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Second kind Chebyshev polynomials differentiation and integration matrices for solving some mathematical models

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ABSTRACT

In the current work, new pseud-spectral differentiation and integration matrices have been constructed via the second kind of Chebyshev polynomials as a basis function. To achieve that purpose, the continuous inner product of the spectral expansion summation is transformed into a discrete one via the Trapezoidal integration technique. Hence, the given problem, differential, integral, or integro-differential equations, is transformed into a system of algebraic equations. Unlike the standard spectral methods, the algebraic system of equations' unknowns are the dependent variables' values at equidistant points. The constructed pseud-spectral differentiation and integration matrices have been tested to approximate the differentiation and integration of known functions and examine their applicability as differentiation and integration operators. In addition, the matrices have been used to approximate the solution of differential, integral equations, and integro-differential equations. Some of the presented differential, integral, and integro-differential equations represent models concerning real-life applications. Log error figures have shown the stability of the results.

1. Introduction

Ordinary differential equations (ODEs) [1] are one of the most important tools to represent several mathematical models in both pure and applied analysis, like physics [2], chemistry [3], biology [4], and heat and fluid flows [5].

Integro-differential equations (IDEs) contain the integral and derivative of an unknown function. IDEs have several applications in the field of engineering. Some applications of these equations can be found in electromagnetic theory [6] and poroelastic solid [7].

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The most common IDE is the Volterra–Fredholm integro-differential equation (VFIDE), which has been examined in [8,9].

The exact solutions to some differential problems have not been found, so we use numerical and approximated methods [10] to obtain their solutions. There are many types of numerical methods, such as finite element [11], finite difference [12], and Runge-Kutta [13] methods. Approximate methods, such as the spectral method [14], are semi-analytic to solve ordinary, partial, and fractional differential or integral equations.

The main idea of the spectral method is to approximate the function $G(s)$ by a finite sum of unknown coefficients and basis polynomials. We can express the solution as follows:

$$G(s) = \sum_{n=0}^{V} e_n P_n(s),$$

which e_n are unknown coefficients and $P_n(s)$ basis function. After using the spectral methods, the differential problem will transform into a system of algebraic equations with unknown coefficients.

The basis function used in the spectral method is usually orthogonal polynomials. The authors used Chebyshev polynomials in [15,16], and the authors in [17] used rational Chebyshev. Also, the authors in [18, 19] used shifted fifth-kind Chebyshev polynomials. In addition, the modified Chebyshev was used in [20], while monic Chebyshev polynomials were used in [21]. Different orthogonal polynomials, such as Legendre polynomials, are shown in [22,23]. On the other hand, the shifted Legendre polynomial has been employed as a basis function in [24].

The spectral methods have three main types: Tau [25], Galerkin [26,27], pseudo-spectral (Collocation) method [17]. The pseudo-spectral method is a form of the spectral method. This paper aims to use the second kind of Chebyshev polynomials as a basis function in the pseudo-spectral method. Hence, new differentiation and integration matrices will be constructed. Consequently, the given differential problem will be transformed into a system of algebraic equations whose unknowns are the dependent variable's values; unlike the regular spectral method, the unknowns are the spectral constants expansion.

The problem in the pseudo-spectral method is how to deal with the integration of the continuous inner product [rev]. We used the trapezoidal numerical integration method to transform the inner product's integration into a summation.

This paper includes six sections. Section 2 introduces some properties of the second kind of Chebyshev polynomials and the main form of the Trapezoidal method. In Section 3, we construct differentiation and integration matrices for the second kind of Chebyshev polynomials. In addition, we show the formula of differential, integral, and integro-differential equations and the solution steps in Section 4. We presented some known test functions and applications of ODEs and IDEs, as well as their approximate solution, to show this method's efficiency in Section 5. Finally, the paper ends with a section for the concluding remarks.

2. Preliminaries

This section presents properties of the second kind of Chebyshev polynomials (CHPs2). In addition, the general form of the Trapezoidal integration technique has been reported.

The second kind of Chebyshev polynomials, $U_n(s)$, The degree n for which the variable s is defined on the interval [-1,1] can be obtained according to the recurrence relation [29].

$$U_{n+1}(s) = 2sU_n(s) - U_{n-1}(s), \; n = 1,2, \ldots, \quad (1)$$

which $U_0(s) = 1$, $U_1(s) = 2s$.

