Statistical Inference for Two Mixture Weibull samples Based on Joint Type-I Censored Scheme

الاستدلال الإحصائي لعينتان من خليط وايبل يعتمد على خطة مراقبة مشتركه من النوع الأول

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ABSTRACT

This study discusses the maximum likelihood estimation for determining the unknown parameters of two mixture Weibull samples under a Joint Type-I censoring scheme. Simulation research is conducted to evaluate the performance of maximal likelihood estimators (MLEs) utilizing R statistical programming software version 4.0.4. The analysis results are displayed in three tables: the first table presents one ordered complete sample of mixed Weibull distributions, the second table exhibits two distinct joint type-I samples from mixed Weibull distributions, and the third table illustrates the maximum likelihood estimates (MLEs) along with their standard errors and 95% confidence intervals. Ultimately, we examine a singular genuine data set for illustrative purposes. The actual data sets were derived from a study on the tensile strength of jute fibers with a modified Weibull distribution, with lengths of 10 mm and 20 mm. Each fiber group included 30 analogous fibers, and the findings were displayed in tables and graphs.

ناقش هذه الدراسة تقدير أقصى احتمال لتحديد المعلمات المجهولة لعينتين من خليط ويبل تحت مخطط الرقابة من النوع الأول المشترك. تم إجراء بحث محاكاة لتقييم أداء مقدرات الاحتمال الأقصى (MLEs) باستخدام برنامج البرمجة الإحصائية R يتم الإصدار 4.0.4. عرض نتائج التحليل في ثلاثة جداول: يعرض الجدول الأول عينة كاملة مرتبة واحدة من توزيعات ويبل المختلطة، ويعرض الجدول الثاني عينتين منفصلتين من النوع الأول المشترك من توزيعات ويبل المختلطة، ويوضح الجدول الثالث تقديرات أقصى احتمال (MLEs) إلى جانب الأخطاء المعيارية وفترات الثقة 95٪. في النهاية، نقوم بفحص مجموعة بيانات أصلية مفردة لأغراض التوضيح. تم استخلاص مجموعات البيانات الفعلية من دراسة حول قوة الشد لألياف الجوت بتوزيع ويبل معدل، بأطوال 10 مم و20 مم. تضمنت كل مجموعة ألياف 30 ليفًا مشابهًا، وتم عرض النتائج في جداول ورسوم بيانية.

Keywords: Joint Type-I censored scheme, mixture Weibull distribution, and Maximum likelihood estimation.

1. Introduction

There exist four categories of joint schemes: joint Type-II, joint progressive censoring Type-II (JPC-II), joint progressive censoring Type-I (JPC-I), and joint Type-I progressive hybrid censoring schemes (JT-I PHCS). Four categories of cooperative plans have been proposed in the literature. Balakrishnan and Rasouli (2008) formulated the likelihood inference for the parameters of two exponential populations subjected to simultaneous Type-II censoring. Rasouli and Balakrishnan (2010) investigated the statistical inference of two exponential populations under JPC-II. Parsi et al. (2011) and Doostparast et al. (2013) examined the censoring sample. Shafay et al. (2013) and Ashour and Abo-Kasem (2014 a,b,c) examined the jointly Type-II censored sample (JT-II CS). Balakrishnan and Feng (2015) expanded upon the research of Balakrishnan and Rasouli (2008) by examining a JT-II CS derived from k separate exponential populations. Furthermore, Balakrishnan et al. (2015) expanded upon the work of Rasouli and Balakrishnan (2010) by examining a JPC-II derived from k independent exponential populations. Ashour and Abo-Kasem (2017) presented the JPC-I scheme, which includes the specific instance of the joint Type-I censored scheme (JT-I CS). They examined statistical inference for two exponential samples under both JPC-I and JT-I CS. Abo-Kasem et al. (2019) introduced a JT-I PHCS technique and examined the estimation issues pertaining to exponential distribution. Abo-Kasem and Nassar (2019) formulated estimation issues for two Weibull populations sharing an identical shape parameter under the JPC-I censoring method. They acquired the maximum likelihood estimators (MLEs) and the approximate confidence intervals. They additionally derived the Bayes estimates utilizing squared error and LINEX loss functions, predicated on the assumption of independent gamma priors. Abo-Kasem and Nassar (2019) examined the maximum likelihood estimators for the parameters of k independent exponential populations, derived from a JT-I PHCS scheme. Recently, numerous scholars have focused on various types and uses of censoring schemes, including Hassan and Nassr (2021).

