مجلة البحوث المالية والتجارية
المجلد (26) – العدد الرابع – أكتوبر 2025

Maximum Likelihood Estimation of The Kumaraswamy Marshal-Olkin Lindely -Lomax Distribution from Type II Censored Samples

By

Ahmed Abd El Rahim Hussein Abou Almaaty

Lecturer at Department of Statistics, Faculty of Commerce, Al-Azhar University, Cairo, Egypt

Ahmedabouelmaaty.22@azhar.edu.eg

Abd El-Hamid Abd El-Hamid Abd El Raouf Eisa

Lecturer at Department of Statistics, Faculty of Commerce, Al-Azhar University, Cairo, Egypt

abdo.easa2012@gmail.com

تاريخ الإرسال	2025-07-27
تاريخ القبول	2025-08-17
رابط المجلة: https://jsst.journals.ekb.eg/	

Abstract

This paper introduces and studies the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution, a new and flexible four-parameter lifetime model capable of capturing a variety of hazard rate shapes, including increasing, decreasing, bathtub, and unimodal forms. Several statistical properties of the KMOLL distribution are derived, including the probability density function (PDF), cumulative distribution function (CDF), survival and hazard functions, skewness, kurtosis, and moments. Estimation of the model parameters is explored through maximum likelihood estimation (MLE) under both complete and Type-II censored data. Additionally, alternative estimation techniques, such as Maximum Product of Spacing (MPS), Least Squares Estimation (LSE), Weighted Least Squares Estimation (WLSE), and Percentile Estimation (PE), are examined. A comprehensive simulation study is conducted to compare the performance of the estimators under various sample sizes and censoring levels. Finally, the applicability of the KMOLL model is illustrated using a real-life dataset, demonstrating its superior fitting capability compared to related models. Furthermore, the study highlights the model's ability to accommodate data with diverse skewness and tail behavior, making it suitable for applications in fields such as reliability engineering, biomedical sciences, and actuarial analysis. The KMOLL distribution also demonstrates strong adaptability when handling censored data scenarios, enhancing its relevance in survival analysis contexts. The inclusion of the Marshall–Olkin and Lindley–Lomax mechanisms provides additional flexibility, enabling the model to capture complex real-world phenomena more accurately. Overall, the KMOLL model offers a robust and versatile statistical framework with significant potential for both theoretical development and practical implementation.

Keywords:

Kumaraswamy distribution; Marshall–Olkin extension; Lindley distribution; Lomax distribution; Flexible lifetime models; Maximum likelihood estimation; Type-II censoring; Hazard function; Simulation study; Real data application.

1. Introduction:

The modeling of lifetime data and reliability systems has always been a central focus in statistical research, given its wide-ranging applications in engineering, medicine, and social sciences. Over time, statisticians have proposed various distributions to capture the complex behavior of real-world data, especially where standard models such as the exponential or Weibull fall short. This has led to the continuous development of more flexible models capable of handling skewness, kurtosis, and censoring. One of the prevailing strategies is to construct new families by compounding or generalizing existing distributions, often through generator functions, mixing techniques, or parameter induction methods (Alzaatreh, Lee, & Famoye, 2013; Nadarajah, Kotz, & safety, 2006).

In this context, several distributions have emerged as foundational building blocks. One such distribution is the Kumaraswamy distribution(Jones, 2009), known for its bounded support and analytical simplicity, which makes it a powerful generator for constructing flexible families. Its probability density function (pdf) and cumulative distribution function (cdf) are frequently used to define new generalizations.

The density function:

$$f(x) = abx^{a-1}(1 - x^a)^{b-1} \quad ,0 < x < 1$$

The class of Kumaraswamy generalized distribution:

$$F(x) = 1 - (1 - x^a)^b \quad ,0 < x < 1$$

The Lindley distribution(Ghitany, Atieh, Nadarajah, & simulation, 2008) extends the classical exponential by offering heavier tail behavior and positive skewness, with pdf and cdf.

The probability density function:

$$g(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \ , \qquad x \geq 0$$

The cumulative distribution function:

$$G(x;\theta) = 1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x} \ , \qquad x \geq 0$$

To enhance its adaptability, Lindley has been utilized as a generator for new distribution families denoted by Lindley–G, where a baseline distribution G(x) is embedded within its structure. Notable examples include the Generalized Odd Lindley–G family (Cakmakyapan, Ozel, & Statistics, 2016) which introduces extra parameters to control tail behavior and provides properties such as quantile functions, moment expressions, and order statistics via mixture representations.

$$f_{Lindley-G}(x;\theta,\varepsilon) = g(x;\varepsilon)\big[1 - [log(1 - G(x;\varepsilon))]\big][1 - G(x;\varepsilon)]^{\theta-1} \frac{\theta^2}{1+\theta}$$

$$F_{Lindley-G}(x;\theta,\varepsilon) = 1 - \left\{1 - \frac{\theta}{1+\theta}[log(1 - G(x;\varepsilon))]\right\}$$

Similarly, the quasi Lindley distribution. (Merovci, Elbatal, & Puka, 2015) demonstrates enhanced flexibility in model fitting for lifetime data using skewness and kurtosis-controlling parameters. These generalizations have shown effectiveness in modeling censored data and improving fit across various applied datasets.

Additionally, the Lomax distribution(Lomax, 1954), often referred to as a Pareto Type II model, is widely applied in modeling economic and actuarial data. Its heavy-tailed nature and simple functional form of the pdf and cdf make it suitable for modeling extreme events.

$$g(x;\alpha,\sigma) = \frac{\alpha}{\sigma}\left(1 + \frac{x}{\sigma}\right)^{-(\alpha+1)} \ , x \geq 0 \ , \quad \alpha > 0 \ , \quad \sigma > 0$$

$$G(x;\alpha,\sigma) = 1 - \left(1 + \frac{x}{\sigma}\right)^{-\alpha} \ , x \geq 0 \ , \quad \alpha > 0 \ , \quad \sigma > 0$$

Lindely-Lomax distribution:

$$F_{LL}(x;\theta,\alpha,\sigma) = 1 - \left[1 + \frac{\theta\alpha}{1+\theta} log\left(1 + \frac{x}{\sigma}\right)\right]\left(1 + \frac{x}{\sigma}\right)^{-\theta\alpha}$$

$$f_{LL}(x;\theta,\alpha,\sigma) = \frac{\theta^2\alpha}{\theta+1}\left[1 + \alpha\, log\left(1 + \frac{x}{\sigma}\right)\right]\left(1 + \frac{x}{\sigma}\right)^{-(\theta\alpha+1)}$$

To further increase model flexibility, the Marshall–Olkin method (Marshall & Olkin, 1997) introduces a shock parameter that adjusts the failure rate behavior of a given distribution. This approach modifies the

baseline pdf and cdf to account for unobserved heterogeneities or sudden risks. Studies such as that by(Alabdulhadi et al., 2023) have shown how such extensions perform under complex censoring schemes, particularly with progressively Type-I censored data.

The CDF of Marshall Olkin's family of distributions is :

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} \qquad , -\infty < x < \infty$$

The probability density function:

$$g(x) = \frac{\alpha f(x)}{(1 - \bar{\alpha}\bar{F}(x))^2} \qquad , -\infty < x < \infty$$

By integrating the Kumaraswamy generator, Marshall–Olkin transformation, and the Lindley–Lomax base, a new flexible model emerges. the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution. Its structure incorporates the pdf and cdf of its underlying components, resulting in a model capable of capturing diverse hazard shapes and tail behaviors.

The Kumaraswamy Marshal-Olkin family of distribution:

$$F_{KMO}(x; a, b, p) = 1 - \left\{1 - \left(\frac{G(x;\varepsilon)}{1 - p\bar{G}(x;\varepsilon)}\right)^{a}\right\}^{b}$$

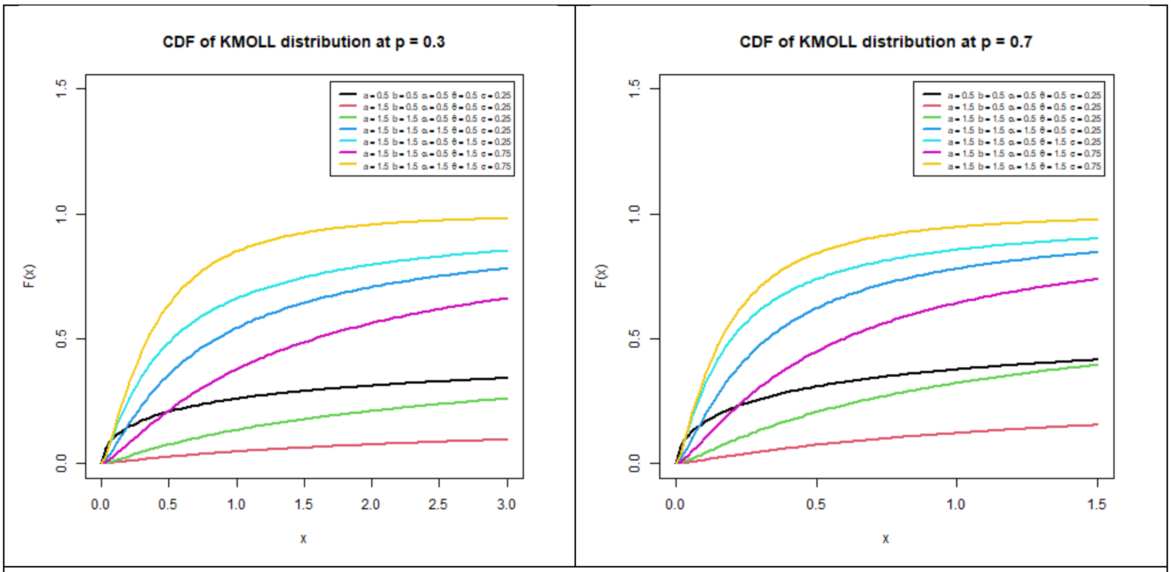

Figure (1): Cdf of KMOLL distribution at various values of parameters

Figure (1) display the cumulative distribution function (CDF) of the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution for various parameter configurations at fixed values of $p = 0.3$ and $p = 0.7$, respectively. The plots clearly illustrate the flexibility of the KMOLL distribution in modeling different data accumulation behaviors.

At both values of p, the CDF curves exhibit diverse growth patterns depending on the parameter values a ,b, α , θ , σ,. For lower values of the parameters, the distribute on accumulates probability mass more gradually, reflecting slower event occurrences in time-to-failure contexts. In contrast, for larger parameter values, the CDF rises steeply, indicating a higher probability of early occurrences.

Comparing the two plots, increasing the value of p from 0.3 to 0.7 shifts the curves upward and makes the distribution accumulate faster for most parameter settings. This confirms that the parameter ppp plays a significant role in controlling the concentration and spread of the distribution.

Overall, the behavior of the KMOLL CDF under different configurations confirms the distribution's capacity to model both light- and heavy-tailed phenomena, early or delayed failure rates, and a wide variety of real-world data structures.