Another form of CHPs2 can be written using trigonometric functions:

$$U_n(s) = \frac{\sin{(n+1)\vartheta}}{sin\vartheta}, \; n = 0,1, \ldots, \quad (2)$$

with $s = cos\vartheta$ and $\vartheta \in [0, \pi]$.

The boundaries of CHPs2 can be determined as the following relations:

$$U_n(-1) = (-1)^n(n+1), \quad (3)$$

$$U_n(1) = (n+1). \quad (4)$$

The relation between the first derivative of the first-kind and second-kind Chebyshev polynomials is defined by:

$$U_n(s) = \frac{1}{n+1} T'_{n+1}(s), \quad n = 0,1, \ldots. \quad (5)$$

The boundary of the first kind of Chebyshev polynomials at $s = \pm 1$ is $T_n(\pm 1) = (\pm 1)^n$.

The orthogonal relation of CHPs2 concerning the weight function $w(s)=\sqrt{1-s^2}$ is [30]:

$$\langle U_n, U_m \rangle_w = \int_{-1}^{1} U_n(s) U_m(s) w(s) ds$$

$$= \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases} \quad (6)$$

The integration of the continuous function $P(s)$ over the interval $[a, b]$ can be approximated using the Trapezoidal rule as:

$$\int_a^b P(s)ds = \frac{h}{2}\left(P(s_0) + 2\sum_{q=1}^{q=v-1} P(s_q) + P(s_v)\right), \quad (7)$$

where $h = \frac{b-a}{v}$. The values of s_q can be calculated from $s_q = s_0 + qh, q = 0,1, \dots, v$.

In the next section, we will use the properties of CHPs2, the Trapezoidal numerical method, and the concept of the pseudo-spectral expansion to get the differentiation and integration matrices.

3. Pseudo-spectral Matrices for the Second Kind of Chebyshev Polynomials

This section will use the pseudo-spectral expression and calculate its coefficients.

Lemma 1. Let G(s) be a continuous function on the interval [-1,1] that can be approximately expanded such that:

$$\mathrm{G}(s) = \sum_{n=0}^{V} \delta_n U_n(s), \quad (8)$$

Then

$$\delta_n = \sum_{i=0}^{K} C_i \sqrt{1 - s_i^2}\, U_n(s_i) G(s_i), \quad (9)$$

which

$$C_i = \begin{cases} \frac{2}{\pi K}, & i = 1, \dots, K-1, \\ \frac{4}{\pi K}, & i = 0, K. \end{cases} \quad (10)$$

while δ_n are coefficients for the expansion of pseudo-spectral and $U_n(s)$ are the basis functions for CHPs2.

Proof. Using the form of pseudo-spectral expansion in Eq. (8), we obtained:

$$\mathrm{G}(s)\mathrm{w}(s)U_n(s) = \sum_{n=0}^{V} \delta_n U_n(s)\, w(s) U_n(s). \quad (11)$$

Via the orthogonal relation of CHPs2 in Eq. (6), we get:

$$\delta_n = \frac{2}{\pi}\int_{-1}^{1} \mathrm{G}(s)\mathrm{w}(s)U_n(s)ds. \quad (12)$$

By solving this integration via the Trapezoidal rule, the proof is complete.

3.1 Differentiation Matrices for the Second Kind of Chebyshev Polynomials

In this subsection, we obtained the pseudo-spectral differentiation matrices of CHPs2 (CHPs2-DMatrix).

Theorem 1. Let $G(s)$ be the function that satisfies Lemma (1). Then:

$$G'(s_j) = \sum_{i=0}^{K} d_{ji} G(s_i), \quad j = 0,1, \dots, K, \quad (13)$$

where:

$$d_{ji} = \sum_{n=0}^{V} C_i \sqrt{1 - s_i^2}\, U_n(s_i)\, U_n{}'(s_j), \quad (14)$$

such that C_i is defined in Eq. (10), and $0 \leq j \leq K$.
Proof. From Lemma (1), by differentiating Eq. (8), we obtained:

$$G'(s) = \sum_{n=0}^{V}\sum_{i=0}^{K} C_i \sqrt{1 - s_i^2}\, U_n(s_i)\, G(s_i)\, U_n{}'(s). \quad (15)$$

Collocate Eq. (15) by the points s_j:

$$G'(s_j) = \sum_{i=0}^{k}\sum_{n=0}^{V} C_i \sqrt{1 - s_i^2}\, U_n(s_i)\, U_n'(s_j)\, G(s_i), \quad (16)$$

which completes the proof.

