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On SC-compact spaces and their relations to semi-compact and S-closed spaces

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ABSTRACT

This study proposes and explores a novel category of topological spaces, referred to as SC-compact spaces. Positioned strictly between the established notions of semi-compact and C-compact spaces, the SC-compact class is shown to satisfy the s-closed property, as conceptualized by Di Maio and Noiri. Utilizing the framework of semi-open and semi-closed sets, the paper offers a detailed investigation of the core characteristics defining SC-compactness and situates it within the wider landscape of generalized compactness concepts. The analysis extends to the behavior of SC-compact spaces under a variety of mappings, such as semi-continuous and irresolute functions, revealing several fundamental interrelations with known topological structures. A series of examples are presented to illustrate these theoretical connections and to highlight the unique role of SC-compactness among other compactness conditions. The findings contribute meaningfully to the theoretical expansion of compactness in topology and suggest promising avenues for further extensions and applications—especially in scenarios where semi-open coverings and neighborhood systems are of central importance. Overall, this work enhances the foundational understanding of topological compactness and opens new perspectives for research in fields where approximation and structural imprecision are intrinsic.

1. Introduction

1.1 Historical Background

Compactness has traditionally played a pivotal role in general topology, serving as a fundamental link between the local properties and the global behavior of topological spaces. Over the years, numerous generalizations have been developed to accommodate

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increasingly complex structural features and to extend the utility of this central notion.

In 1987, Di Maio and Noiri [1] introduced the concept of s-closedness, offering a refined perspective on compactness that lies between semi-compactness [2] and S-closedness [3]. This contribution enriched the classical framework of compactness-related properties and inspired subsequent studies, including those by Khan et al. [4]. Later, Viglino [5] proposed the concept of C-compact spaces, a specialized subset of quasi-H-closed spaces [6], further advancing the theoretical understanding of compactness through intermediate concepts grounded in semi-open sets.

Concurrently, various mathematical paradigms were developed to model real-world problems involving uncertainty and imprecision, including probability theory, fuzzy sets, rough set theory, and decision-making methodologies. Among these, Pawlak's seminal 1982 work [7] introduced rough set theory, which models uncertainty using lower and upper approximations based on equivalence relations. However, the limitations imposed by strict equivalence motivated efforts to adopt more general relational frameworks.

As a response, topological approaches were incorporated into rough set theory, enabling the replacement of equivalence relations with broader binary relations. This led to enhanced models with wide-ranging applications such as clustering [8–10], decision-making problems [11–14], and medical diagnosis [15–18]. A major development came with the work of Abd El-Monsef et al. [19], who introduced a method for generating multiple topologies from neighborhood systems derived from binary relations—thereby establishing the κ-neighborhood framework. This model broadened Pawlak's structure by incorporating right and left neighborhoods [8], as well as novel types of neighborhoods formed through intersections and unions [19].

More recently, Nawar et al. [20] proposed θβ-rough approximations, which integrate ideals into Pawlak's model and refine the κ-neighborhood structure. This model was further generalized by El-Bably et al. [21], who developed new algorithms and theoretical results to validate and extend the framework, reinforcing its originality and applicability.

1.2 Objectives and Motivations

Motivated by these developments, this paper presents and examines a novel class of topological spaces, designated as SC-compact spaces. This class occupies a position firmly between semi-compact and C-compact spaces, inheriting essential features from the s-closed framework introduced by Di Maio and Noiri [1]. Through the use of semi-open and semi-closed sets, SC-compact spaces offer fresh insights into the structure of compactness.

The core aims of this study are as follows:

- To formally define SC-compact spaces and contextualize them within the existing hierarchy of generalized compactness.
- To explore the principal properties of SC-compact spaces and examine their relationships with other compactness generalizations.
- To analyze how SC-compactness behaves under different types of mappings, particularly irresolute and semi-continuous functions.
- To identify potential applications and future research directions, especially those related to rough set theory and generalized neighborhood systems.

This work seeks to contribute meaningfully to the evolving discourse on topological compactness by offering a new structural perspective and by establishing connections between classical topological ideas and modern models of uncertainty and approximation.

2. Some Preliminary Concepts

Throughout this paper, unless otherwise stated, all spaces are assumed to be topological spaces in the sense of general topology.

Let (X,τ) be a topological space. A subset $S \subseteq X$ is defined as semi-open [22] if $S \subseteq Cl(\text{Int}(S))$, where $Cl(S)$ and $\text{Int}(S)$ denote the closure and interior of S, respectively.