The density function:

$$f_{KMO}(x; a, b, p) = \frac{ab(1-p)g(x;\varepsilon)G(x;\varepsilon)^{a-1}}{1-p\bar{G}(x;\varepsilon)^{a+1}}\left\{1-\left(\frac{G(x;\varepsilon)}{1-p\bar{G}(x;\varepsilon)}\right)^{a}\right\}^{b-1}$$

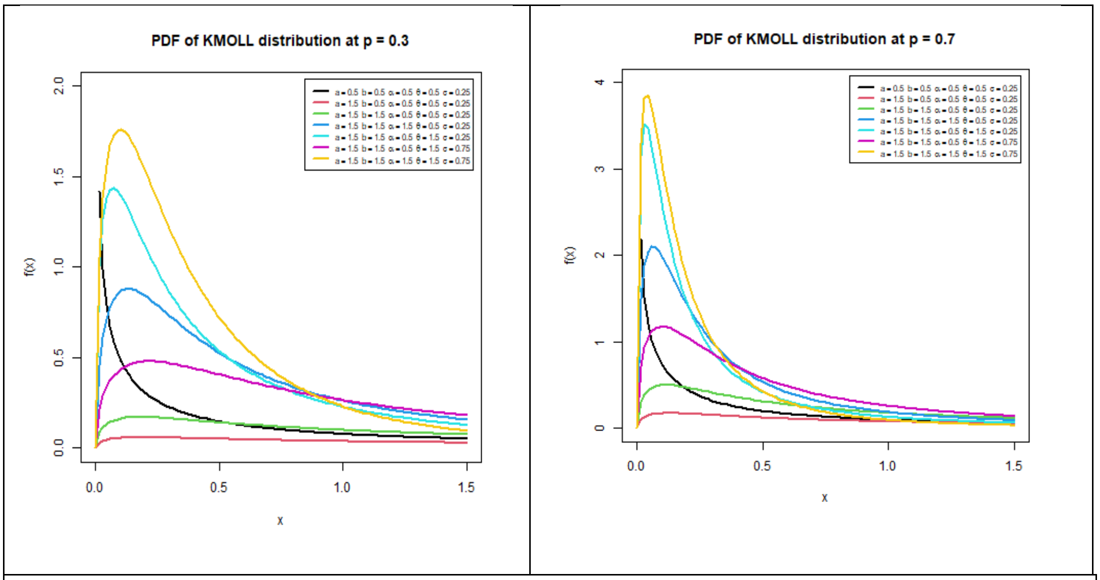

Figure (2): Pdf of KMOLL distribution at various values of parameters

Figure (2) illustrates the probability density function (PDF) of the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution for a fixed value of $p = 0.3$ and $p = 0.7$ under various parameter combinations. The figure demonstrates the remarkable flexibility of the KMOLL distribution in modeling a wide range of data shapes.

Depending on the chosen values of the parameters $a, b, \alpha, \theta, \sigma$ the PDF exhibits diverse behaviors. These include monotonic decreasing patterns, unimodal (single-peaked) shapes, and heavy-tailed structures. Such characteristics enable the KMOLL distribution to capture complex phenomena such as early-life failures, aging effects, and high-variability risks.

The sharpness and height of the peaks in the curves are influenced primarily by the shape parameters a and b, while the scale and tail behavior are regulated by α, θ, and σ. Notably, for higher values of these parameters, the PDF tends to become more peaked and concentrated near the origin, indicating a higher likelihood of early observations.

This versatility confirms the KMOLL distribution as a powerful model for applications in reliability engineering, survival analysis, and lifetime data modeling, where accurately capturing the shape of the underlying distribution is essential.

In many real-world applications, data are subject to Type-II censoring, where the observation process is terminated after a specific number of events. Several works have demonstrated how maximum likelihood estimation performs under such schemes in various distributions, including exponential, Weibull, and Lindley-based models(Balakrishnan & Aggarwala, 2000; Gupta, Kundu, & simulation, 2001). This study derives the maximum likelihood estimators for the parameters of the KMOLL distribution under Type-II censored samples. The objective is to support statistical modeling with a robust and adaptable framework for analyzing censored lifetime data.

2. Kumaraswamy marshal-Olkin Lindely -Lomax distribution and its properties:

THE KUMARASWAMY MARSHAL-OLKIN LINDELY -LOMAX DISTRIBUTION

$$F_{KMOLL}(x;t) = 1 - \left\{ 1 - \left[\frac{1-\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]}{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]} \right]^{a} \right\}^{b} \quad ,x \geq 0$$

$$t = (a,b,p,\alpha,\theta,\sigma) \quad , \quad a,b,p,\alpha, and\ \sigma > 0$$

$$f_{KMOLL}(x;t) = \frac{ab(1-p)\frac{\theta^2\alpha}{\theta+1}\left[1+\alpha\, log\left(1+\frac{x}{\sigma}\right)\right]\left(1+\frac{x}{\sigma}\right)^{-(\theta\alpha+1)}\left\{1-\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\right\}^{a-1}}{\left\{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]\right\}^{a+1}} \cdot \left\{ 1 - \left[\frac{1-\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]}{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]} \right]^{a} \right\}^{b-1} \quad ,x \geq 0$$

2.1.The survival function:

The survival function, defined as $S(x) = 1 - f(x)$

, is a fundamental concept in survival analysis and reliability theory. It represents the probability that a system, component, or individual will survive beyond a specific time . Traditional lifetime models such as the exponential and Weibull distributions have been widely used due to their simplicity and analytical tractability. However, their limited flexibility in capturing complex survival patterns has motivated the development of more general models (Lawless, 2011; Meeker, Escobar, & Pascual, 2021).

The recently proposed Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution enhances flexibility in survival modeling by integrating several generalization techniques. Specifically, it combines the Kumaraswamy-G transformation (Cordeiro, De Castro, & simulation, 2011), the Marshall–Olkin shock model (Marshall & Olkin, 1997), and the Lindley–Lomax distribution (Lindley, 1958; Lomax, 1954) as the baseline. This construction allows the KMOLL survival function to accommodate a wide

range of behaviors such as slow or fast decay, heavy or light tails, and delayed failure patterns.

Graphical representations of the KMOLL survival function under different parameter configurations reveal that small values of the shape parameters a and b tend to yield flatter survival curves, indicating higher likelihoods of long-term survival. Conversely, larger values produce sharper drops, reflecting early failure behavior. The parameter p, inherited from the Kumaraswamy generator, also significantly affects the curvature and steepness of the survival function, thereby offering additional control over tail weight and resilience modeling.

Due to these properties, the KMOLL distribution presents itself as a valuable model in fields such as biomedical sciences, industrial reliability, and actuarial science, where accurately describing time-to-failure is crucial.

$$S_{KWMOLL}(x;t) = 1 - f_{KWMOLL}(x;t)$$

$$f_{KWMOLL}(x;t) =$$

$$\left\{1 - \left[\frac{1-\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]}{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]}\right]^{a}\right\}^{b}$$

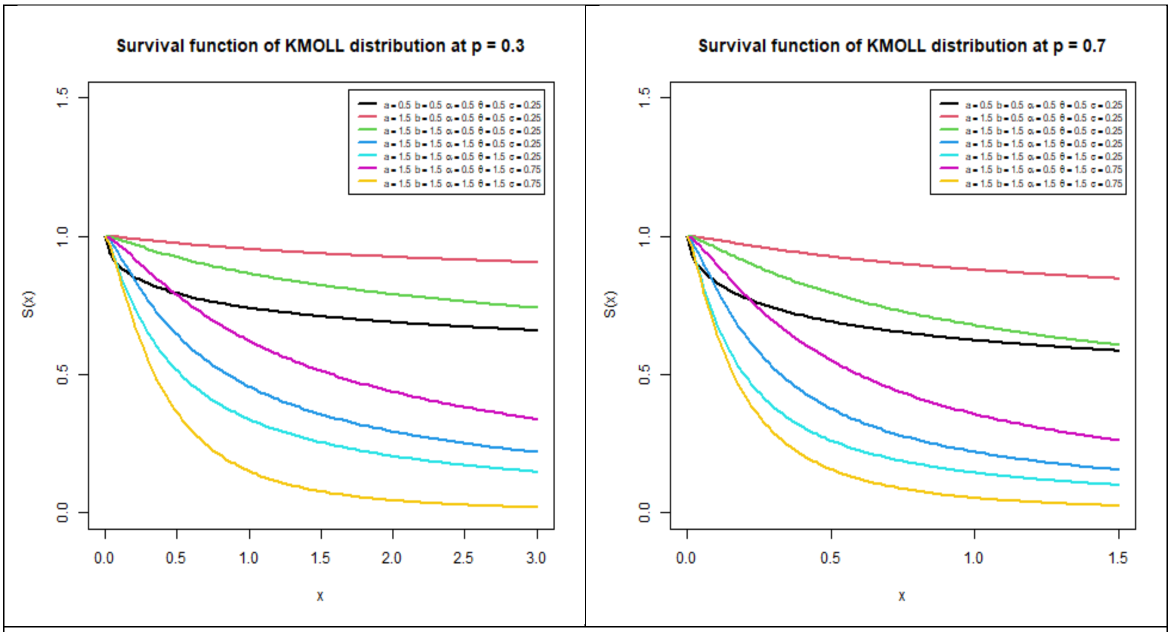

Figure (3): Survival of KMOLL distribution at various values of parameters

Figure (3) Survival function curves of the KMOLL distribution for different parameter combinations at fixed values $p = 0.3$ and $= 0.7$. The plots demonstrate the distribution's flexibility in modeling survival behavior. For $p = 0.3$, the curves show slower decay, indicating longer survival times, while at $p = 0.7$, the survival probabilities drop more rapidly, reflecting shorter lifetimes. These differences highlight the role of parameter ppp in controlling the heaviness of the tail and the failure rate of the distribution.

2.2.The hazard function:

The hazard function, denoted by $h(x) = \frac{f(x)}{S(x)}$

, is a key tool in survival analysis and reliability engineering. It quantifies the instantaneous risk of failure at time x, given that the subject has survived up to that time. Unlike the cumulative distribution or density functions, the hazard function provides deeper insight into the aging behavior of components and systems. Classical distributions such as exponential or Weibull offer limited hazard shapes (constant or monotonic), which often fall short when modeling complex real-life data(Lawless, 2011; Meeker et al., 2021).

The Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution introduces a flexible framework that generates a wide variety of hazard rate shapes, including increasing, decreasing, bathtub-shaped, and unimodal behaviors. These shapes are controlled by the parameters a, b, p, α, θ and σ, as well as the Kumaraswamy parameter p. For example, increasing hazard rates are suited to aging systems, while decreasing rates model early failures. The bathtub shape, which combines both, is common in industrial applications where infant mortality and wear-out phases are observed.