We can write the expression form of CHPs2-DMatrix as $G' = \boldsymbol{D}G$, such that:

$$\boldsymbol{D} = \begin{pmatrix} d_{00} & \cdots & d_{0K} \\ \vdots & \ddots & \vdots \\ d_{K0} & \cdots & d_{KK} \end{pmatrix}.$$

The order of differentiation $l \in \mathbb{N}$ for DMatrix can be introduced as $\boldsymbol{D^l}$. Such that we can express to $d^{(l)}$as:

$$d_{ji}^{(l)} = \sum_{n=0}^{V} C_i \sqrt{1 - s_i^2}\, U_n(s_i)\, U_n^{(l)}(s_j). \quad (17)$$

3.2 Integration Matrices for the Second Kind of Chebyshev Polynomials

In this subsection, we obtained the pseudo-spectral integration matrices of CHPs2 (CHPs2-BMatrix).

Theorem 2. Let $G(s)$ be the function that is defined in Lemma (1). Then:

$$\int_{-1}^{s_j} G(s)ds = \sum_{i=0}^{K} b_{ji} G(s_i), \quad j = 0,1, \dots, K, \quad (18)$$

where:

$$b_{ji}=\sum_{n=0}^{V}\frac{1}{n+1}C_i\sqrt{1-s_i^2}U_n(s_i)\big(T_{n+1}(s_j)-(-1)^{n+1}\big). \quad (19)$$

Proof. From Lemma (1), by integrating Eq. (8), we get:

$$\int_{-1}^{s_j}G(s)ds=\sum_{n=0}^{V}\sum_{i=0}^{K}C_i\sqrt{1-s_i^2}U_n(s_i)G(s_i)\int_{-1}^{s_j}U_n(s)ds, \quad (20)$$

where $j=0,1,\dots,K$.

Using the relation in Eq. (5), and boundary of $T(s)$ to determine this integration, we obtained:

$$\int_{-1}^{s_j}G(s)ds=\sum_{n=0}^{V}\sum_{i=0}^{K}\frac{1}{n+1}C_i\sqrt{1-s_i^2}U_n(s_i)\big(T_{n+1}(s_j)-(-1)^{n+1}\big)G(s_i). \quad (21)$$

This completes the proof.

We can write the expression form of CHPs2-BMatrix as $\int_{-1}^{s}G(s)ds=\boldsymbol{B}G$, such that:

$$\boldsymbol{B}=\begin{pmatrix} b_{00} & \cdots & b_{0K}\\ \vdots & \ddots & \vdots\\ b_{K0} & \cdots & b_{KK}\end{pmatrix}.$$

The next section presents the problems formulation of ODEs and IDEs and how we can solve them via the DMatrix and BMatrix of CHPs2.

4. Problem Formulation and the Algorithm of Solution

We will discuss the problem statements of ordinary differential, integral, and integro-differential equations. Then, algorithms for approximating the solution via the CHPs2-DMatrix and CHPs2-BMatrix have been designed.

4.1 CHPs2-DMatrix for Solving the Ordinary Differential Equation

Consider the formula of linear or nonlinear ODEs as:

$$\mathcal{F}\big(s,G(s),G'(S),\dots,G^{(l)}(s)\big)=0, \quad s\in[-1,1], \quad (22)$$

which the initial and boundary conditions can be expressed as:

$$G(-1)=g_0, G'(-1)=g_1,\dots,G^{(p)}(-1)=g_p, \quad (23)$$
$$G(1)=h_0, G'(1)=h_1,\dots,G^{(r)}(1)=h_r, \quad (24)$$

such that $r,p\in\mathbb{N}$, $g_{0,\dots,}g_p, h_{0,\dots,}h_r\in\mathbb{R}$.

Using the approximated expression of Eq. (8), the definition of the CHPs2-DMatrix Eq. (17), and collocating Eq. (22) by the equidistant points to obtain:

$$\mathcal{F}\left(s_j,G(s_j),\sum_{i=0}^{K}d_{ji}\,G(s_i),\dots,\sum_{i=0}^{K}d_{ji}^{(l)}\,G(s_i)\right)=0, \quad 0\le j\le K, \quad (25)$$

with the initial and boundary conditions:

$$G(-1)=g_0,\quad \sum_{i=0}^{K}d_{0i}\,G(s_i)=g_1,\dots,\ \sum_{i=0}^{K}d_{0i}^{(p)}\,G(s_i)=g_p \quad (26)$$
$$G(1)=h_0,\quad \sum_{i=0}^{K}d_{Ni}\,G(s_i)=h_1,\dots,\ \sum_{i=0}^{K}d_{Ni}^{(r)}\,G(s_i)=h_r. \quad (27)$$

Eqs. (25,26,27) form an algebraic system of linear or nonlinear equations that are easily solved for the unknown functions $G(s)$. Algorithm (1) shows steps for solving the initial boundary value problem (IBVP) Eqs. (22,23,24) via CHPs2-DMatrix.