Probability distributions are fundamental in understanding and modeling random events in various fields. They describe how data is spread and help predict future outcomes. In economics, distributions are used to study income levels, market risks, and consumer behavior. In medicine, they help model patient survival times, disease progression, and treatment effects. Engineering relies on distributions for analyzing system reliability, product lifetimes, and quality control. These applications make distributions valuable tools for solving practical problems. By using the right distribution, researchers and professionals can make better, data-informed decisions see for example, Hassan et al. (2020), Hassan and Nassr (2020), Hassan et al. (2021).

This research will examine JT-I CS applied to mixed samples. We consider inference of the parameters of two mixture Weibull samples under JT-I CS. The significance of employing the mixture Weibull distribution arises from its widespread use and adaptability in modeling lifetime data characterized by increasing or decreasing failure rates.

The subsequent sections of this work are structured as follows: Section 2 contains the model description and nomenclature. The maximum likelihood estimation is presented in section 3. Section 4 presents a numerical example to elucidate the suggested estimation and analysis of an actual dataset. The final segment comprises a succinct conclusion.

2. Model Description and Notation

Suppose that the lifetime of *m* units of product *A*, $X_1, \ldots, X_m$ be IID random variables from mixture distribution made up from two subpopulations $(s = 2)$ with unknown proportions p_j with distribution function $F_j(x_{ji})$ and density function $f_j(x_{ji})$ where $j = 1, \ldots, s$ and $i = 1, \ldots, r_j$,and $Y_1, \ldots, Y_n$ the lifetime of *n* units of product *B*, are IID random variables from mixture distribution made up from two subpopulations with unknown proportions φ_j with distribution function $G_j(y_{ji})$ and density function $g_j(y_{ji})$, further, suppose $W_{(1)} < W_{(2)} < \cdots < W_{(N)}$ denote the order statistics of the *N*= *m* + *n* random variables $\{X_1, \ldots, X_m; Y_1, \ldots, Y_n\}$. Then under JT-I CS in equation (1), (see Ashour and Abo-Kasem ,2017)

$$L = \frac{m!n!}{(m-m_r)!(n-n_r)!}\prod_{i=1}^{r}\{[f(w_i)]^{z_i}[g(w_i)]^{(1-z_i)}\}\{\overline{F}(T)\}^{m-m_r}\{\overline{G}(T)\}^{n-n_r} \quad (1)$$

With using separation of subpopulations between the two samples of *N* units placed on a test. After the test is terminated *r* number of failure units occur, r_j from subpopulation j and $\sum_{j=1}^{s} r_j = r$. (see Afify,2014) the likelihood function in general, can be written as

$$L(\vartheta|\mathbf{w}) = C \prod_{j=1}^{s}\prod_{i=1}^{r_j}\left[p_j f_j(w_{ji})\right]^{z_i}\left[\varphi_j g_j(w_{ji})\right]^{(1-z_i)}\left[\bar{F}^*(T)\right]^{m-m_r}\left[\bar{G}^*(T)\right]^{n-n_r}, \tag{2}$$

where $\bar{F}^*(T) = \sum_{j=1}^{s} p_j \bar{F}_j(T)$ and. $\bar{G}^*(T) = \sum_{j=1}^{s} \varphi_j \bar{G}_j(T)$