The KMOLL hazard function inherits flexibility from the Kumaraswamy-G structure (Cordeiro et al., 2011) and the Marshall–Olkin extension(Marshall & Olkin, 1997), while also incorporating the tail properties of the Lindley–Lomax baseline(Lindley, 1958; Lomax, 1954). This makes it particularly suitable for modeling time-to-failure data in reliability studies, medical survival analysis, and risk assessment contexts.

$$h_{KMOLL}(x;t) = \frac{f_{KMOLL}(x;t)}{s_{KMOLL}(x;t)}$$

$$h_{KMOLL}(x;t)=\frac{ab(1-p)\frac{\theta^2\alpha}{\theta+1}\left[1+\alpha\log\left(1+\frac{x}{\sigma}\right)\right]\left(1+\frac{x}{\sigma}\right)^{-(\theta\alpha+1)}\left\{1-\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\right\}^{a-1}}{\left\{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]\right\}^{a+1}\left\{1-\left[\frac{1-\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]}{1-p\left(1+\frac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\frac{\theta\alpha}{1+\theta}\log\left(1+\frac{x}{\sigma}\right)\right]}\right]^{a}\right\}}$$

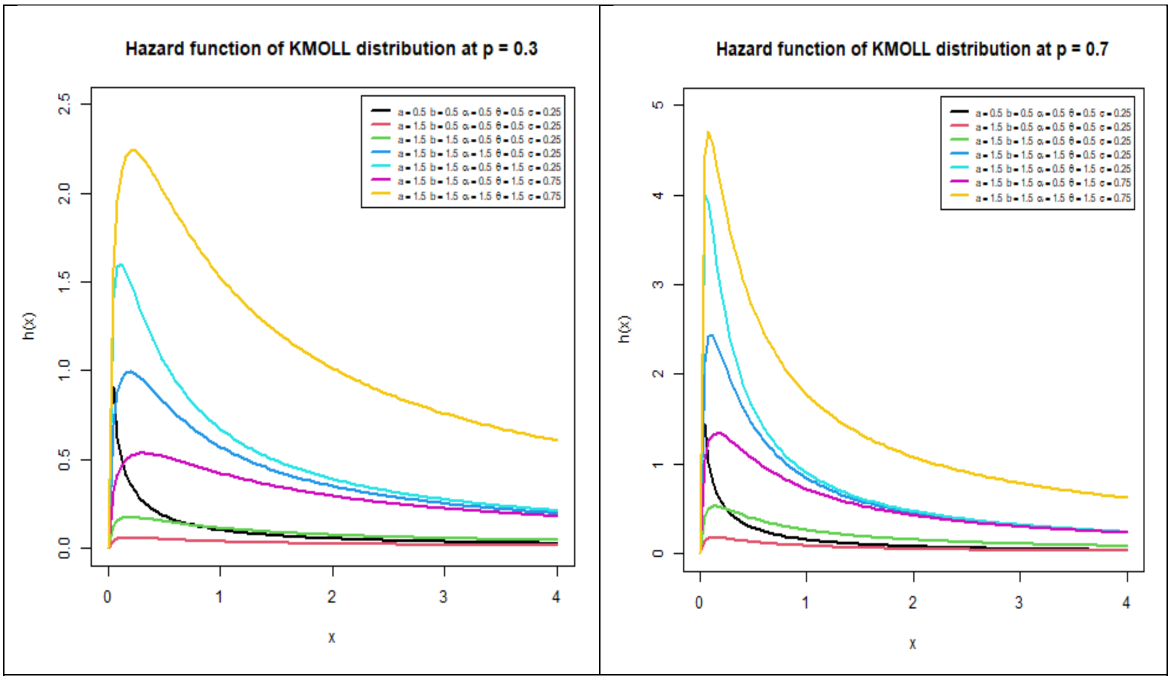

Figure (4): Hazard of KMOLL distribution at various values of parameters

Figure (4) Hazard function plots of the KMOLL distribution for various parameter combinations at fixed values of $p = 0.3$ and $p = 0.7$. The curves exhibit a variety of shapes, including increasing, decreasing, and bathtub-shaped hazard rates. This illustrates the high flexibility of the KMOLL distribution in capturing different failure behaviors, such as early-life failures, aging effects, and mixed risk profiles. The parameter p notably influences the steepness and structure of the hazard rate, allowing better control over the failure dynamics.

2.3. Expansions for the cumulative and density functions:

Here, we give simple expansions for the KMOLL cumulative distribution. by using the generalized binomial theorem $(for\ 0 < a < 1)$

$$(1+a)^v=\sum_{i=0}^{\infty}\binom{v}{i}a^i$$

$$(1-Z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j$$

where

$$\binom{n}{i} = \frac{n(n-1)\ldots(n-j+1)}{i!}$$

2.4.Moments:

Moments play a central role in summarizing the characteristics of a probability distribution. The r -th moment about the origin, denoted $\mu_r^\backslash$ is defined as:

$$\mu_r^\backslash = E[x^r] = \int_0^\infty x^r f_{KMOLL}(x;t)\, dx$$

where $f_{KMOLL}(x;t)$ is the probability density function (PDF) of the distribution. Moments provide information about the location, spread, skewness, and kurtosis of the distribution. In particular, the first moment gives the mean, the second central moment gives the variance, the third standardized moment measures skewness, and the fourth standardized moment captures kurtosis (Johnson, Kotz, & Balakrishnan, 1995).

For the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution, the analytical expression of the moments can be quite involved due to the complexity of its PDF. In many cases, a closed-form expression for the rrr-th moment may not be tractable. Therefore, numerical integration or simulation-based techniques (e.g., Monte Carlo methods) are often employed to approximate the moments (Lai & Balakrishnan, 2009).

Moreover, the moments of the KMOLL distribution can also be used to derive other important quantities such as the moment-generating function (MGF) (if it exists), as well as to compute coefficients of variation, skewness, and kurtosis, which further help in understanding the shape and behavior of the distribution under various parameter settings (Kenney & Keeping, 1962).

2.5.Order statistics:

Order statistics play a fundamental role in statistical theory and applications, particularly in reliability analysis, nonparametric inference, and estimation under censoring. Given a random sample $x_1, x_2, \ldots, x_n$ from a continuous distribution, the order statistics are the sorted values of the sample, x_1, x_2 represents the minimum (first order statistic), and x_{n-1}, x_n represents the maximum (last order statistic) (Arnold, Balakrishnan, & Nagaraja, 2008; David & Nagaraja, 2025).

Order statistics are particularly important in the context of Type-II censoring, where only the first r smallest observations are observed, and the remaining

n − r are censored. They are also essential in the derivation of the sampling distributions of extremes, medians, and quantiles.

The distribution of an individual order statistic depends on the parent distribution and the sample size. is given by:

$$f_{r:n}(x;t) = \frac{1}{\beta(r,n-r+1)} f(x;t)[f(x;t)]^{r-1}[1-f(x;t)]^{n-r}$$

$$f_{r:n}(x;t) = \frac{1}{\beta(r,n-r+1)} f(x;t) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x;t)]^{i+r-1}$$

$for\ n > 0$ **non- real integer, we obtain**

$$[1-f(x;t)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x;t)]^i$$

2.6. Skewness and Kurtosis:

Skewness and kurtosis are two fundamental measures used to describe the shape characteristics of probability distributions. These higher-order moments, beyond the first (mean) and second (variance), offer insights into the asymmetry and tail behavior of a distribution, which are particularly valuable when assessing the flexibility and descriptive power of new statistical models such as the KMOLL distribution (Johnson et al., 1995).

Skewness assesses the degree of asymmetry of a distribution around its mean. A skewness value of zero indicates perfect symmetry. Positive skewness suggests a distribution with a longer right tail, while negative skewness reflects a longer left tail. The standardized measure of skewness is expressed as:

$$\gamma_1 = \frac{E[(X-\mu)^3]}{\sigma^3}$$

where μ is the mean and σ is the standard deviation (Kenney & Keeping, 1962).

Kurtosis, meanwhile, quantifies the "tailedness" or the concentration of probability in the tails relative to the center. A kurtosis value equal to 3 corresponds to the normal distribution. Distributions with kurtosis greater than 3 (leptokurtic) have heavier tails and sharper peaks, while those with kurtosis less than 3 (platykurtic) are flatter and have lighter tails. Kurtosis is computed as:

$$\gamma_2 = \frac{E[(X-\mu)^3]}{\sigma^3}$$

According to Joanes and Gill (Joanes & Gill, 1998), the empirical estimation of skewness and kurtosis is crucial for assessing model fit, particularly when normality cannot be assumed. In the context of the KMOLL distribution, these shape measures provide valuable diagnostics on how the model responds to parameter changes and whether it can adequately describe real-world data sets characterized by asymmetric or heavy-tailed behavior.

Because of the complexity of the KMOLL density function, closed-form expressions for skewness and kurtosis may not be readily available. Therefore, numerical integration or Monte Carlo simulation techniques are often employed to approximate these values under various parameter combinations(Lai & Balakrishnan, 2009) .

2.7. Special distribution:

KUMARASWAMY LINDLEY LOMAX DISTRIBUTION

If $b = 1$, the kumaraswamy lindley lomax distribution reduces to

$$f_{KMOLL}(x;t) = \frac{a(1-p)\dfrac{\theta^2\alpha}{\theta+1}\left[1+\alpha\log\left(1+\dfrac{x}{\sigma}\right)\right]\left(1+\dfrac{x}{\sigma}\right)^{-(\theta\alpha+1)}\left\{1-\left[1+\dfrac{\theta\alpha}{1+\theta}\log\left(1+\dfrac{x}{\sigma}\right)\right]\left(1+\dfrac{x}{\sigma}\right)^{-\theta\alpha}\right\}^{a-1}}{\left\{1-p\left(1+\dfrac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\dfrac{\theta\alpha}{1+\theta}\log\left(1+\dfrac{x}{\sigma}\right)\right]\right\}^{a+1}}$$

MARSHAL – OLKIN LINDLEY LOMAX DISTRIBUTION

If $a = b = 1$, the Marshal – Olkin Lindley Lomax distribution reduces to

$$f_{MOLL}(x;t) = \frac{(1-p)\dfrac{\theta^2\alpha}{\theta+1}\left[1+\alpha\log\left(1+\dfrac{x}{\sigma}\right)\right]\left(1+\dfrac{x}{\sigma}\right)^{-(\theta\alpha+1)}}{\left\{1-p\left(1+\dfrac{x}{\sigma}\right)^{-\theta\alpha}\left[1+\dfrac{\theta\alpha}{1+\theta}\log\left(1+\dfrac{x}{\sigma}\right)\right]\right\}^2}$$

2.8. Maximum likelihood Estimation of KMOLL:

The maximum likelihood estimation (MLE) method is one of the most widely used techniques for estimating the parameters of statistical distributions due to its desirable asymptotic properties, such as consistency, efficiency, and asymptotic normality(Casella & Berger, 2002) . For the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution, let $x_1, x_2, \ldots, x_n$ be a random sample from the KMOLL model with parameter a, b, p, α, θ and σ.

The log-likelihood function $l(t)$ is constructed by taking the natural logarithm of the joint probability density function (PDF) of the sample. Due to the complex structure of the KMOLL PDF, the resulting log-likelihood function is nonlinear and does not yield closed-form solutions.

2.8.1. Maximum likelihood estimators based on complete sample :

In this subsection, we consider the estimation of the unknown parameters of the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution based on a complete sample.

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the KMOLL model with parameter a, b, p, α, θ and σ.

The joint likelihood function is constructed from the product of the individual probability density functions (PDFs), and the log-likelihood function is then derived by taking the natural logarithm.

To obtain the maximum likelihood estimators (MLEs) of the parameters, the log-likelihood function is maximized with respect to each parameter. This involves computing the score functions by taking partial derivatives of the log-likelihood with respect to a, b, p, α, θ and σ, and setting them equal to zero. However, due to the complexity and nonlinearity of the KMOLL log-likelihood function, closed-form solutions are not available.

