0.50002	75	0.51531	31	0.57223
	20	11657	0.50004	505	0.50004	99	0.51178	41	0.56268
	30	17297	0.50002	749	0.50010	147	0.50716	61	0.55128
0.75	0	782	0.25008	34	0.25138	6	0.30972	2	0.50000
	5	4186	0.25004	180	0.25432	35	0.25690	14	0.29052
	10	7342	0.25009	317	0.25069	61	0.26491	25	0.27062
	15	10425	0.25009	450	0.25110	87	0.26103	35	0.30379
	20	13469	0.25003	581	0.25228	113	0.25388	46	0.27574
	30	19486	0.25003	841	0.25194	163	0.26215	67	0.26928
0.99	0	2596	0.01000	111	0.01003	20	0.01163	7	0.01562
	5	7390	0.01000	316	0.01022	59	0.01079	22	0.01330
	10	11357	0.01001	487	0.01011	92	0.01012	35	0.01215
	15	15078	0.01000	647	0.01014	122	0.01096	47	0.01295
	20	18664	0.01001	802	0.01004	152	0.01056	59	0.01237
	30	25599	0.01000	1101	0.01002	210	0.01015	82	0.01283

5. Real Data Analysis

In the quest for reliable renewable energy sources to mitigate CO2 emissions, a comprehensive analysis of global CO2 emissions per person in 2020 across 152 countries was conducted. This dataset, featuring countries with capita CO2 emissions exceeding 1, was sourced from [https://www.statista.com/statistics/270508/co2-emissions-per-capita-by-country/].

The intensified burning of carbon-based fuels since the industrial revolution has significantly elevated atmospheric carbon dioxide concentrations, accelerating global warming and contributing to anthropogenic climate change. This phenomenon further induces ocean acidification through the dissolution of CO2 in water, disrupting the earth's radiative balance and resulting in rising surface temperatures, impacting climate, sea levels, and global agriculture. CO2 emissions stem from various sources, including the combustion of fossil fuels (oil, coal, gas), burning wood and waste, and industrial activities like cement manufacturing. Recent research by Hassan *et al.* (2022) applied the weighted Weibull exponential model to this dataset.

To assess the suitability of the PITLP distribution for this dataset, we employed MLEs of parameters and evaluated goodness-of-fit metrics. Negative log-likelihood (NLL), Akaike's information criterion (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn IC (HQIC), and Kolmogorov–Smirnov (K–S) test statistics with P-values were computed. Before analysis, a scaled-TTT plot, as recommended by Aarset (1987), was used to visually confirm the model's correctness, revealing a concave scaled-TTT plot indicating an increasing hazard rate function (HRF).

Parameter estimates, standard errors (SE), and goodness-of-fit statistics for the PITLP distribution are presented in Table 9. Plots of estimated PDF, CDF, and Probability-Probability (PP) plots of estimated densities are illustrated in Figure 2. The comprehensive evaluation in Table 10 and Figure 2 suggests that the PITLP distribution serves as a suitable model for the examined CO2 emissions dataset. The P-value exceeding 0.05 and the visual assessments in Figure 2 reinforces the conclusion that the PITLP distribution provides a superior fit for the considered data.

Table 10: The MLEs and SEs of the PITLP model parameters and goodness of fit measures

MLE	SE	NLL	AIC	BIC	CAIC	HQIC	K-S	P-value
$\hat{\delta} = 0.320$	0.186	406.545	819.089	828.161	819.251	822.774	0.051	0.822
$\hat{\phi} = 9.933$	7.284							
$\hat{\gamma} = 73.076$	158.361							