Algorithm 1: Steps for solving ODEs.
Step 1: Input $V,K\in\mathbb{N}$. Step 2: Use Eq. $s_i=s_0+ih$ to calculate the equidistance point. Step 3: Use Eq.(17) to construct the CHPs2-Dmatrix's elements $d_{ji}^{(l)}$. Step 4: Use steps (1-3) to substitute into the Eqs. (22,23,24) to get the system Eqs. (25,26,27). Step 5: Solve the algebraic system from step 4 to get the unknown functions $G(s_i)$.

4.2 CHPs2-BMatrix for Solving the Integral Equation

Consider the following integral Equation:

$$G(s)=\mathcal{Q}(s)-\int_{-1}^{s}\mathcal{F}\big(s,t,G(s)\big)G(t)dt=0, \quad (28)$$

where $\mathcal{V}(s)$, and $\mathcal{Q}(s)$ are given continuous functions. To solve this problem via the integration matrices for CHPs2, similar procedures to those done for the IBVP can be executed.

$$\mathrm{G(s_j)}=\mathcal{Q}(s_j)-\sum_{i=0}^{j}b_{ji}\mathcal{F}(s_j,t_i,G(s_j))G(t_i)=0, \quad j=0,1,\dots,K. \quad (29)$$

Algorithm (2) shows steps for solving the integral Equation via CHPs2-BMatrix.

Algorithm 2: Steps for solving an integral Equation.
Step 1: Input $V, K \in \mathbb{N}$.
Step 2: Use Eq. $s_i = s_0 + ih$ to calculate the points of CHPs2.
Step 3: Use Eq. (19) to construct the CHPs2-Bmatrix's elements b_{ji}.
Step 4: Use steps (1-3) to substitute into Eq. (28) to get Eq. (29).
Step 5: Solve Eq. (29) from step 4 to get the unknown functions $G(s)$.

4.3 BMatrix and DMatrix for Solving IDEs

Integro-differential equations contain a derivative part and an integral part. For this reason, we will use both matrices of CHPs2.

Consider the problem of IDEs:

$$\mathcal{F}_1\big(s, G(s), G'(S), \dots, G^{(l)}(s)\big) = \int_{-1}^{1} \mathcal{F}_2\big(s, t, G(t)\big)dt + \int_{-1}^{s} \mathcal{F}_3\big(s, t, G(t)\big)dt,\ s \in [-1,1], \tag{30}$$

subject to a sufficient number of initial and boundary conditions. Substitute with the DMatrix and BMatrix of CHPs2 in Eqs. (13) and (18), we obtained:

$$\mathcal{F}_1(s_j, G(s_j), \sum_{i=0}^{K} d_{ji}\, G(s_i), \dots, \sum_{i=0}^{K} d_{ji}^{(l)}\, G(s_i)) = \sum_{i=0}^{K} b_{Ki} \mathcal{F}_2(s_j, t_i, G(s_i)) + \sum_{i=0}^{j} b_{ji} \mathcal{F}_3(s_j, t_i, G(s_i)),$$
$$j = 0,1,\dots,K. \tag{31}$$

The obtained algebraic system will be solved to determine the unknown function values. Algorithm (3) shows the steps for solving IDE via DMatrix and BMatrix for CHPs2.

Algorithm 3: Steps for solving IDEs.
Step 1: Input $V, K \in \mathbb{N}$.
Step 2: Use Eq. $s_i = s_0 + ih$ to calculate the points of CHPs2.
Step 3: Use Eq. (17) to construct the CHPs2-Dmatrix's elements $d_{ji}^{(l)}$.
Step 4: Use Eq. (19) to construct the CHPs2-Bmatrix's elements b_{Ki}.
Step 5: Use Eq. (19) to construct the CHPs2-Bmatrix's elements b_{ji}.
Step 6: Use steps (1-5) to substitute into Eqs. (30,23,24) to get the system Eqs. (31,26,27).
Step 7: Solve the system from step 6 to get the unknown function $G(s)$.