As a result, numerical optimization methods such as the Newton–Raphson algorithm, the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm, or other quasi-Newton methods are employed to find the MLEs (Casella & Berger, 2002; Nocedal & Wright, 2006). These methods are implemented in statistical software like R, MATLAB, or Python. Once the MLEs are obtained, the observed Fisher information matrix is used to estimate the asymptotic variances and construct confidence intervals for the parameters(Meeker et al., 2021) .

MLEs based on complete samples are consistent, asymptotically normal, and efficient under standard regularity conditions, which makes them a powerful tool in parametric statistical inference(Lehmann & Casella, 1998).

.

$$l(t) = n\log a + n\log b + n\log(1-p) + n[\log(\theta^2\alpha) - \log(\theta+1)] + \sum_{i=0}^{n}[1+\alpha\log w(x_i)] - (\alpha\theta+1)\sum_{i=1}^{n}\log w(x_i) + (a-1)\sum_{i=1}^{n} log\{1-v(x_i)w(x_i)^{-\alpha\theta}\} - (a+1)\sum_{i=1}^{n} log\{1-pv(x_i)w(x_i)^{-\alpha\theta}\} + (b-1)\sum_{i=1}^{n} log\left\{1-\left[\frac{1-v(x_i)w(x_i)^{-\alpha\theta}}{1-pv(x_i)w(x_i)^{-\alpha\theta}}\right]\right\}$$

$$w(x_i) = \left(1+\frac{x}{\sigma}\right)$$

$$v(x_i) = \left[1+\frac{\theta\alpha}{1+\theta} log\left(1+\frac{x}{\sigma}\right)\right]$$

$$n(x_i) = \frac{1-v(x_i)w(x_i)^{-\alpha\theta}}{1-pv(x_i)w(x_i)^{-\alpha\theta}}$$

$$\frac{\partial l}{\partial a} = \frac{n}{\theta} - \theta\sum_{i=1}^{n}\log w(x_i) + \sum_{i=1}^{n} log\{1-v(x_i)w(x_i)^{-\alpha\theta}\} - \sum_{i=1}^{n} log\{1-pv(x_i)w(x_i)^{-\alpha\theta}\} + (1-b)\sum_{i=1}^{n}\frac{n(x_i)^a logn(x_i)}{1-n(x_i)^{-a}}$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n-r}\log(1-n(x_i)^a)$$

$$\frac{\partial l}{\partial p} = \frac{-n}{1-p} + (a+1)\sum_{i=1}^{n}\left[\frac{v(x_i)w(x_i)^{-\alpha\theta}}{1-pv(x_i)w(x_i)^{-\alpha\theta}}\right] + (b-1)\sum_{i=1}^{n}\frac{-an(x_i)^{a-1}v(x_i)w(x_i)^{-\alpha\theta}(1-v(x_i)w(x_i)^{-\alpha\theta})}{1-n(x_i)^{a}[pv(x_i)w(x_i)^{-\alpha\theta}]^{2}}$$

$$\frac{\partial l}{\partial \theta} = \frac{2n\theta}{\theta^2\alpha} - \frac{n}{\theta+1} - a\sum_{i=1}^{n}\log w(x_i) - (a-1)\sum_{i=1}^{n}\frac{w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial\theta} - \alpha v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)}{1-v(x_i)w(x_i)^{-\alpha\theta}} + p(a+1)\sum_{i=1}^{n}\frac{\alpha v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i) + w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial\theta}}{1-p\,v(x_i)w(x_i)^{-\alpha\theta}} - a(b-1)\sum_{i=1}^{n}\frac{n(x_i)^{a-1}\frac{\partial n(x_i)}{\partial\theta}}{1-n(x_i)^{a}}$$

$$\frac{\partial v(x_i)}{\partial\theta} = \frac{\alpha(\theta+1)\log w(x_i) - \theta\alpha\log w(x_i)}{(\theta+1)^2}$$

$$\frac{\partial n(x_i)}{\partial\theta} = \frac{[1-pv(x_i)w(x_i)^{-\alpha\theta}]\left\{\alpha v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i) + w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial\theta}\right\}}{[1-p\,v(x_i)w(x_i)^{-\alpha\theta}]^2} - \frac{[1-v(x_i)w(x_i)^{-\alpha\theta}]\left\{-p\left[w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial\theta} + \alpha v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)\right]\right\}}{[1-p\,v(x_i)w(x_i)^{-\alpha\theta}]^2}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n\theta^2}{\theta^2\alpha} + \sum_{i=1}^{n} \log w(x_i) - (a - 1)\sum_{i=1}^{n} \frac{\left\{\left[\frac{\theta w(x_i)^{-\alpha\theta}\log w(x_i)}{\theta+1} - \theta v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)\right]\right\}}{1 - v(x_i)w(x_i)^{-\alpha\theta}} + p(a + 1)\sum_{i=1}^{n} \frac{\left\{\left[\frac{\theta w(x_i)^{-\alpha\theta}\log w(x_i)}{\theta+1} - \theta v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)\right]\right\}}{1 - pv(x_i)w(x_i)^{-\alpha\theta}} + a(1-b)\sum_{i=1}^{n} \frac{n(x_i)^{a-1}\frac{\partial n(x_i)}{\partial \alpha}}{1 - n(x_i)^a}$$

$$\frac{\partial n(x_i)}{\partial \alpha} = \frac{[1 - pv(x_i)w(x_i)^{-\alpha\theta}]\left[\frac{\theta w(x_i)^{-\alpha\theta}\log w(x_i)}{\theta+1} + \theta v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)\right]}{[1 - p\, v(x_i)w(x_i)^{-\alpha\theta}]^2} - \frac{[1 - v(x_i)w(x_i)^{-\alpha\theta}]\left\{-p\left[\frac{\theta w(x_i)^{-\alpha\theta}\log w(x_i)}{\theta+1} - \theta v(x_i)w(x_i)^{-\alpha\theta}\log w(x_i)\right]\right\}}{[1 - p\, v(x_i)w(x_i)^{-\alpha\theta}]^2}$$

$$\frac{\partial l}{\partial \sigma} = -(\alpha + \alpha\theta + 1)\sum_{i=1}^{n} \frac{x_i}{w(x_i)\sigma^2} - (a-1)\sum_{i=1}^{n} \frac{\left\{w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial \sigma} + \alpha\theta v(x_i)w(x_i)^{-\alpha\theta-1}\frac{x_i}{\sigma^2}\right\}}{1 - v(x_i)w(x_i)^{-\alpha\theta}} + p(a+1)\sum_{i=1}^{n} \frac{\left\{w(x_i)^{-\alpha\theta}\frac{\partial v(x_i)}{\partial \sigma} + \alpha\theta\, v(x_i)w(x_i)^{-\alpha\theta-1}\frac{x_i}{\sigma^2}\right\}}{1 - pv(x_i)w(x_i)^{-\alpha\theta}} + a(1-b)\sum_{i=1}^{n} \frac{n(x_i)^{\mathrm{a}-1}\frac{\partial n(x_i)}{\partial \sigma}}{1 - n(x_i)^{\mathrm{a}}}$$

$$\frac{\partial v(x_i)}{\partial \sigma} = \frac{\alpha\theta x_i}{\left(1 + \frac{x_i}{\sigma}\right)\,(\theta+1)^{\sigma^2}}$$

$$\frac{\partial n(x_i)}{\partial \sigma} = \frac{[1 - p\upsilon(x_i)w(x_i)^{-\alpha\theta}]\left\{w(x_i)^{-\alpha\theta}\frac{\partial \upsilon(x_i)}{\partial \sigma} + \alpha\theta\, \upsilon(x_i)w(x_i)^{-\alpha\theta-1}\frac{x}{\sigma^2}\right\}}{[1 - p\,\upsilon(x_i)w(x_i)^{-\alpha\theta}]^2}$$

$$- \frac{[1 - \upsilon(x_i)w(x_i)^{-\alpha\theta}]\left\{-p\left[\alpha\theta w(x_i)^{-\alpha\theta-1}\,\upsilon(x_i)\frac{x}{\sigma^2} + w(x_i)^{-\alpha\theta}\frac{\partial \upsilon(x_i)}{\partial \sigma}\right]\right\}}{[1 - p\,\upsilon(x_i)w(x_i)^{-\alpha\theta}]^2}$$

Maximum Likelihood Estimation Based on Type II Censored Sample:

In many practical applications, complete data may not be available due to time or resource limitations, especially in reliability and survival studies. One common form of incomplete data is Type-II censoring, where an experiment is terminated after a pre-specified number of failures, say r , out of n total units under observation (Lawless, 2011) .

For a Type-II right-censored sample drawn from the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution, the likelihood function is modified accordingly.

As with the complete sample case, the log-likelihood function under Type-II censoring does not yield closed-form solutions for the parameter estimates due to the nonlinear structure of the KMOLL distribution. Therefore, numerical optimization techniques must be applied to obtain the maximum likelihood estimates (MLEs). These estimates can then be used for further inference, including hypothesis testing and confidence interval construction(Klein & Moeschberger, 2006; Meeker et al., 2021) .

The resulting log-likelihood function is given by:

$$l(t) = k\log + k\log + k\log(1-p) + k[\log(\theta^2\alpha) - \log(\theta+1]$$
$$+ \sum_{i=1}^{k}[1+\alpha\log w(x_i)]$$
$$-(\alpha\theta+1)\sum_{i=1}^{k}\log w(x_i)$$
$$+(a-1)\sum_{i=1}^{k} log\{1 - v(x_i)w(x_i)^{-\alpha\theta}\} - (a$$
$$+1)\sum_{i=1}^{k}\log\{1 - pv(x_i)w(x_i)^{-\alpha\theta}\}$$
$$+(b-1)\sum_{i=1}^{k} log\left\{1 - \left[\frac{1 - v(x_i)w(x_i)^{-\alpha\theta}}{1 - pv(x_i)w(x_i)^{-\alpha\theta}}\right]\right\}$$
$$+(n-k)b\sum_{i=1}^{k} log\left\{1 - \left[\frac{1 - v(x_i)w(x_i)^{-\alpha\theta}}{1 - pv(x_i)w(x_i)^{-\alpha\theta}}\right]^{a}\right\}$$

$$w(t_i) = \left(1 + \frac{t_i}{\sigma}\right)$$

$$v(t_i) = \left[1 + \frac{\theta\alpha}{1+\theta} log\left(1 + \frac{t_i}{\sigma}\right)\right]$$

$$n(t_i) = \frac{1 - v(t_i)w(t_i)^{-\alpha\theta}}{1 - pv(t_i)w(t_i)^{-\alpha\theta}}$$

$$\frac{\partial l}{\partial a} = \frac{n}{\theta} - \theta\sum_{i=1}^{k}\log w(t_i)$$
$$+\sum_{i=1}^{k} log\{1 - v(t_i)w(t_i)^{-\alpha\theta}\}$$
$$-\sum_{i=1}^{k}\log\{1 - pv(t_i)w(t_i)^{-\alpha\theta}\} + (1-b)\sum_{i=1}^{k}\frac{n(t_i)^{a} logn(t_i)}{1 - n(t_i)^{-a}}$$
$$+(n-k)b\sum_{i=1}^{k}\frac{n(t_k)^{a} logn(t_k)}{1 - n(t_k)^{-a}}$$