Let $s = 2$

$$L(\vartheta|\mathbf{w}) \propto (p_1 p_2)^{r_1}\prod_{i=1}^{r_1}\left[f_1(w_{1i})f_2(w_{2i})\right]^{z_i}\left[\bar{F}^*(T)\right]^{m-m_r} \times(\varphi_1\varphi_2)^{r_2}\prod_{i=1}^{r_2}\left[g_1(w_{1i})g_2(w_{2i})\right]^{(1-z_i)}\left[\bar{G}^*(T)\right]^{n-n_r}. \tag{3}$$

where $\bar{F}^*(T) = p_1\bar{F}_1(T) + p_2\bar{F}_2(T)$ and $\bar{G}^*(T) = \varphi_1\bar{G}_1(T) + \varphi_2\bar{G}_2(T)$.

Note this $p_1, p_2, \varphi_1,$ and φ_2 are non-negative under conditions $p_1 + p_2 = 1$ and $\varphi_1 + \varphi_2 = 1$. If we set $p_1, \varphi_1 = 1$ or $p_2, \varphi_2 = 1$,The mixture JT-I CS in equation (3) reduces to JT-I CS in equation (1).

3. Maximum Likelihood Estimators

Suppose that the lifetimes of *m* units of product *A*, be IID random variables from Mixed Weibull Distributions (MWD) with two parameters (α_j, β_j) where $\alpha_j > 0$ is a scale parameter and $\beta_j > 0$ is a shape parameter, with the density function $f_j(w_{ji})$. Similarly, let the lifetimes of *n* units of product *B*, be IID random variables from Mixed Weibull Distributions with two parameters (α_j, β_j) where $\alpha_j > 0$ is a scale parameter and $\beta_j > 0$ is a shape parameter, with the density function $g_j(w_{ji})$. In this case, the likelihood function in Equation (3) becomes

$$
\begin{aligned}
L(\vartheta|\mathbf{w}) \propto & \left(p_1(1-p_1)\right)^{r_1} \prod_{i=1}^{r_1}\left[f_1(w_{1i};\alpha_1,\beta_1) f_2(w_{2i};\alpha_2,\beta_2)\right]^{z_i} \\
& \times\left(\varphi_1(1-\varphi_1)\right)^{r_2} \prod_{i=1}^{r_2}\left[g_1(w_{1i};\alpha_1,\beta_1) g_2(w_{2i};\alpha_2,\beta_2)\right]^{(1-z_i)} \\
& \times\left[p_1 \exp\left(-(T/\alpha_1)^{\beta_1}\right)+(1-p_1)\exp\left(-(T/\alpha_2)^{\beta_2}\right)\right]^{m-m_r} \\
& \times\left[\varphi_1 \exp\left(-(T/\alpha_1)^{\beta_1}\right)+(1-\varphi_1)\exp\left(-(T/\alpha_2)^{\beta_2}\right)\right]^{n-n_r},
\end{aligned} \quad (4)
$$

where $\vartheta=\left(\alpha_j,\beta_j,p_j,\varphi_j\right)^{\mathbf{T}},\ j=1,2$ is parameter vector,

$$
f_j(w_{ji};\alpha_j,\beta_j)=\frac{\beta_j}{\alpha_j}\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j-1}\exp\left(-\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j}\right),\ i=1,2,
$$

$$
g_j(w_{ji};\alpha_j,\beta_j)=\frac{\beta_j}{\alpha_j}\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j-1}\exp\left(-\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j}\right),\ i=1,2,
$$