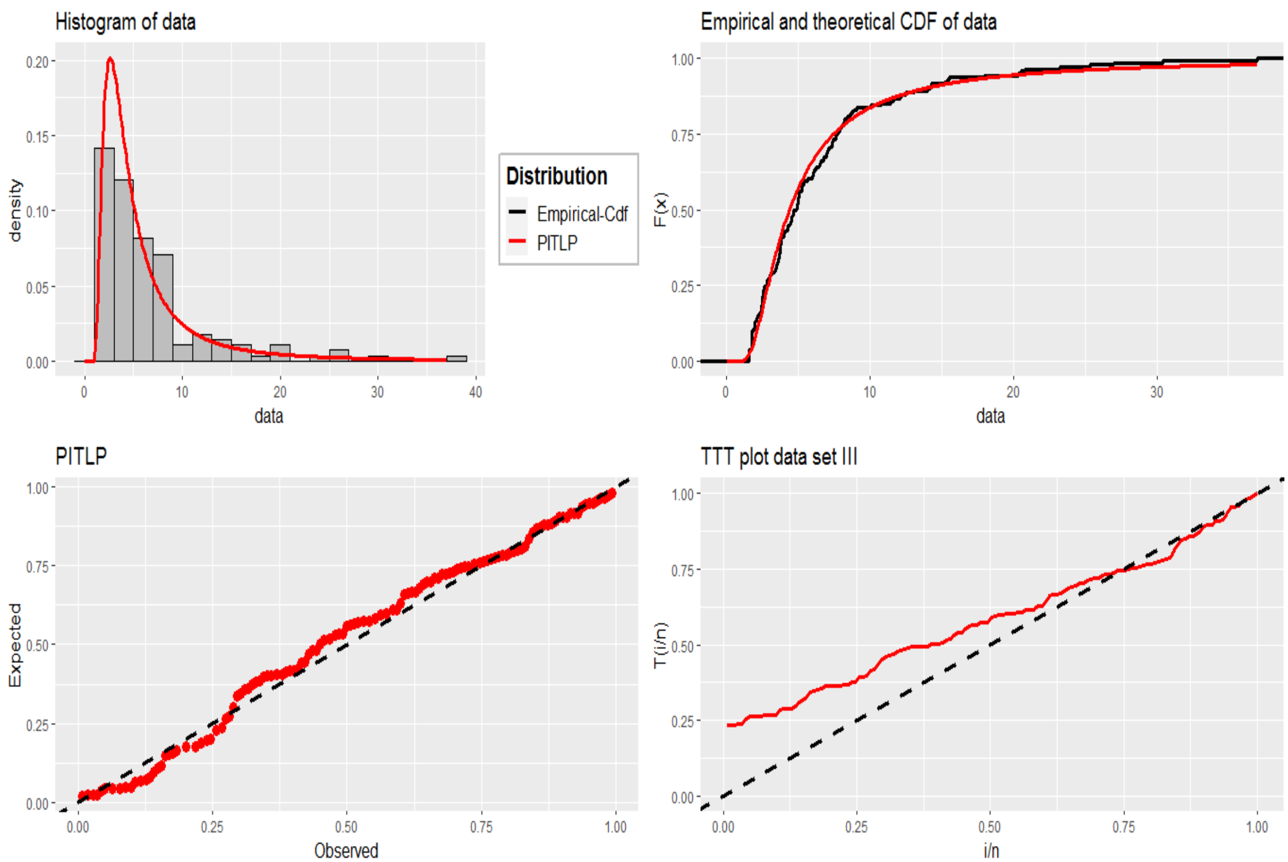

Figure 2: Estimated PDF, CDF, TTT, and PP Plots, of the PITLP model

6. Conclusion

In this study, we present a novel class of lifetime models known as the power inverted Topp-Leone power series. This class incorporates the inverted Topp-Leone power series distributions as a distinctive sub-class alongside various innovative compounding models. Crafted through the fusion of PITL and power series distributions, the proposed class encompasses a range of statistical aspects, including the density function, moments, incomplete moments, quantile function, and entropy. All statistical properties are applicable to a single selected model within the class. We focus on the ML estimation of parameters for the power inverted Topp-Leone Poisson distribution. The study also introduces an acceptance sampling plan for the PITLP distribution. Real-world data analysis in the engineering and renewable energy sources domain demonstrates the applicability and utility of the PITLP model compared to alternative models. Future studies may explore the challenge of Bayesian estimation, as proposed by Riad *et al.* (2023).

Data availability

The data is available in this article.

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