$$\frac{\partial l}{\partial b}=\frac{k}{b}+\sum_{i=1}^{k}\log(1-n(t_i)^a)+\sum_{i=1}^{k}\log(1-n(t_k)^a)$$

$$\frac{\partial l}{\partial p}=\frac{-k}{1-p}+(a+1)\sum_{i=1}^{k}\left[\frac{\upsilon(t_i)w(t_i)^{-\alpha\theta}}{1-p\upsilon(t_i)w(t_i)^{-\alpha\theta}}\right]+(b-1)\sum_{i=1}^{k}\frac{-an(t_i)^{a-1}\upsilon(t_i)w(t_i)^{-\alpha\theta}(1-\upsilon(t_i)w(t_i)^{-\alpha\theta})}{1-n(t_i)^a[p\upsilon(t_i)w(t_i)^{-\alpha\theta}]^2}+(n-k)b\sum_{i=1}^{k}\frac{-an(t_k)^{a-1}\upsilon(t_k)w(t_k)^{-\alpha\theta}(1-\upsilon(t_k)w(t_k)^{-\alpha\theta})}{1-n(t_k)^a[p\upsilon(t_k)w(t_k)^{-\alpha\theta}]^2}$$

$$\frac{\partial l}{\partial \theta}=\frac{2k\theta}{\theta^2\alpha}-\frac{k}{\theta+1}-a\sum_{i=1}^{k}\log w(t_i)-(a-1)\sum_{i=1}^{k}\frac{w(t_i)^{-\alpha\theta}\frac{\partial\upsilon(t_i)}{\partial\theta}-a\upsilon(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)}{1-\upsilon(t_i)w(t_i)^{-\alpha\theta}}+p(a+1)\sum_{i=1}^{k}\frac{\alpha\upsilon(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)+w(t_i)^{-\alpha\theta}\frac{\partial\upsilon(t_i)}{\partial\theta}}{1-p\,\upsilon(t_i)w(t_i)^{-\alpha\theta}}-a(b-1)\sum_{i=1}^{k}\frac{n(t_i)^{a-1}\frac{\partial n(t_i)}{\partial\theta}}{1-n(t_i)^a}-ab(n-k)\sum_{i=1}^{k}\frac{n(t_k)^{a-1}\frac{\partial n(t_k)}{\partial\theta}}{1-n(t_k)^a}$$

where

$$\frac{\partial\upsilon(t_i)}{\partial\theta}=\frac{\alpha(\theta+1)\log w(t_i)-\theta\alpha\log w(t_i)}{(\theta+1)^2}$$

$$\frac{\partial n(t_i)}{\partial\theta}=\frac{[1-p\upsilon(t_i)w(t_i)^{-\alpha\theta}]\left\{\alpha\upsilon(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)+w(t_i)^{-\alpha\theta}\frac{\partial\upsilon(t_i)}{\partial\theta}\right\}}{[1-p\,\upsilon(t_i)w(t_i)^{-\alpha\theta}]^2}-\frac{[1-\upsilon(t_i)w(t_i)^{-\alpha\theta}]\left\{-p\left[w(t_i)^{-\alpha\theta}\frac{\partial\upsilon(t_i)}{\partial\theta}+\alpha\upsilon(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)\right]\right\}}{[1-p\,\upsilon(t_i)w(t_i)^{-\alpha\theta}]^2}$$

$$\frac{\partial l}{\partial \alpha} = \frac{k\theta^2}{\theta^2\alpha} + \sum_{i=1}^{k} \log w(t_i) - (a - 1)\sum_{i=1}^{k} \frac{\left\{\left[\frac{\theta w(t_i)^{-\alpha\theta}\log w(t_i)}{\theta+1} - \theta v(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)\right]\right\}}{1 - v(t_i)w(t_i)^{-\alpha\theta}}$$
$$+ p(a + 1)\sum_{i=1}^{k} \frac{\left\{\left[\frac{\theta w(t_i)^{-\alpha\theta}\log w(t_i)}{\theta+1} - \theta v(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)\right]\right\}}{1 - pv(t_i)w(t_i)^{-\alpha\theta}}$$
$$+ a(1-b)\sum_{i=1}^{k} \frac{n(t_i)^{a-1}\frac{\partial n(t_i)}{\partial \alpha}}{1 - n(t_i)^a} + ab(n - k)\sum_{i=1}^{k} \frac{n(t_k)^{a-1}\frac{\partial n(t_k)}{\partial \alpha}}{1 - n(t_k)^a}$$

$$\frac{\partial n(t_i)}{\partial \alpha} = \frac{[1 - pv(t_i)w(t_i)^{-\alpha\theta}]\left[\frac{\theta w(t_i)^{-\alpha\theta}\log w(t_i)}{\theta+1} + \theta v(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)\right]}{[1 - p\,v(t_i)w(t_i)^{-\alpha\theta}]^2}$$
$$- \frac{[1 - v(t_i)w(t_i)^{-\alpha\theta}]\left\{-p\left[\frac{\theta w(t_i)^{-\alpha\theta}\log w(t_i)}{\theta+1} - \theta v(t_i)w(t_i)^{-\alpha\theta}\log w(t_i)\right]\right\}}{[1 - p\,v(t_i)w(t_i)^{-\alpha\theta}]^2}$$

$$\frac{\partial l}{\partial \sigma} = -(\alpha + \alpha\theta + 1)\sum_{i=1}^{k} \frac{t_i}{w(t_i)\sigma^2} - (a-1)\sum_{i=1}^{k} \frac{\left\{w(t_i)^{-\alpha\theta}\frac{\partial v(t_i)}{\partial \sigma} + \alpha\theta v(t_i)w(t_i)^{-\alpha\theta-1}\frac{t_i}{\sigma^2}\right\}}{1 - v(t_i)w(t_i)^{-\alpha\theta}}$$
$$+ p(a+1)\sum_{i=1}^{k} \frac{\left\{w(t_i)^{-\alpha\theta}\frac{\partial v(t_i)}{\partial \sigma} + \alpha\theta\, v(t_i)w(t_i)^{-\alpha\theta-1}\frac{t_i}{\sigma^2}\right\}}{1 - pv(t_i)w(t_i)^{-\alpha\theta}}$$
$$+ a(1-b)\sum_{i=1}^{k} \frac{n(t_i)^{\mathrm{a}-1}\frac{\partial n(t_i)}{\partial \sigma}}{1 - n(t_i)^{\mathrm{a}}}$$

$$\frac{\partial v(t_i)}{\partial \sigma} = \frac{\alpha\theta t_i}{\left(1+\frac{t_i}{\sigma}\right)\ (\theta+1)^{\sigma^2}}$$

$$\frac{\partial n(t_i)}{\partial \sigma} = \frac{[1 - pv(t_i)w(t_i)^{-\alpha\theta}]\left\{w(t_i)^{-\alpha\theta}\frac{\partial v(t_i)}{\partial \sigma} + \alpha\theta\, v(t_i)w(t_i)^{-\alpha\theta-1}\frac{t_i}{\sigma^2}\right\}}{[1 - p\,v(t_i)w(t_i)^{-\alpha\theta}]^2} - \frac{[1 - v(t_i)w(t_i)^{-\alpha\theta}]\left\{-p\ \left[\alpha\theta w(t_i)^{-\alpha\theta-1}\, v(t_i)\frac{t_i}{\sigma^2} + w(t_i)^{-\alpha\theta}\frac{\partial v(t_i)}{\partial \sigma}\right]\right\}}{[1 - p\,v(t_i)w(t_i)^{-\alpha\theta}]^2}$$

2.9. Simulation Study

To assess the performance of various estimation methods for the parameters of the Kumaraswamy Marshall–Olkin Lindley–Lomax (KMOLL) distribution, a comprehensive simulation study was conducted. The simulation was based on 1000 replications for different sample sizes, under both complete and Type-II censored samples

1. **Number of replications = 1000**
2. **Sample sizes are:** $n = 50, 75, 150, 200.$
3. **Methods of estimation are: MLE, MPS, LSE, WLSE, PE**
4. **Parameters of KMOLL distribution are:**
   a. **Case-I:** $p = 0.3$
      i. $a = 0.50, b = 0.5, \alpha = 0.5, \theta = 0.5, \sigma = 0.25$
      ii. $a = 1.50, b = 1.5, \alpha = 0.5, \theta = 1.5, \sigma = 0.75$
   b. **Case-II:** $p = 0.7$
      i. $a = 0.50, b = 0.5, \alpha = 0.5, \theta = 0.5, \sigma = 0.25$
      ii. $a = 1.50, b = 1.5, \alpha = 0.5, \theta = 1.5, \sigma = 0.75$

5. **Computed measure: Average (Avg.) and root mean square error (RMSE)**
6. **Type-II censoring: Number of failure item are computed as** $r = f * n$ **where:** $f = 40\%, 60\%, 80\%$ **and** $n = 100, 150, 200, 500$

Part I: Complete Case

Table (1.a): Average and RMSE for different estimation methods of KMOLL distribution at different sample sizes and $a = 0.50, b = 0.50, \alpha = 0.50, \theta = 0.50, \sigma = 0.25$, Case-I: $p = 0.3$

n	Parm	Method of Estimation									
		MLE		MPS		LSE		WLSE		PE	
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE
50	a	1.7761	1.7210	2.0680	1.7802	0.0969	0.4147	1.0836	0.9382	0.5246	0.1340
	b	2.2644	2.5842	1.9392	1.9270	0.8093	0.4843	0.5112	0.3458	0.5285	0.1083
	p	0.1089	0.2066	0.1442	0.1634	0.3768	0.2492	0.2216	0.1478	0.2949	0.1569
	α	1.2776	1.3798	1.0650	1.0106	0.1103	0.4433	0.9397	0.6578	0.5321	0.1747
	θ	1.0898	1.2110	1.0329	1.1349	0.5203	0.2331	1.3582	1.1889	0.5248	0.0625
	σ	0.5557	1.5030	0.2916	0.9472	0.3555	0.2538	0.1977	0.3026	0.2548	0.1250
75	a	2.1042	2.1382	2.0942	1.9388	0.1033	0.4080	1.3058	1.0529	0.4746	0.1169
	b	2.2253	2.1059	2.1224	2.4894	0.7693	0.4375	0.3173	0.3244	0.5050	0.0911
	p	0.1521	0.1808	0.1485	0.1436	0.4429	0.3469	0.4040	0.4183	0.3249	0.1525
	α	1.1283	1.1385	0.9738	0.8534	0.0983	0.4472	1.1866	0.9482	0.4978	0.1394
	θ	0.6848	0.6114	0.9263	0.9104	0.4427	0.2147	1.5997	1.2525	0.5429	0.0957
	σ	0.5517	1.0365	0.4274	0.9572	0.4042	0.3352	0.1780	0.3262	0.2625	0.0994
150	a	1.8425	1.7913	1.6220	1.4612	0.1009	0.4054	1.6680	1.4042	0.5237	0.1078
	b	2.7073	2.5065	2.3794	2.2890	0.8366	0.5140	0.2539	0.3043	0.5195	0.0923
	p	0.1832	0.1255	0.2857	0.0906	0.4630	0.3247	0.1425	0.1986	0.2954	0.1282
	α	1.0091	1.2802	0.9310	0.8721	0.1098	0.4493	1.2164	0.9000	0.5215	0.1183
	θ	1.1393	1.2002	0.8894	0.8284	0.4786	0.2210	1.7516	1.3614	0.5322	0.0866
	σ	0.5690	1.4020	0.5314	1.1127	0.2684	0.2184	0.0372	0.2317	0.2769	0.1589
200	a	1.4127	1.2967	1.4503	1.4094	0.0982	0.4052	1.6826	1.4074	0.4950	0.1092
	b	1.9692	1.8149	2.5321	2.5163	0.8847	0.5738	0.2591	0.3021	0.5191	0.0707
	p	0.0976	0.2124	0.2370	0.2647	0.4285	0.2794	0.1513	0.1881	0.3145	0.1581
	α	0.9795	0.8148	0.7556	0.5840	0.0849	0.4568	1.2606	0.9494	0.5121	0.1238
	θ	0.7890	0.5877	1.2306	1.3018	0.4965	0.2137	1.8264	1.4910	0.5232	0.0813
	σ	0.7272	1.3990	1.1790	2.2044	0.2809	0.2647	0.0422	0.2278	0.2596	0.1029