Dually, a subset S is termed semi-closed if its complement X\S is semi-open. Equivalently, this means

$$\text{Int}(Cl(S)) \subseteq S.$$

For any subset $A \subseteq X$, the semi-closure of A, symbolized by $sCl(A)$, is defined as the smallest semi-closed set containing A.

Similarly, the semi-interior of A, symbolized $s\text{Int}(A)$, is the largest semi-open set contained in A. For any subset $A \subseteq X$, it is well known that:
$sCl(A) = A \cup \text{Int}(Cl(A))$ and $s\text{Int}(A) = A \cap Cl(\text{Int}(A))$.

We denote by so (X, τ) the collection of all semi-open subsets of the space (X, τ).

Definition 2.1. Let (X, τ) be a topological space. The space is characterized as follows:

- It is called s-closed [1] if for every collection $\{V_\lambda : \lambda \in \Lambda\}$ of semi-open subsets of X such that $X = \bigcup_{\lambda\in\Lambda} V_\lambda$a, there exists a finite subcollection $\Lambda_0 \subseteq \Lambda$ satisfying $X = \bigcup_{\lambda\in\Lambda_0} sCl(V_\lambda)$, where $sCl(V_\lambda)$ denotes the semi-closure of V_λ.
- It is termed quasi-H-closed [6] if any open cover of X admits a finite subclass whose closures together cover X.
- It is said to be semi-compact [2] if any semi-open cover of X contains a finite subcover.
- The space is called C-compact [5] if each closed subset of X is quasi-H-closed with respect to the topology τ.
- It is defined as p-closed [23, 24] if for every preopen cover of X, a finite subcollection can be found such that the union of their preclosures equals X.

Definition 2.2. Let (X, τ) be a topological space. Then:

- The subset A is called s-closed relative a subset [1] if each semi-open cover of A, consisting of semi-open subsets of (X, τ), contains a finite subcollection whose semi-closures collectively cover A. More precisely, for any family $\{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $A \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ satisfying $A \subseteq \bigcup_{\lambda\in\Lambda_0} sCl(V_\lambda)$, where $sCl(V_\lambda)$ denotes the semi-closure of V_λ.
- quasi-H-closed relative to $(\boldsymbol{X}, \boldsymbol{\tau})$ [6] if every open cover of A has a finite subcollection whose closures cover A. That is, given any collection $\{U_\lambda : \lambda \in \Lambda\}$ of open sets with $A \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $A \subseteq \bigcup_{\lambda\in\Lambda_0} Cl(U_\lambda)$.

3. Main Results

In this section, we present the central theoretical contributions of this study, formally defining SC-compact spaces and establishing their place within the hierarchy of generalized compactness notions in topology. We begin by introducing the definition of SC-compactness and clarifying its relationship with existing concepts such as semi-compactness, C-compactness, and s-closedness. Through precise theorems and carefully constructed examples, we demonstrate how SC-compactness naturally extends classical compactness properties while maintaining distinctive characteristics that differentiate it from other compactness conditions.

We further provide necessary and sufficient conditions for a space to be SC-compact, examine its preservation under standard topological constructions, and illustrate these results with explicit counterexamples that show the limitations of the converses. This rigorous development not only situates SC-compact spaces within the broader landscape of general topology but also lays the groundwork for their potential applications and deeper study.

Definition 3.1. Let (X, τ) be a topological space, then it is called SC-compact if each semi-closed subset of X is s-closed relative to the topology τ. That is, each semi-closed subset can be covered by finitely many semi-closures of semi-open sets that together form a cover of the subset.

It is evident that each SC-compact space (respectively, C-compact space) satisfies the condition of being s-closed (respectively, quasi-H-closed). Furthermore, it is straightforward to verify that all semi-compact spaces are necessarily SC-compact. Moreover, since the semi-closure of any open set V coincides with its closure, i.e., $sCl(V) = Cl(V)$, it follows that every SC-compact space necessarily satisfies the condition for C-compactness.

The diagram below illustrates the hierarchical relationships between SC-compact spaces and several other important classes of generalized compact spaces.

$$\begin{array}{ccccc} semi-compact & \Rightarrow & sc-compact & \Rightarrow & c-compact \\ \Downarrow & & \Downarrow & & \Downarrow \\ compact & & s-closed & \Rightarrow & quasi-H-Closed \end{array}$$

In the subsequent discussion, we demonstrate that the converse of each of these implications does not necessarily hold. This is substantiated by Example 4.8(d) from [25], Example 2.10 from [26], and an additional illustrative example provided below.