The corresponding log likelihood function will be

$$
\begin{aligned}
\ell(\vartheta|\mathbf{w}) \propto\ & r_1\log\left(p_1(1-p_1)\right)+\sum_{i=1}^{r_1} z_i\log\left(f_1(w_{1i};\alpha_1,\beta_1)\right)+\sum_{i=1}^{r_1} z_i\log\left(f_2(w_{2i};\alpha_2,\beta_2)\right) \\
& +r_2\log\left(\varphi_1(1-\varphi_1)\right)+\sum_{i=1}^{r_2}(1-z_i)\log\left(g_1(w_{1i};\alpha_1,\beta_1)\right)+\sum_{i=1}^{r_2}(1-z_i)\log\left(g_2(w_{2i};\alpha_2,\beta_2)\right) \\
& +(m-m_r)\log\left[p_1\exp\left(-(T/\alpha_1)^{\beta_1}\right)+(1-p_1)\exp\left(-(T/\alpha_2)^{\beta_2}\right)\right] \\
& +(n-n_r)\log\left[\varphi_1\exp\left(-(T/\alpha_1)^{\beta_1}\right)+(1-\varphi_1)\exp\left(-(T/\alpha_2)^{\beta_2}\right)\right].
\end{aligned}
$$

(5)

Differentiation (5) with respect to $\beta_1,\beta_2,\alpha_1,\alpha_2,p_1$ and φ_1 and equating the first derivations to zero and solving to get the MLEs $\hat{p}_1,\hat{\varphi}_1,\hat{\beta}_1,\hat{\beta}_2,\widehat{\alpha_1}$ *and* $\widehat{\alpha_2}$, we get the following equations respectively.

$$\frac{\partial \ln L}{\partial p_1}=\frac{r_1}{p_1}-\frac{r_1}{1-p_1}+\frac{m-m_r}{p_1}-\frac{n-n_r}{1-p_1}=0$$

(6)

$$\frac{\partial \ln L}{\partial \varphi_1}=\frac{r_2}{\varphi_1}-\frac{r_2}{1-\varphi_1}+\frac{m-m_r}{\varphi_1}-\frac{n-n_r}{1-\varphi_1}=0$$

$$\frac{\partial \ln L}{\partial \beta_1}=\frac{2}{\beta_1}+\sum_{i=1}^{r_1} z_i \ln\left(\frac{w_{1i}}{\alpha_1}\right)-\left(\frac{\sum_{i=1}^{r_1} z_i w_{1i}}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{\sum_{i=1}^{r_1} z_i w_{1i}}{\alpha_1}\right)-(m-m_r)\left(\frac{T}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{T}{\alpha_1}\right)$$

$$+\sum_{i=1}^{r_2}(1-z_i)\ln\left(\frac{w_{1i}}{\alpha_1}\right)-\left(\frac{\sum_{i=1}^{r_2}(1-z_i)w_{1i}}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{\sum_{i=1}^{r_2}(1-z_i)w_{1i}}{\alpha_1}\right)-(n-n_r)\left(\frac{T}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{T}{\alpha_1}\right)=0$$

(7)

$$\frac{\partial \ln L}{\partial \beta_2}=\frac{2}{\beta_2}+\sum_{i=1}^{r_1} z_i \ln\left(\frac{w_{2i}}{\alpha_2}\right)-\left(\frac{\sum_{i=1}^{r_1} z_i w_{2i}}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{\sum_{i=1}^{r_1} z_i w_{2i}}{\alpha_2}\right)-(m-m_r)\left(\frac{T}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{T}{\alpha_2}\right)$$

$$+\sum_{i=1}^{r_2}(1-z_i)\ln\left(\frac{w_{2i}}{\alpha_2}\right)-\left(\frac{\sum_{i=1}^{r_2}(1-z_i)w_{2i}}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{\sum_{i=1}^{r_2}(1-z_i)w_{2i}}{\alpha_2}\right)-(n-n_r)\left(\frac{T}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{T}{\alpha_2}\right)=0$$