Table (1.b): Average and RMSE for different estimation methods of KMOLL distribution at different sample sizes and $a = 0.50, b = 0.50, \alpha = 0.50, \theta = 0.50, \sigma = 0.25$, Case-II: $p = 0.7$

n	Parm	Method of Estimation									
		MLE		MPS		LSE		WLSE		PE	
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE
50	a	1.5990	1.5214	1.5397	1.3339	0.0588	0.4502	0.6708	0.7421	0.4887	0.1401
	b	1.9208	1.8670	1.8860	2.3960	0.2473	0.3749	0.4378	0.2830	0.4946	0.1023
	p	0.3731	0.4168	0.2778	0.4573	0.6755	0.2751	0.9713	0.2757	0.7098	0.1876
	α	1.2457	1.4628	1.1837	1.3708	0.5234	0.4822	0.9076	0.9010	0.4337	0.1922
	θ	0.9800	0.9204	1.1971	1.2182	0.9613	1.0212	0.4061	0.4490	0.5464	0.1146
	σ	0.2657	0.4331	0.3720	0.7842	0.5399	0.9094	1.0816	2.1549	0.3075	0.1068
75	a	1.4769	1.2748	1.8008	1.4913	0.0259	0.4752	0.7020	0.3497	0.4749	0.0977
	b	3.4533	3.8211	2.3172	2.6182	0.3719	0.4606	0.7288	0.2673	0.5493	0.1440
	p	0.2673	0.4698	0.4915	0.3162	0.7153	0.2056	0.7325	0.2865	0.6599	0.1696
	α	1.2360	1.4235	1.1534	1.3812	0.4797	0.4497	0.0375	0.4625	0.5427	0.1888
	θ	0.8014	0.8013	1.1543	1.1686	0.5664	0.4265	1.3715	0.9980	0.5192	0.0886
	σ	0.2214	0.6415	0.1860	0.8926	0.4430	0.3843	0.0344	0.2179	0.3020	0.1739
150	a	0.9054	0.4819	1.0058	0.6356	0.6673	0.2296	0.7389	0.3095	0.4025	0.1037
	b	0.4806	0.2006	0.5007	0.2463	0.5149	0.2160	0.3894	0.1761	0.5708	0.0743
	p	0.6491	0.1991	0.6196	0.2277	0.6102	0.2211	0.5477	0.2657	0.7338	0.0389
	α	0.5980	0.2442	0.6209	0.2922	0.5474	0.2010	0.6520	0.2397	0.5261	0.0376
	θ	0.6810	0.3206	0.7018	0.3662	0.6692	0.3112	0.7091	0.3095	0.5680	0.0712
	σ	0.1945	0.2603	0.1726	0.2249	0.2203	0.1237	0.1609	0.1633	0.2390	0.0128
200	a	0.9174	0.4902	0.9886	0.5915	0.5981	0.1758	0.7566	0.2875	0.3768	0.1254
	b	0.5042	0.1871	0.5043	0.2260	0.5320	0.1642	0.4782	0.1583	0.5891	0.0897
	p	0.5979	0.2218	0.6111	0.2024	0.4980	0.2962	0.5719	0.2375	0.7388	0.0437
	α	0.5970	0.2247	0.6034	0.2585	0.5119	0.1452	0.5500	0.1562	0.5174	0.0358
	θ	0.6656	0.3230	0.6904	0.3348	0.6583	0.2531	0.6787	0.2470	0.5877	0.0886
	σ	0.1637	0.2669	0.1438	0.2229	0.2988	0.3514	0.1527	0.1345	0.2346	0.0157

Table (2.a): Average and RMSE for different estimation methods of KMOLL distribution at different sample sizes and $a = 1.50, b = 1.50, \alpha = 0.50, \theta = 1.50, \sigma = 0.75$, Case-I: $p = 0.3$

n	Parm	Method of Estimation									
		MLE		MPS		LSE		WLSE		PE	
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE
50	a	3.0421	2.6331	2.9782	1.9060	1.6101	0.7069	1.6824	0.6768	1.0146	0.7839
	b	1.9603	1.5479	2.4942	2.0490	1.8640	1.1342	1.5372	0.8414	2.5215	2.1342
	p	0.4470	0.2806	0.3365	0.2198	0.4097	0.3344	0.3222	0.2024	0.2938	0.2040
	α	1.0221	0.9176	0.8832	0.6689	0.6699	0.3507	0.6733	0.3654	1.0685	1.0029
	θ	1.5104	0.9706	1.8940	1.4945	1.9225	1.0858	1.9335	0.9529	1.5360	0.6872
	σ	0.6062	0.6098	0.5008	0.6521	1.2318	0.8199	0.8932	0.5937	2.3605	2.5865
100	a	2.9720	1.8983	2.9443	1.8897	1.7069	0.7055	1.7419	0.3572	1.0326	0.6896
	b	1.6719	1.1552	2.4473	1.8985	1.5049	0.8956	1.5451	0.4059	2.3958	1.9416
	p	0.3831	0.2338	0.3619	0.2001	0.2799	0.1998	0.2343	0.0982	0.1675	0.1883
	α	0.8528	0.7177	0.7707	0.5742	0.7066	0.3124	0.6226	0.3240	0.9943	0.8780
	θ	1.8359	1.0155	1.8378	1.4410	1.8986	1.6087	1.6711	0.3374	1.4186	0.5681
	σ	0.5372	0.4993	0.4165	0.5532	1.1616	1.1396	0.7531	0.2215	2.0449	2.0157
150	a	2.9984	2.2744	3.0476	1.9115	1.6960	0.6472	1.9768	0.6158	0.7791	0.8585
	b	2.0080	1.3685	2.2589	1.7329	1.4696	0.5905	1.2429	0.3967	2.6207	2.1734
	p	0.4260	0.2619	0.4273	0.2940	0.2238	0.1933	0.3892	0.2331	0.1769	0.2058
	α	0.7822	0.6618	0.8688	0.6521	0.6752	0.3172	0.7921	0.3792	1.3297	1.1411
	θ	1.8928	1.1273	1.5681	0.9621	1.7891	0.6420	1.4915	0.2951	1.1819	0.9757
	σ	0.5990	0.7297	0.5411	0.8540	0.8880	0.4032	0.8728	0.4038	3.1415	3.3983
200	a	3.0023	1.9150	3.1067	1.9670	1.7339	0.5132	2.1203	0.8231	0.7736	0.8672
	b	2.2738	1.9052	2.1816	1.6599	1.4517	0.5288	1.2002	0.4599	2.7168	2.7211
	p	0.3597	0.2137	0.4470	0.3173	0.2362	0.1911	0.2369	0.1799	0.0112	0.2888
	α	0.7301	0.5614	0.8743	0.6225	0.6219	0.2236	0.7546	0.3984	1.3368	1.1403
	θ	1.9500	1.1794	1.8866	1.3456	1.7424	0.5198	1.6426	0.3441	1.2116	0.6448
	σ	0.3713	0.5338	0.5447	0.7179	0.8484	0.3512	0.6482	0.3021	3.2388	3.8491

Table (2.b): Average and RMSE for different estimation methods of KMOLL distribution at different sample sizes and $a = 1.50, b = 1.50, \alpha = 0.50, \theta = 1.50, \sigma = 0.75$, Case-II: $p = 0.7$

n	Parm	Method of Estimation									
		MLE		MPS		LSE		WLSE		PE	
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE	AVG	RMSE
50	a	2.1943	1.2753	2.5095	1.6204	1.4379	0.6492	1.7292	0.7941	1.0798	0.7712
	b	1.5544	0.6468	1.4987	0.8571	2.1871	1.6929	1.5512	0.6744	1.9274	1.1857
	p	0.4168	0.3620	0.3736	0.3960	0.3870	0.3982	0.3906	0.3827	0.3983	0.3694
	α	0.7952	0.5498	1.0528	1.0594	0.7590	0.6475	0.8992	0.8511	1.2265	1.2736
	θ	1.8818	0.9675	1.8525	0.9716	1.9071	0.7784	1.6305	0.5416	1.6671	0.7585
	σ	0.6153	0.4106	0.6344	0.6419	1.2854	0.9432	0.8037	0.4175	1.5669	1.3326
75	a	2.4588	1.2760	2.7731	1.7852	1.5125	0.6480	1.9669	0.8104	1.1329	0.8033
	b	1.5110	0.7251	1.5140	1.0908	2.2446	2.1007	1.5176	0.8212	1.9492	1.2904
	p	0.4226	0.3769	0.3949	0.3811	0.4393	0.3665	0.5185	0.2796	0.3770	0.3949
	α	0.7986	0.5481	1.0554	1.0160	0.8062	0.5455	0.7652	0.3881	1.1265	0.9098
	θ	1.7418	0.6754	1.9682	1.1226	1.6058	0.6627	1.6528	0.6892	1.6076	0.8582
	σ	0.5222	0.4652	0.7072	1.5460	1.1430	1.0287	0.6071	0.3089	1.5675	1.3365
150	a	2.4319	1.3718	2.6729	1.8534	1.6646	0.6373	2.1087	0.8446	0.8826	0.7561
	b	1.3068	0.6320	1.2550	0.7378	1.6390	0.4973	1.2587	0.4678	2.0483	1.5837
	p	0.3573	0.3899	0.3722	0.3786	0.4867	0.2885	0.4068	0.3637	0.3264	0.4152
	α	0.8043	0.5750	0.9136	0.7557	0.7408	0.3642	0.7635	0.4200	1.4221	1.3721
	θ	1.9444	0.7732	2.0798	1.0098	1.5766	0.3834	2.0004	0.8764	1.3926	1.2851
	σ	0.4436	0.4414	0.5112	0.5263	0.8417	0.4194	0.6090	0.5447	1.9862	2.0529
200	a	2.5124	1.4571	2.5282	1.4429	1.6928	0.4251	2.0075	0.7432	0.8491	0.8618
	b	1.2276	0.6192	1.1929	0.7532	1.6972	0.3991	1.2511	0.4450	2.1775	1.7183
	p	0.3781	0.3999	0.3784	0.3860	0.5592	0.2597	0.3787	0.3817	0.2862	0.4503
	α	0.8614	0.6046	0.8682	0.6933	0.6746	0.3173	0.8524	0.5404	1.4314	1.3854
	θ	2.0232	0.8556	2.1802	1.0578	1.4187	0.3661	1.6813	0.5024	1.3879	0.6733
	σ	0.6652	0.8828	0.5471	0.6160	0.7520	0.3176	0.5012	0.3242	2.3749	2.5149