Example 3.1. This set of examples illustrates that the implications between various compactness conditions discussed earlier are not reversible.

(i) Consider the topological space (X, τ), where $X = \mathbb{R}$, and $\tau = \{\phi, \{0\}, X\}$. It is easy to verify that this space is s-closed. However, it fails to be SC-compact, since not every semi-closed subset is s-closed relative to X.

(ii) Let $X = \mathbb{Z}$, and define a topology τ on X with a basis consisting of all singleton sets $\{n\}$, for $n \in \mathbb{Z} - \{0\}$, along with all sets $T \subseteq X$ such that $0 \in T \in \tau$ and $X \setminus T$ is finite. This space is compact but not semi-compact. Indeed, the collection $\{\{n, 0\}: n \in \mathbb{Z} \setminus \{0\}\}$ forms a cover of X by semi-open sets, yet no finite subcollection of this family covers the space—therefore violating semi-compactness.

(iii) There exist spaces that are s-closed but fail to be C-compact, and consequently, they are not SC-compact. A prominent example of such a space is the Katětov extension $K\mathbb{N}$ of the set of natural numbers $\mathbb{N}$. This space is constructed by adjoining to $\mathbb{N}$ all free ultrafilters on $\mathbb{N}$, with the topology defined by:

- For every $n \in \mathbb{N}$, the singleton set $\{n\}$ is declared open.
- For any point $\mu \in K\mathbb{N} \setminus \mathbb{N}$, a basic open neighborhood is given by $\{\mu\} \cup G$, where $G \subseteq \mathbb{N}$ and $G \in \mu$.

As established in [27] and [28], the space $K\mathbb{N}$ is both p-closed and s-closed. However, it fails to be compact and, as a result, is not semi-compact. Moreover, it does not fulfill the requirements for C-compactness and thus cannot be categorized as SC-compact.

Recall that a subset $A \subseteq X$ in a topological space (X, τ) is termed semi-regular open [1] if it satisfies the condition:

$$A = \text{sInt}(sCl(A)).$$

This condition implies that A coincides with the semi-interior of its semi-closure.

It follows directly that a subset $A \subseteq X$ is semi-regular open if and only if it can be expressed as the semi-interior of some semi-closed set in (X, τ).

Furthermore, suppose $W \subseteq X$ is semi-open, and define $H = s\text{Int}(sCl(W)))$. Then it holds that:

$$Cl(W) = sCl(H),$$

indicating that the semi-closure of W is equal to the semi-closure of the corresponding semi-regular open set H.

Theorem 3.1. Let (X, τ) be a topological space. The subsequent statements are equivalent:

(1) (X, τ) is SC-compact.

(2) For every semi-closed subset $A \subseteq X$, and every collection $\{D_\lambda: \lambda \in \nabla\}$ of semi-closed sets such that $(\cap \{D_\lambda, \text{ bda } : \lambda \in \nabla\}) \cap A = \phi$, there exists a finite subcollection $\nabla_o \subseteq \nabla$ such that

$$A \subseteq \cup \{sCl(W_\lambda): \lambda \in \nabla_o\} - \cup \{sCl(U_\lambda): \lambda \in \nabla_o\}.$$

(3) For any semi-closed set $A \subseteq X$ and any semi-regular open cover $\{U_\lambda: \lambda \in \nabla\}$ of A, there exists a finite subcollection $\nabla_o \subseteq \nabla$such that $A \subseteq U\{sCl(U_\lambda): \lambda \in \nabla_o\}$.

Proof.

(1) ⇒ (2): Immediate from the definition of SC-compactness.

(1) ⇒ (3): Also follows directly, since semi-regular open sets are formed via semi-closures and semi-interiors, consistent with the structure of SC-compactness.

(3) ⇒ (1): Assume condition (3) holds. Consider a semi-closed subset $A \subseteq X$ and $\{U_\lambda: \lambda \in \nabla\}$ is a semi-open cover of A. For every $\lambda \in \nabla$, define $W_\lambda = \text{sInt}\big(sCl(U_\lambda)\big)$.