(8)

$$\frac{\partial \ln L}{\partial \alpha_1}=\frac{-2}{\alpha_1}-\frac{\sum_{i=1}^{r_1} z_i(\beta_1-1)}{\alpha_1}+\beta_1(\alpha_1)^{-\beta_1-1}\left(\sum_{i=1}^{r_1} z_i w_{1i}\right)^{\beta_1}-\frac{\sum_{i=1}^{r_2}(1-z_i)(\beta_1-1)}{\alpha_1}+$$
$$\beta_1(\alpha_1)^{-\beta_1-1}\left(\sum_{i=1}^{r_2}(1-z_i)w_{1i}\right)^{\beta_1}+(m-m_r)\beta_1(\alpha_1)^{-\beta_1-1}(T)^{\beta_1}+(n-n_r)\beta_1(\alpha_1)^{-\beta_1-1}(T)^{\beta_1}=0$$

(9)

$$\frac{\partial \ln L}{\partial \alpha_2}=\frac{-2}{\alpha_2}-\frac{\sum_{i=1}^{r_1} z_i(\beta_2-1)}{\alpha_2}+\beta_2(\alpha_2)^{-\beta_2-1}\left(\sum_{i=1}^{r_1} z_i w_{2i}\right)^{\beta_2}-\frac{\sum_{i=1}^{r_2}(1-z_i)(\beta_2-1)}{\alpha_2}+$$
$$\beta_2(\alpha_2)^{-\beta_2-1}\left(\sum_{i=1}^{r_2}(1-z_i)w_{2i}\right)^{\beta_2}+(m-m_r)\beta_2(\alpha_2)^{-\beta_2-1}(T)^{\beta_2}+(n-n_r)\beta_2(\alpha_2)^{-\beta_2-1}(T)^{\beta_2}=0$$

(10)

These equations do not yield any explicit solutions for MLE of the parameters ($\beta_1, \beta_2, \alpha_1, \alpha_2, p_1$and φ_1). Therefore, these are to be solved numerically using R software

Special cases of the mixed Weibull distribution when the shape parameter $\beta_j = 1$ the likelihood function of the mixture joint Weibull Type-I censored in equation (4) became the mixture joint exponential Type-I censored with scal parameter α_j.

$$\begin{aligned} L(\vartheta|\mathbf{w}) \propto & \left(p_1(1-p_1)\right)^{r_1} \prod_{i=1}^{r_1}\left[f_1(w_{1i};\alpha_1)f_2(w_{2i};\alpha_2)\right]^{z_i} \\ & \times\left(\varphi_1(1-\varphi_1)\right)^{r_2} \prod_{i=1}^{r_2}\left[g_1(w_{1i};\alpha_1)g_2(w_{2i};\alpha_2)\right]^{(1-z_i)} \\ & \times\left[p_1 \exp-\left(T/\alpha_1\right)+(1-p_1)\exp-\left(T/\alpha_2\right)\right]^{m-m_r} \\ & \times\left[\varphi_1 \exp-\left(T/\alpha_1\right)+(1-\varphi_1)\exp-\left(T/\alpha_2\right)\right]^{n-n_r}, \end{aligned}$$

(11)

where $\vartheta=\left(\alpha_j, p_j, \varphi_j\right)^{\mathrm{T}},\ j=1,2$ is parameter vector,

$$f_j(w_{ji};\alpha_j)=\frac{1}{\alpha_j}\exp-\left(\frac{w_{ji}}{\alpha_j}\right),\ i=1,2$$

$$g_j(w_{ji};\alpha_j)=\frac{1}{\alpha_j}\exp-\left(\frac{w_{ji}}{\alpha_j}\right),\ i=1,2$$

If we set $p_1, \varphi_1 = 1$ or $p_2,\ \varphi_2 = 1$, The mixture joint exponential distribution Type-I censored in equation (11) reduces to joint exponential distribution Type-I censored.

Another special case when the shape parameter $\beta_j = 2$ the likelihood function in equation (4) became the mixture joint Rayleigh Type-I censored.