Part II: Censoring case (Type-II)

Table (3.a): Average and RMSE for MLE method of KMOLL distribution under Type-II censoring at different sample sizes and $a = 0.50, b = 0.50, \alpha = 0.50, \theta = 0.50, \sigma = 0.25$, Case-I: $p = 0.3$

	Parm	Number of failure (r)					
		$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
		AVG	RMSE	AVG	RMSE	AVG	RMSE
100	a	0.7150	0.4298	0.7001	0.4113	0.6788	0.3945
	b	0.8848	1.0377	0.6508	0.5875	0.5716	0.3763
	p	0.5952	0.4401	0.5592	0.4316	0.5434	0.3712
	α	0.7044	0.6931	0.6978	0.4899	0.5712	0.3465
	θ	0.7437	0.7040	0.5971	0.4291	0.5814	0.2948
	σ	0.6025	1.0678	0.4392	0.7088	0.4126	0.6402
150	a	0.8155	0.6143	0.6896	0.3954	0.5918	0.2645
	b	0.9673	1.1239	0.7048	0.5862	0.5149	0.2852
	p	0.5684	0.4290	0.5398	0.4050	0.4449	0.3258
	α	0.6779	0.5149	0.5627	0.3240	0.5676	0.2746
	θ	0.6934	0.6349	0.5729	0.3582	0.5788	0.2825
	σ	0.4575	0.8298	0.3268	0.3902	0.3819	0.5137
200	a	0.6654	0.4147	0.6118	0.2497	0.5577	0.1665
	b	0.8068	0.8816	0.6378	0.5098	0.5386	0.2988
	p	0.5307	0.3593	0.5058	0.3554	0.4128	0.2663
	α	0.5953	0.3680	0.5401	0.2977	0.5374	0.2547
	θ	0.5672	0.3560	0.5389	0.2695	0.5481	0.2195
	σ	0.3510	0.4606	0.3093	0.3679	0.3334	0.3613
500	a	0.5454	0.1474	0.5577	0.1323	0.5329	0.1161
	b	0.6081	0.4432	0.5787	0.3034	0.5182	0.1867
	p	0.5197	0.3731	0.4446	0.3204	0.3862	0.2302
	α	0.5676	0.2823	0.5019	0.2187	0.5235	0.1594
	θ	0.5470	0.3037	0.5683	0.2611	0.5244	0.1733
	σ	0.2939	0.2330	0.2618	0.2061	0.2731	0.1853

Table (3.b): Average and RMSE for MLE method of KMOLL distribution under Type-II censoring at different sample sizes and $a = 0.50, b = 0.50, \alpha = 0.50, \theta = 0.50, \sigma = 0.25$, Case-II: $p = 0.7$

n	Parm	Number of failure (r)					
		$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
		AVG	RMSE	AVG	RMSE	AVG	RMSE
100	a	0.9098	0.7360	0.7487	0.4642	0.5928	0.2162
	b	1.3874	1.6581	0.8962	1.0628	0.6536	0.6127
	p	0.6260	0.3742	0.6575	0.3152	0.6767	0.3052
	α	0.6008	0.4965	0.6729	0.5332	0.5606	0.3494
	θ	0.8081	0.7599	0.6768	0.5424	0.5984	0.4044
	σ	0.3842	0.7270	0.2735	0.3272	0.3180	0.3400
150	a	0.6675	0.4617	0.6645	0.4278	0.5636	0.1694
	b	0.8211	1.2059	0.7751	0.7552	0.5967	0.3951
	p	0.7227	0.2506	0.6530	0.3079	0.7390	0.2263
	α	0.6691	0.5141	0.5786	0.3689	0.5343	0.2576
	θ	0.6739	0.4869	0.6236	0.4319	0.5223	0.2706
	σ	0.3252	0.4740	0.2721	0.3238	0.3272	0.2790
200	a	0.5784	0.2417	0.6075	0.2625	0.5699	0.1635
	b	1.0341	1.4553	0.7806	0.8451	0.5689	0.2682
	p	0.7322	0.2983	0.6671	0.2862	0.6436	0.2662
	α	0.7294	0.7736	0.5561	0.3158	0.5035	0.2151
	θ	0.6928	0.6208	0.6314	0.5099	0.5880	0.3109
	σ	0.4266	0.6256	0.2919	0.3207	0.2736	0.2725
500	a	0.5206	0.1113	0.5491	0.1129	0.5205	0.0792
	b	0.8271	0.7980	0.7413	0.5920	0.5263	0.1654
	p	0.7124	0.2629	0.6340	0.2853	0.7005	0.1976
	α	0.5516	0.2726	0.5071	0.2594	0.5001	0.1818
	θ	0.6007	0.3587	0.5433	0.2991	0.5193	0.1960
	σ	0.3971	0.4082	0.2558	0.1896	0.2938	0.1843

Table (4.a): Average and RMSE for MLE method of KMOLL distribution under Type-II censoring at different sample sizes and $a = 1.50, b = 1.50, \alpha = 0.50, \theta = 1.50, \sigma = 0.75$, Case-I: $p = 0.3$

n	Parm	Number of failure (r)					
		$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
		AVG	RMSE	AVG	RMSE	AVG	RMSE
100	a	1.6973	0.9566	1.6208	0.7680	1.6797	0.7263
	b	2.7523	3.8582	2.3685	3.6623	1.5361	0.9436
	p	0.4052	0.2969	0.3839	0.3070	0.4232	0.2776
	α	1.0066	1.1513	0.7560	0.6874	0.6249	0.3353
	θ	2.1189	2.0680	1.6987	1.1001	1.6887	0.7834
	σ	1.7269	2.4477	1.2291	1.4704	0.9871	0.7341
150	a	1.5793	0.6742	1.6768	1.5104	1.6667	0.5143
	b	2.1273	3.0389	1.7647	1.3142	1.4406	0.5572
	p	0.3510	0.2568	0.3835	0.2863	0.3726	0.2658
	α	0.8562	0.9035	0.6499	0.4435	0.5650	0.2262
	θ	1.9652	1.1496	1.7801	0.9933	1.6752	0.5645
	σ	1.6337	2.1469	1.1350	0.9438	0.8228	0.4732
200	a	1.5067	0.4905	1.5308	0.4871	1.6787	0.6380
	b	2.6398	4.0074	1.8259	1.5284	1.5234	0.9486
	p	0.3312	0.2384	0.3391	0.2354	0.4030	0.2921
	α	0.9096	1.1528	0.6084	0.2913	0.5629	0.2430
	θ	1.8173	1.2266	1.7991	1.0475	1.6195	0.5947
	σ	1.7050	2.4786	1.1918	1.0951	0.8861	0.5318
500	a	1.4961	0.3130	1.4986	0.2706	1.5544	0.2600
	b	1.8980	1.8317	1.5593	0.5947	1.4796	0.4712
	p	0.3284	0.2080	0.3129	0.2006	0.3827	0.2469
	α	0.6388	0.4164	0.5481	0.1707	0.5123	0.1557
	θ	1.7431	1.0752	1.6554	0.4026	1.6155	0.4829
	σ	1.1636	1.0881	0.9153	0.4225	0.8270	0.3320

Table (4.b): Average and RMSE for MLE method of KMOLL distribution under Type-II censoring at different sample sizes and $a = 1.50, b = 1.50, \alpha = 0.50, \theta = 1.50, \sigma = 0.75$, Case-II: $p = 0.7$

n	Parm	Number of failure (r)					
		$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
		AVG	RMSE	AVG	RMSE	AVG	RMSE
100	a	1.5662	0.5586	1.4805	0.6866	1.5919	0.6213
	b	2.2739	2.0294	2.1663	1.8148	1.4524	0.6268
	p	0.6094	0.3041	0.4984	0.3343	0.5001	0.3261
	α	1.1306	1.4440	0.9621	1.2107	0.7223	0.3741
	θ	1.9625	1.9351	1.8068	0.9530	1.6367	0.7104
	σ	1.4958	1.7890	1.4127	1.5153	0.7924	0.4357
150	a	1.5868	0.5477	1.5408	0.6741	1.5243	0.4076
	b	2.1604	2.1781	2.3144	2.3468	1.4359	0.6041
	p	0.5950	0.2893	0.5282	0.3151	0.5068	0.3048
	α	1.0038	1.1547	0.8599	0.8738	0.7227	0.3601
	θ	1.8122	1.2047	2.0482	2.5864	1.7554	0.6434
	σ	1.5245	2.1314	1.3127	1.6187	0.8772	0.4154
200	a	1.4896	0.5062	1.4801	0.6338	1.5732	0.3903
	b	2.9381	3.0304	2.3686	3.3363	1.4239	0.6040
	p	0.5630	0.2813	0.5404	0.3119	0.5524	0.2873
	α	1.0830	1.4511	0.8121	0.6510	0.6030	0.2646
	θ	1.9275	2.2409	1.8436	1.3816	1.7043	0.6522
	σ	1.5743	2.0155	1.2271	1.1474	0.7376	0.3569
500	a	1.4612	0.2885	1.4809	0.2707	1.5497	0.2371
	b	2.3209	1.9779	1.7210	0.8271	1.3587	0.4198
	p	0.5806	0.2602	0.5754	0.2443	0.6002	0.2184
	α	0.7546	0.5747	0.5896	0.2626	0.6023	0.2537
	θ	1.7650	1.7431	1.6470	0.4922	1.6770	0.5162
	σ	1.2432	1.0910	0.9075	0.5044	0.7559	0.3205

Comment on complete case: It is observed that as the sample size nnn increases, the Root Mean Square Error (RMSE) values decrease consistently across all estimation methods. This indicates improved estimation accuracy with larger sample sizes, as expected due to the law of large numbers. Additionally, the average estimates of the parameters show clear convergence toward the true parameter values, reflecting the consistency of the estimators.

When comparing the different estimation methods, it is observed that the Maximum Likelihood Estimator (MLE) generally provides the most

accurate and stable estimates, especially for larger sample sizes. The Maximum Product of Spacings (MPS) and Weighted Least Squares Estimator (WLSE) also perform well in terms of both average and RMSE values, and in some cases, they exhibit comparable or even slightly superior behavior to MLE in smaller samples.

In contrast, the Least Squares Estimator (LSE) and Percentile Estimator (PE) tend to show relatively higher RMSE values, particularly when the sample size is small, indicating a lower efficiency under such conditions. However, as nnn increases, the differences in performance between methods tend to narrow.