In the next section, we will test the differentiation and integration matrices for CHPs2 via the well-known functions to prove the stability and accuracy of the present matrices and solve real-life applications.

5. Applications of CHPs2-DMatrix and CHPs2-BMatrix

The following section will apply the differentiation and integration matrices to differentiate and integrate some test known functions. Then, some different types of applications have been solved. The results have been presented to show the accuracy and efficiency of the investigated CHPs2-DMatrix and CHPs2-BMatrix.

5.1 Test of known functions

Example 1: Consider the following different test functions:

$$G_1(s) = s^2, G_2(s) = e^s, G_3(s) = \sin s.$$

Tables (1) and (2) present the point-wise absolute error (point-wise AE) and the max absolute error (MAE) for differentiating the functions using the differentiation matrix for CHPs2 for different values of V at $K = 1000$. While Tables (3) and (4) show the point-wise AE and MAE of the integration. The results prove the efficiency and stability of the present method.

Table 1: Point-wise AE for the approximate differentiation at $V = 5$ and $K = 1000$ of Example 1.

s	$G_1(s)$	$G_2(s)$	$G_3(s)$
-1.0	$7.48 * 10^{-04}$	$5.69 * 10^{-03}$	$1.39 * 10^{-02}$
-0.8	$2.91 * 10^{-04}$	$1.97 * 10^{-04}$	$3.11 * 10^{-03}$
-0.6	$3.85 * 10^{-04}$	$1.32 * 10^{-03}$	$6.77 * 10^{-04}$
-0.4	$5.98 * 10^{-04}$	$2.82 * 10^{-04}$	$8.07 * 10^{-04}$
-0.2	$5.56 * 10^{-04}$	$9.56 * 10^{-04}$	$1.77 * 10^{-04}$
0	$4.99 * 10^{-04}$	$9.28 * 10^{-04}$	$6.86 * 10^{-04}$
0.2	$5.56 * 10^{-04}$	$4.50 * 10^{-04}$	$1.77 * 10^{-04}$
0.4	$5.98 * 10^{-04}$	$1.95 * 10^{-03}$	$8.07 * 10^{-04}$
0.6	$3.85 * 10^{-04}$	$6.00 * 10^{-04}$	$6.77 * 10^{-04}$
0.8	$2.91 * 10^{-03}$	$8.79 * 10^{-03}$	$3.11 * 10^{-03}$
1.0	$7.48 * 10^{-03}$	$3.41 * 10^{-02}$	$1.39 * 10^{-02}$

Table 2: MAE for the approximate differentiation at $K = 1000$ of Example 1.

V	$G_1(s)$	$G_2(s)$	$G_3(s)$
2	$8.02 * 10^{-04}$	$5.69 * 10^{-01}$	$3.78 * 10^{-01}$
3	$8.02 * 10^{-04}$	$1.34 * 10^{-01}$	$1.97 * 10^{-02}$
4	$7.50 * 10^{-03}$	$3.65 * 10^{-02}$	$1.97 * 10^{-02}$
5	$7.50 * 10^{-03}$	$3.41 * 10^{-02}$	$1.39 * 10^{-02}$

Table 3: Point-wise AE for the approximate integration at $V = 5$ and $K = 1000$ of Example 1.

s	$G_1(s)$	$G_2(s)$	$G_3(s)$
-1.0	0	0	0
-0.8	$1.19 * 10^{-04}$	$1.74 * 10^{-05}$	$6.05 * 10^{-05}$
-0.6	$1.14 * 10^{-04}$	$4.36 * 10^{-05}$	$6.12 * 10^{-05}$
-0.4	$8.90 * 10^{-05}$	$2.41 * 10^{-05}$	$4.43 * 10^{-04}$
-0.2	$8.44 * 10^{-05}$	$4.31 * 10^{-06}$	$3.97 * 10^{-05}$
0	$1.00 * 10^{-04}$	$9.44 * 10^{-07}$	$5.12 * 10^{-05}$
0.2	$1.16 * 10^{-04}$	$3.86 * 10^{-05}$	$6.28 * 10^{-05}$
0.4	$1.12 * 10^{-04}$	$5.75 * 10^{-05}$	$5.81 * 10^{-05}$
0.6	$8.65 * 10^{-05}$	$7.34 * 10^{-06}$	$4.09 * 10^{-05}$
0.8	$8.20 * 10^{-05}$	$4.04 * 10^{-05}$	$4.20 * 10^{-05}$
1.0	$2.01 * 10^{-04}$	$3.16 * 10^{-04}$	$1.02 * 10^{-04}$

Table 4: MAE for the approximate integration at $K = 1000$ of Example 1.