Each W_λ is semi-regular open by construction, and $\{W_\lambda: \lambda \in \nabla\}$ still covers A, since $U_\lambda \subseteq sCl(U_\lambda)$ and hence $U_\lambda \subseteq W_\lambda$. By the assumption, there exists a finite subcollection $\nabla_o \subseteq \nabla$ such that:

$$A \subseteq \cup_{\lambda \in \nabla_o} sCl(W_\lambda).$$

But since $sCl(W_\lambda) = sCl(U_\lambda)$, we conclude:

$$A \subseteq \cup_{\lambda \in \nabla_o} sCl(U_\lambda).$$

This demonstrates that A is s-closed relative to X, thereby proving that (X, τ) is SC-compact.

Theorem 3.2. Every infinite topological space Y can be embedded as a closed subspace within an SC-compact topological space X.

Proof.

Let Z be an arbitrary infinite T_1-space. Define two disjoint copies of Z as follows:

$$Z_1 = Zx\{1\} \text{ and } Z_2 = Zx\{z\}.$$

Assume that the original space Y is disjoint from $Z_1 \cup Z_e$, i.e., $Y \cap (Z_1 \cup Z_e) = \phi$.

Now define the space X as the union: $X = Y \cup Z_1 \cup Z_2$.

For $i = 1,2$, define a subbase β_i for the topology on Z_i as:

$\beta_i = \{W_i \subseteq Z_i: W_i = vx\{i\}$ for some open subset v of $Z\}$.

Next, define a family β_3 consisting of subsets of X of the form:

$\beta_3 = \{G \subseteq X : G = U \cup C_1 \cup C_2$, where U is open in

Y, and C_1, C_2 are cofinite in Z_1 and Z_2, respectively}.

The topology on X is then generated by the union of these three collections: $\beta_1 \cup \beta_2 \cup \beta_3$.

Under this topology, it can be shown that X is SC-compact and that the subspace $Y \subseteq X$ is closed.

It can be readily verified that the collection $\beta_1 \cup \beta_2 \cup \beta_3$ serves as a basis for a topology on X. Under this topology, both Z_1 and Z_2 are clearly open subsets of X, while the original space Y becomes a closed subspace of X.

For each $i = 1,2$, it follows directly that the closure of Z_i in X satisfies: $Cl_X(Z_1) = Y \cup Z_1$ and the semi-closure of Z_i is simply: $sCl_X Z_i = Z_i$

Now, consider any point $y \in Y$, and define the subset $S_y = Z_1 \cup \{Y\}$. Then S_y is semi-open in X, and its semi-closure is: $sCl_X\left(S_y\right) = S_y$.

Thus, the collection $\left\{S_y : y \in Y\right\} \cup \{Z_z\}$ forms a semi-open cover of X that lacks any finite subfamily whose semi-closures collectively cover X.

As established in [29], this shows that the space X is not s-closed, and consequently, it fails to be SC-compact.

4. Some Applications

In this section, we explore key applications and consequences of the concept of SC-compactness by examining how it interacts with various classes of functions between topological spaces. We introduce and compare different forms of continuity—such as almost s-continuity, strongly semi-continuity, irresoluteness, and semi-continuity—and investigate their connections to the structure and behavior of SC-compact spaces.

Fundamental results are established concerning the preservation of SC-compactness under continuous and irresolute mappings, the properties of function graphs, and the implications for factor spaces in product topologies. These results demonstrate how SC-compactness provides a flexible framework for analyzing function spaces and contributes to a richer understanding of generalized compactness within broader topological contexts.

Definition 4.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function between two topological spaces. The function f is classified according to the following types of continuity:

- Almost s-continuous [30]:

  The function f is called almost s-continuous if, for every point $x \in X$ and every semi-open set $V \subseteq Y$ containing $f(x)$, there exists an open neighborhood $U \subseteq X$ of x such that $f(U) \subseteq sCl(V)$.

- s-continuous (or strongly semi-continuous) [23, 31]:

  The function f is said to be s-continuous if, for each $x \in X$ and for every semi-open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ with $x \in U$ such that: $f(U) \subseteq V$.

- Irresolute [32]:

  The function f is called irresolute if for every $x \in X$ and every semi-open set $V \subseteq Y$ containing $f(x)$, there exists a semi-open neighborhood $U \subseteq X$ of x such that: $f(U) \subseteq V$.

- Semi-continuous [22]:

  The function f is semi-continuous if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists a semi-open subset $U \subseteq X$ with $x \in U$ such that: $f(U) \subseteq V$.