Overall, the simulation confirms that MLE, MPS, and WLSE are more reliable choices for parameter estimation of the KMOLL distribution under complete sample scenarios.

Comment on Type-II censoring: It is observed that as the level of censoring (r) increases, the RMSE value decreases. Similarly, as the sample size (n) increases, the RMSE also decreases. Moreover, the average estimate of the parameter tends to converge to the true (assumed) value. In the case of Type-II censored samples, the number of observed failure items r was determined by $r = f * n$, where the censoring proportion f takes values 40% ,60% and 80%, and the sample sizes n were set to 100,150,200,100, and 500. This setting allows for evaluating the robustness of estimation techniques under varying degrees of censoring.

The results of the simulation provide insight into the bias, efficiency, and robustness of each estimator under different scenarios, and highlight the flexibility of the KMOLL model in modeling data with different characteristics.

2.10. Real Data Analysis

The dataset presented below represents the number of cycles to failure for 60 electrical appliances in a life test experiment(Lawless, 2011). Notably, this dataset has also been used by(Kim, Kim, & Safety, 2017; Kohansal, Pérez-González, & Fernández, 2023). The dataset is given below:

0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100, 4.106, 4.116, 4.315, 4.510, 4.580, 5.267, 5.299, 5.583, 6.065, 9.701

We begin by assessing the goodness-of-fit of the data to the proposed KMOLL distribution using the Kolmogorov–Smirnov test based on the MLEs. The estimated parameters of the distribution are as follows: $a = 0.7048, b = 77.0585, p = 0.0234, \alpha = 148.9359, \theta = 0.0177, \sigma = 122.3321$. The results of the exact one-sample Kolmogorov–Smirnov test showed a test statistic of D = 0.0635 with a p-value = 0.9563, indicating that the data conforms well to the proposed KMOLL distribution. This conclusion is further supported graphically using both the histogram and the empirical cumulative distribution curve, where it is clearly observed that the data aligns closely with the proposed distribution.

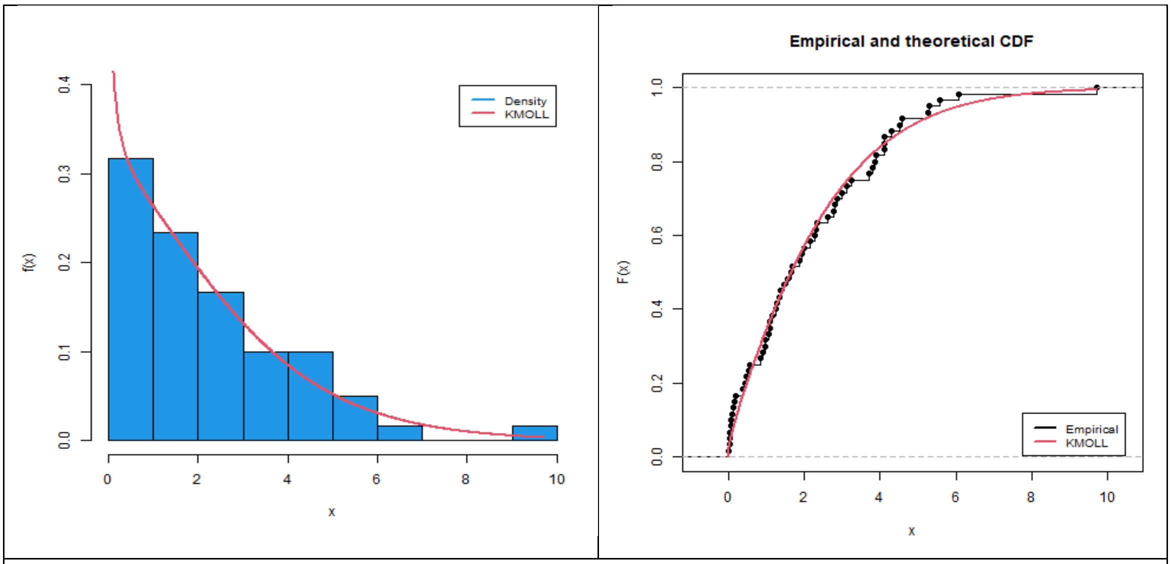

Figure (5): Histogram and ECDF for KMOLL distribution with given real dataset

The following presents the estimation of the parameters of the KMOLL distribution using real data based on various estimation methods. The parameter estimates (Est.) are accompanied by their standard errors (St.Er). We then estimated the distribution parameters using the (MLE) method under Type-II censoring, assuming three levels of censoring (r). As in the simulation study, three sample size levels were considered: 40%, 60%, and 80%. The standard error for each parameter estimate was also calculated.

Table (5.a): Estimates and standard errors for different estimation methods of KMOLL distribution under given real data

Parm	Method of Estimation									
	MLE		MPS		LSE		WLSE		PE	
	Est.	St.Er	Est.	St.Er	Est.	St.Er	Est.	St.Er	Est.	St.Er
a	0.7616	0.1161	0.7332	0.6859	0.6623	0.6598	0.6859	0.0342	0.6660	0.0682
b	151.3715	NA	76.8207	2001.1290	76.2308	1246.3376	78.3577	27.2733	87.7683	101.7585
p	0.0043	NA	0.9535	1.5078	0.0427	NA	0.0192	NA	0.0338	NA
α	393.6707	341.5182	149.7664	504.6339	155.5029	4981.6060	160.8034	196.0499	162.6557	NA
θ	0.0234	0.0266	0.0043	0.0097	0.0134	0.0261	0.0171	0.0019	0.0086	0.0008
σ	622.2920	185.6186	121.4946	538.6213	121.8764	4505.3640	146.9598	205.3149	80.4899	NA

The parameter estimates for the KMOLL distribution based on real data show noticeable variation across estimation methods. MLE and WLSE generally provide more stable and reliable estimates with relatively smaller standard errors for most parameters, particularly a, θ, and σ. In contrast, LSE and PE yield higher variability, especially for shape parameters. The MPS method gives reasonable estimates but with large standard errors in some cases. Notably, the estimate of p differs significantly across methods, indicating sensitivity to the estimation technique. Overall, MLE and WLSE appear to be the most efficient under the applied dataset.

Table (5.b): Estimates and standard errors for MLE method of KMOLL distribution under Type-II censoring under given real data

Parm	Number of failure (r)					
	$r = 40\%n$		$r = 60\%n$		$r = 80\%n$	
	Est.	St.Er	Est.	St.Er	Est.	St.Er
a	0.6584	0.1379	0.7048	0.1267	0.7090	0.0915
b	94.1985	0.0016	78.9202	0.0881	101.5098	0.0001
p	0.8798	0.6625	0.0234	3.2922	0.0900	0.0004
α	137.2059	0.0016	148.9359	0.0774	101.7651	0.0001
θ	0.0049	0.0154	0.0177	0.0368	0.0218	0.0096
σ	148.1294	0.0015	129.7789	0.0874	154.8963	0.0001

The MLE estimates of the KMOLL distribution under Type-II censoring reveal consistent patterns as the number of observed failures increases (from 40% to 80%). Overall, the standard errors decrease with more observed data, indicating improved precision in parameter estimation.

The parameter a shows increasing stability with higher failure proportions, and parameters
θ and σ also reflect improved accuracy at 80% censoring. Notably, some parameters—especially
b, α, and σ exhibit extremely small standard errors, suggesting strong identifiability under higher r.

However, the parameter p shows instability, particularly at 60% failure (high standard error of 3.29), indicating sensitivity to censoring level. Overall, the MLE performs better as the censoring level decreases, especially in estimating key shape and scale parameters.

References

Alabdulhadi, M. H., El-Saeed, A. R., Elgarhy, M., Eisa, A. E.-H., Abdo, D. A. J. A. M., & Sciences, I. (2023). Statistical Inference of Marshall-Olkin Extended Exponential Distribution Based on Progressively Type-I Censored Data. 17(6), 1033-1046.

Alzaatreh, A., Lee, C., & Famoye, F. J. M. (2013). A new method for generating families of continuous distributions. 71(1), 63-79.

Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (2008). A first course in order statistics: SIAM.

Balakrishnan, N., & Aggarwala, R. (2000). Progressive censoring: theory, methods, and applications: Springer Science & Business Media.

Cakmakyapan, S., Ozel, G. J. H. J. o. M., & Statistics. (2016). The Lindley family of distributions: properties and applications. 46(6), 1113-1137.

Casella, G., & Berger, R. L. J. P. G., CA. (2002). Statistical inference duxbury press.

Cordeiro, G. M., De Castro, M. J. J. o. s. c., & simulation. (2011). A new family of generalized distributions. 81(7), 883-898.

David, H. A., & Nagaraja, H. N. (2025). Order Statistics. In International Encyclopedia of Statistical Science (pp. 1838-1840): Springer.

Ghitany, M. E., Atieh, B., Nadarajah, S. J. M., & simulation, c. i. (2008). Lindley distribution and its application. 78(4), 493-506.

Gupta, R. D., Kundu, D. J. J. o. s. c., & simulation. (2001). Generalized exponential distribution: different method of estimations. 69(4), 315-337.

Joanes, D. N., & Gill, C. A. J. J. o. t. R. S. S. S. D. (1998). Comparing measures of sample skewness and kurtosis. 47(1), 183-189.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). Continuous univariate distributions, volume 2 (Vol. 2): John wiley & sons.

Jones, M. J. S. m. (2009). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. 6(1), 70-81.

Kenney, J. F., & Keeping, E. J. M. o. s. (1962). Linear regression and correlation. 1, 252-285.

Kim, H., Kim, P. J. R. E., & Safety, S. (2017). Reliability models for a nonrepairable system with heterogeneous components having a phase-type time-to-failure distribution. 159, 37-46.

Klein, J. P., & Moeschberger, M. L. (2006). Multivariate Survival Analysis. In Survival Analysis: Techniques for Censored and Truncated Data (pp. 425-441): Springer.

Kohansal, A., Pérez-González, C. J., & Fernández, A. J. J. B. o. t. M. M. S. S. (2023). Multi-component reliability inference in modified Weibull extension distribution and progressive censoring scheme. 46(2), 61.

Lai, C. D., & Balakrishnan, N. (2009). Continuous bivariate distributions: Springer.

Lawless, J. F. (2011). Statistical models and methods for lifetime data: John Wiley & Sons.

Lehmann, E. L., & Casella, G. (1998). Theory of point estimation: Springer.

Lindley, D. V. J. J. o. t. R. S. S. S. B. (1958). Fiducial distributions and Bayes' theorem. 102-107.

Lomax, K. S. J. J. o. t. A. s. a. (1954). Business failures: Another example of the analysis of failure data. 49(268), 847-852.

Marshall, A. W., & Olkin, I. J. B. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. 84(3), 641-652.

Meeker, W. Q., Escobar, L. A., & Pascual, F. G. (2021). Statistical methods for reliability data: John Wiley & Sons.

Merovci, F., Elbatal, I., & Puka, L. J. A. U. A. (2015). The McDonald quasi lindley distribution and its applications. 45, 87-105.

Nadarajah, S., Kotz, S. J. R. e., & safety, s. (2006). The beta exponential distribution. 91(6), 689-697.

Nocedal, J., & Wright, S. J. (2006). Numerical optimization: Springer.