V	$G_1(s)$	$G_2(s)$	$G_3(s)$
2	$1.37 * 10^{-04}$	$1.08 * 10^{-02}$	$9.84 * 10^{-03}$
3	$1.34 * 10^{-04}$	$1.30 * 10^{-03}$	$1.46 * 10^{-04}$
4	$2.01 * 10^{-04}$	$3.16 * 10^{-04}$	$1.46 * 10^{-04}$
5	$2.01 * 10^{-04}$	$3.16 * 10^{-04}$	$1.21 * 10^{-04}$

5.2 Numerical Examples

This subsection approximates solutions of some applications using the CHPs2-DMatrix and CHPs2-BMatrix to illustrate our method's efficiency.

Example 2: The formula of the nonlinear Riccati Equation [31]:

$$G'(s) - G^2(s) = 1, \qquad 0 \le s \le 1,$$

with the initial condition $G(0) = 0$, and the exact solution is $G(s) = \tan s$. After shifting the domain of s from [0,1] to [-1,1]. The MAE equals $8.509 * 10^{-01}$ at $V = 4, K = 1000$. Table (5) presents the point-wise AE. Moreover, Figure (1) shows the Log Error for different values of V and $K = 1000$.

Table 5: Point-wise AE at $V = 4$ and $K = 1000$ of Example 2.

s	$G(s)$
-1.0	$1.33 * 10^{-09}$
-0.8	$3.29 * 10^{-01}$
-0.6	$3.52 * 10^{-01}$
-0.4	$3.82 * 10^{-01}$
-0.2	$3.01 * 10^{-01}$
0	$3.50 * 10^{-01}$
0.2	$3.88 * 10^{-01}$
0.4	$4.34 * 10^{-01}$
0.6	$5.02 * 10^{-01}$
0.8	$6.08 * 10^{-01}$
1.0	$7.74 * 10^{-01}$

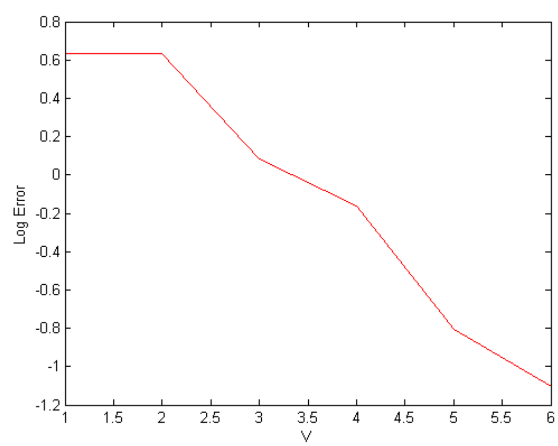

Figure 1: Log error for Example 2.

Example 3: Consider the stable population model [32]:

$$G(s) = e^s - \int_0^s (s-t)G(t)dt, \quad s \in [0,1].$$

The exact solution is $G(s) = \frac{1}{2}(e^s + \cos s + \sin s)$, such that $(s-t)$ is the net maternity function of females of class age t at time s. e^s is the contribution of birth due to females already present at time s, with G being the number of female births. After shifting the domain of s from [0,1] to [-1,1]. The MAE can be determined using CHPs2-BMatrix and equals $6.03 * 10^{-05}$ at $V = 7, K = 1000$. The MAE and point-wise AE are presented in Tables (6) and (7). In addition, Figure (2) shows the Log error, such that, by increasing the value of V, the accuracy of the approximation increases.

Table 6: MAE at $K = 1000$ of Example 3.