4. Data Applications

This section illustrates the usefulness of the proposed estimators of the unknown parameters α_j , β_j , p_j and φ_j for $j=1,2$ of the mixed Weibull distributions obtained in previous sections by analyzing two data sets; one based on simulated data and the other based on real-life data, in order to show the applicability of the proposed model in practice scenario.

4.1 Application (1): Simulated Data

Using one simulated data generated from the mixed Weibull distributions, the MLEs of the unknown parameters α_j , β_j , p_j and φ_j for $j=1,2$ are calculated in presence of JT-I CS data set. To conduct the experiment based on a JT-I CS from two mixed Weibull populations, we propose the subsequent procedure:

Step-1: Set the parameter values of α_j , β_j , p_j and φ_j for $j=1,2$.

Step-2: Set the values of m , n , and T

Step-3: Generate X and Y independent observations of sizes m and n from MWD(α_j,β_j,p_j) and MWD($\alpha_j,\beta_j,\varphi_j$) for $j=1,2$, respectively.

Step-4: Merge the two created samples of sizes m and n , then arrange them in ascending order.

Step-5: Generate an ordinary JT-I CS for particular values of m , n and T .

Now, when the true value of α and β are taken as $\alpha_j = \beta_j = 1,\ j = 1,2$ and $p = \varphi = 0.5$, two complete samples from the mixed Weibull distributions when $T = (0.5, 1.0)$ and $m = n = 20$ (say) are generated and reported in Table 1. All required computations have performed using R statistical programming software version 4.0.4.

Table 1. One ordered complete samples from mixed Weibull distributions.

0.0571, 0.1059, 0.1215, 0.1326, 0.1541, 0.1765, 0.2435, 0.2437, 0.2556, 0.2932, 0.3147, 0.3355, 0.3634, 0.3801, 0.5101, 0.5274, 0.5498, 0.6345, 0.6740, 0.7023, 0.7518, 0.8291, 0.8340, 0.8732, 0.8992, 0.9223, 1.2215, 1.2924, 1.3144, 1.3875, 1.4480, 1.4685, 1.6730, 2.1415, 2.2418, 2.2943, 2.3123, 3.0851, 3.7616, 4.4836

Table 2. Two different JTI samples from mixed Weibull distributions.

Sample	T	(m_x^*, n_y^*)	$w(z)$
I	0.5	(5,9)	0.0571(0), 0.1059(1), 0.1215(0), 0.1326(0), 0.1541(1), 0.1765(1), 0.2435(0), 0.2437(0), 0.2556(0), 0.2932(0), 0.3147(1), 0.3355(1), 0.3634(0), 0.3801(0)
II	1.0	(15,11)	0.0571(0), 0.1059(1), 0.1215(0), 0.1326(0), 0.1541(1), 0.1765(1), 0.2435(0), 0.2437(0), 0.2556(0), 0.2932(0), 0.3147(1), 0.3355(1), 0.3634(0), 0.3801(0), 0.5101(1), 0.5274(1), 0.5498(0), 0.6345(1), 0.6740(1), 0.7023(1), 0.7518(1), 0.8291(0), 0.8340(1), 0.8732(1), 0.8992(1), 0.9223(1)

Using the samples I and II reported in Table 2, the observed variance-covariance matrix of the MLEs $\hat{\alpha}_j$, $\hat{\beta}_j$, $\hat{p}_j$ and $\hat{\varphi}_j$ for $j = 1,2$ of α_j, β_j, p_j and φ_j for $j = 1,2$, respectively, become

$$\mathbf{I}_1^{-1}\left(\hat{\vartheta}\right)=\begin{bmatrix} 0.00139 & 0.00053 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00053 & 0.23580 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.12290 & -0.06246 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & -0.06246 & 0.07268 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00665 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00761 \end{bmatrix}_{\left(\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2,\hat{p}_1,\hat{\varphi}_1\right)}$$

and

$$\mathbf{I}_2^{-1}\left(\hat{\vartheta}\right)=\begin{bmatrix} 0.06134 & -0.02398 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ -0.02398 & 0.05526 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.04565 & -0.05267 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & -0.05267 & 0.13423 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.01168 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.02635 \end{bmatrix}_{\left(\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2,\hat{p}_1,\hat{\varphi}_1\right)}$$