- s-closed [31]:

  The function f is said to be s-closed if for every semi-closed subset $A \subseteq X$, the image $f(A)$ is closed in Y; that is: A semi-closed in $X \implies f(A)$ is closed in Y.

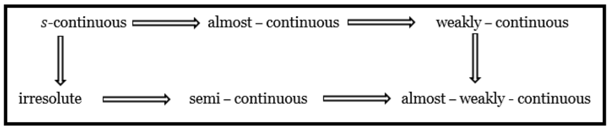
Remak 4.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is called weakly continuous [33] if for every point $x \in X$ and every open set $V \subseteq Y$ with $f(x) \in V$, there exists an open neighborhood $U \subseteq X$ of x such that:

$$f(U) \subseteq Cl(V).$$

A function $f : (X, \tau) \to (Y, \sigma)$ is called almost weakly continuous [34] if for every open set $V \subseteq Y$, the following condition is satisfied: $f^{-1}(V) \subseteq \mathrm{Int}\left(Cl(f^{-1}(Cl(V)))\right)$.

Theorem 3.1 in [35] established that a function f is almost weakly continuous if and only if, for every $\in X$ and every open neighborhood $V \subseteq Y$ containing $f(x)$, there exists a preopen set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq Cl(V)$.

Furthermore, we note the following relationships:



The following definition introduces a closure-related property for the graph of a function:

Definition 4.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. The graph of f, denoted by $G(f)\subseteq X\times X$, is said to be strongly s-closed if for every $(x,y)\in(X\times X)\setminus G(f)$, there exists an open neighborhood $U\subseteq X$ of x and a semi-open set $V\subseteq Y$ containing y such that:

$(U\times sC(V))\cap G(f)=\phi$ or equivalently, $F(U)\cap sCl(V)=\phi$.

Note that a topological space (X,τ) is referred to as semi-Urysohn [1] if, for any pair of distinct points $x,y\in U$ with $x\neq y$, there exist semi-open subsets $U,V\subseteq X$ such that:

$x\in U, y\in V$, and $sCl(U)\cap sCl(V)=\phi$.

Theorem 4.1. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following hold:

(i) If f is almost s-continuous and the codomain (Y,σ) is a semi-Urysohn space, then the graph $G(f)$ is strongly s-closed.

(ii) If $G(f)$ is strongly s-closed, then for every subset $K\subseteq Y$ that is s-closed relative to (Y,σ), the preimage $f^{-1}(K)$ is a closed subset of X.

Proof.

(i) Let $(x,y)\in(X\times Y)\setminus G(f)$, meaning $f(x)\neq y$. Since (Y,σ) is assumed to be semi-Urysohn, there exist semi-open sets $V,W\subseteq Y$ such that $f(x)\in V$, $y\in W$ and $sCl(V)\cap sCl(W)=\phi$.

Given that f is almost s-continuous, there exists an open neighborhood $U\subseteq X$ of x for which:

$$f(U)\subset sCl(V).$$

This implies:

$$f(U)\cap sCl(W)=\phi,$$

and thus:

$$(U\times sCl(W))\cap G(f)=\phi.$$

Hence, the graph $G(f)$ is strongly s-closed.

(ii) Suppose $K\subseteq Y$ is s-closed in (Y,σ), and let $x\in X\setminus f^{-1}(K)$. Then $f(x)\notin K$, and for every $y\in K$, the point (x,y) does not belong to the graph $G(f)$. Due to the strong s-closedness of $G(f)$, for each $y\in K$, there exist:

- an open neighborhood $U_y\subseteq X$ of x, and
- a semi-open set $V_y\subseteq Y$ containing y, such that: $f(U)\cap sCl(V_y)=\phi$.

Since K is s-closed, there exists a finite subset $K_1\subseteq K$ satisfying:

$$K\subseteq\bigcup_{y\in K_1} sCl(V_y).$$

Now, define:

$$U=\bigcap_{y\in K_1} U_y.$$

Then, U is an open neighborhood of x, and $f(U)\cap K\subseteq f(U)\cap\bigcup_{y\in K_1} sCl(V_y)=\phi$. Therefore, $U\cap f^{-1}(K)=\phi$, implying that $f^{-1}(K)$ is closed in (X,τ).