V	$G(s)$
1	$2.19 * 10^{-02}$
2	$4.03 * 10^{-04}$
3	$7.59 * 10^{-05}$
4	$4.30 * 10^{-05}$
5	$4.62 * 10^{-05}$
6	$6.00 * 10^{-05}$
7	$6.03 * 10^{-05}$
8	$7.37 * 10^{-05}$

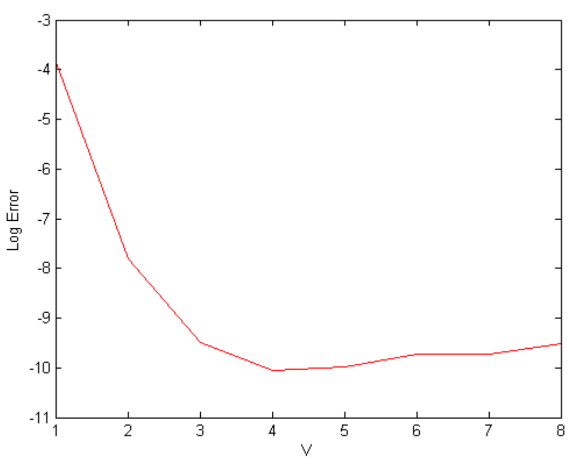

Figure 2: Log error for Example 3.

Table 7: Point-wise AE at $V = 7$ and $K = 1000$ of Example 3.

s	$G(s)$
-1.0	0
-0.8	$1.01 * 10^{-05}$
-0.6	$1.91 * 10^{-05}$
-0.4	$3.40 * 10^{-05}$
-0.2	$3.02 * 10^{-05}$
0	$3.09 * 10^{-05}$
0.2	$4.46 * 10^{-05}$
0.4	$5.18 * 10^{-05}$
0.6	$4.69 * 10^{-05}$
0.8	$5.33 * 10^{-05}$
1.0	$5.28 * 10^{-05}$

Example 4: Consider the following first-order Volterra IDE [33]:

$$G'(s) + \int_0^s G(t)dt = 1, \quad s \in [0,1],$$

with the initial condition $G(0) = 1$, and the exact solution is $G(s) = \sin s$. After shifting the domain to [-1,1] using the relation $\mu = \frac{1}{2}(s+1)$. Table (8) presents the point-wise AE at $V = 5, K = 500$.

Example 5: Consider the following second-order Volterra-Fredholm IDE [34]:

$$G''(s) + G(s) = 2 + e^s - \mathrm{e} + \int_0^s G(t)dt + \int_0^1 G(t)dt, \quad 0 \le s \le 1,$$

where the initial conditions are $G(0) = G'(0) = 1$, the exact solution is $G(s) = e^s$. The MAE equals $1.46 * 10^{-03}$ at $V = 5, K = 1500$ and the point-wise AE has been expressed in Table (9).

Table 8: Point-wise AE at $V = 5$ and $K = 500$ of Example 4.

s	$\mathrm{G}(s)$
-1.0	$1.06 * 10^{-22}$
-0.8	$5.57 * 10^{-01}$
-0.6	$5.52 * 10^{-01}$
-0.4	$5.08 * 10^{-01}$
-0.2	$5.04 * 10^{-01}$
0	$4.55 * 10^{-01}$
0.2	$4.94 * 10^{-01}$
0.4	$3.47 * 10^{-01}$
0.6	$3.92 * 10^{-01}$
0.8	$2.84 * 10^{-01}$
1.0	$8.41 * 10^{-01}$

Table 9: point-wise AE at $V = 5$ and $K = 1500$ of Example 5.

s	$G(s)$
-1.0	$2.03 * 10^{-05}$
-0.8	$1.05 * 10^{-03}$
-0.6	$1.14 * 10^{-03}$
-0.4	$1.21 * 10^{-03}$
-0.2	$1.18 * 10^{-03}$
0	$1.07 * 10^{-03}$
0.2	$1.05 * 10^{-03}$
0.4	$1.23 * 10^{-03}$
0.6	$1.45 * 10^{-03}$
0.8	$1.11 * 10^{-03}$
1.0	$7.90 * 10^{-04}$

6. Conclusion

This paper uses the Trapezoidal integration rule to introduce new differentiation and integration matrices for solving ODEs and IDEs via the pseudo-spectral expansion method. The basis function of pseudo-spectral is the second kind of Chebyshev polynomial. The constructed matrices convert the given differential problem into a system of algebraic equations. The unknown values of the algebraic system are the values of the dependent variable. In addition, algorithms for solving the initial boundary value problem and integral and integro-differential equations have been designed. Different well-known tests, like polynomial, exponentiation, and trigonometric functions, are used to prove the constructed matrices' reliability. Moreover, the matrices have been applied to approximate the solution of some real-life applications.

Ethics approval

Not applicable.

Availability of data and material

Not applicable.

Conflict of interest

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