Corollary 4.1. Let $f:(X,\tau)\to(Y,\sigma)$ be a function, where the codomain (Y,σ) is semi-Urysohn and SC-compact. Thus, the subsequent statements are equal:

(1) The function f is strongly semi-continuous,

(2) The function f is almost s-continuous,

(3) The graph $G(f)$ is strongly s-closed,

(4) The preimage $f^{-1}(K)$ is closed in X for every subset $K\subseteq Y$ that is s-closed relative to (Y,σ).

Proof. The implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)$ follow directly from Theorem 4.1.

To establish $(4)\Rightarrow(1)$: Let $F\subseteq Y$ be a semi-closed set. Since (Y,σ) is assumed to be SC-compact, it follows that F is s-closed relative to Y. By assumption (4), the inverse image $f^{-1}(K)$ is a closed subset of X. Thus, by the definition of strongly semi-continuous functions, f is strongly semi-continuous.

Theorem 4.2. Let (X,τ) be an SC-compact topological space and suppose that $f:(X,\tau)\to(Y,\sigma)$ is a surjective function which is irresolute (respectively, semi-continuous). Then (Y,σ) is SC-compact (respectively, C-compact).

Proof. Let $F\subseteq Y$ be a semi-closed (respectively closed) subset. Since f is assumed to be irresolute (respectively, semi-continuous), the preimage $f^{-1}(F)\subseteq X$ is semi-closed. By the SC-compactness of X, this implies that $f^{-1}(F)$ is s-closed relative to X. It then follows from the surjectivity of f that:

$F = f(f^{-1}(F))$ is s-closed (respectively, quasi-H-closed) relative to Y. Consequently, Y is SC-compact (respectively, C-compact).

Corollary 4.2. If the product space $\prod_{\alpha\in\nabla} X_\alpha$ is SC-compact, then every factor. space (X_α, τ_α) is SC-compact.
Proof. Each projection mapping from the product space onto a factor space is a continuous and open surjection. Therefore, it is irresolute. Applying Theorem 4.2, the SC-compactness of the product implies the SC-compactness of each coordinate space. (By using the results in Viglino [5]).

Theorem 4.3. Let $f: (X,\tau) \to (Y,\sigma)$ be a semi-continuous function, where (X,τ) is SC-compact and (Y,σ) is a Hausdorff space. Then f is s-closed.
Proof. Consider a semi-closed set $F \subseteq X$. Since X is SC-compact, F is s-closed relative to X. From the semi-continuity of f, it follows that $f(F) \subseteq Y$ is quasi-H-closed relative to Y. Given that Y is Hausdorff, every quasi-H-closed set is closed. Thus, $f(F)$ is closed in Y, and so f is s-closed.

5. Conclusion and Future Work

In this study, we introduced and investigated the concept of SC-compact spaces, a novel class of topological spaces situated between semi-compact and C-compact spaces. These spaces are inherently s-closed in the sense of Di Maio and Noiri, and they contribute to a refined hierarchy of compactness-related structures within general topology.

We systematically explored the initial properties of SC-compact spaces and examined their preservation under various classes of mappings, including irresolute and semi-continuous functions. A range of illustrative examples was provided to clarify their connections with existing notions such as s-closed, semi-compact, and C-compact spaces. This framework offers a richer understanding of compactness through the interplay of semi-open and semi-closed sets within neighborhood systems.

Future Research Directions
The development of SC-compact spaces suggests several promising avenues for further exploration:

- Topological Rough Set Theory: Investigating the interface between SC-compactness and generalized rough set approximations, particularly those based on topological neighborhood systems and semi-open sets, such as [11-14], initial-rough sets [16, 17], and fuzzy topological spaces [36].
- Decision-Theoretic Models: Applying SC-compactness in frameworks where semi-open covers and neighborhood-based reasoning are intrinsic, such as decision-making under uncertainty by using Primal approximation spaces [37].
- Mapping Extensions and Fixed-Point Theory: Extending the analysis to broader classes of mappings, including weakly semi-continuous and almost weakly continuous functions, and studying fixed-point results within SC-compact frameworks.
- Applications in Data-Driven Contexts: Exploring the utility of SC-compactness in fields such as artificial intelligence, data mining, and information systems, where topological notions aligned with uncertainty, approximation, and granularity play a critical role.

This research lays a foundation for extending the theoretical boundaries of compactness in topology and highlights the relevance of SC-compact spaces in both pure and applied mathematical contexts.

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Availability of data and material

Not applicable.

Conflict of interest

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