Similarly, using both samples I and II reported in Table 2, the MLEs with their estimated standard errors (ESEs) and corresponding two-sided 95% ACIs of the unknown parameters of α_j , β_j , p_j and φ_j for $j=1,2$, are computed and reported in Table 3.

Table 3. The MLEs with their ESEs (in parenthesis) and 95% ACIs.

Sample	Parameter	MLE(ESE)	95% ACI
I	α_1	0.2370(0.0374)	(0.1637,0.3102)
	β_1	1.9556(0.4856)	(1.0038,2.9073)
	α_2	1.0554(0.3506)	(0.3683,1.7426)
	β_2	1.0745(0.2696)	(0.5462,1.6029)
	p_1	0.2032(0.0816)	(0.0434,0.3630)
	φ_1	0.3140(0.0872)	(0.1430,0.4849)
II	α_1	0.8091(0.2477)	(0.3237,1.2946)
	β_1	1.2377(0.2351)	(0.7770,1.6984)
	α_2	0.5851(0.2137)	(0.1663,1.0038)

	β_2	1.4232(0.3664)	(0.7051,2.1413)
	p_1	0.5323(0.1081)	(0.3205,0.7441)
	φ_1	0.5752(0.1623)	(0.2571,0.8934)

4.2 Application (2): Real Data

The actual data sets were obtained from Xia et al. (2009), which depict the breaking strength of jute fiber with gauge lengths of 10 mm and 20 mm. Each fiber group of 30 analogous fibers. The data are provided for convenient reference.

Data set 1 Gauge length 10 mm. and data set 2 (Gauge length 20 mm.) as follow: 43.93 (36.75), 50.16 (45.58), 101.15 (48.01), 108.94 (71.46), 123.06 (83.55), 141.38 (99.72), 151.48 (113.85), 163.40 (116.99), 177.25 (119.86), 183.16 (145.96), 212.13 (166.49), 257.44 (187.13), 262.90 (187.85), 291.27 (200.16), 303.90 (244.53), 323.83 (284.64), 353.24 (350.70), 376.42 (375.81), 383.43 (419.02), 422.11 (456.60), 506.60 (547.44), 530.55 (578.62), 590.48 (581.60), 637.66 (585.57), 671.49 (594.29), 693.73 (662.66), 700.74 (688.16), 704.66 (707.36), 727.23 (756.70), 778.17 (765.14).

We divide both data sets by 1000, which will not impact the inference procedure. We fit a two-parameter generalized exponential distribution to each data set. Table 4 presents the maximum likelihood estimates (MLEs) and the Kolmogorov-Smirnov (K-S) distance between empirical distribution functions and fitted distributions, together with the related p-values for both datasets.

Table 4. MLEs and (K-S) distance.

Data set	Estimates	SE	KS	P-value
X1	345.3351	44.1557	0.1592	0.3912
	1.4574	0.2277		
X2	371.9737	52.5878	0.1492	0.4720
	1.3607	0.2030		
X	390.8096	35.8528	0.1162	0.3643
	1.4822	0.1561		

We note that distance (D) value of k-s test (0.1592, 0.1492, and 0.1162) is less than the p-value (0.3912, 0.4720, and 0.3643) Therefore, the null hypothesis of nothingness is not rejected, that is the proposed model is proportional to the data. Where (X1 = Data set 1, X2 = Data set 2 and X = the mixed data).

Table 5 shows four other measures, AIC-CAIC-BIC-QHIC from which we derive that the Data set 2 gets the lowest four values (410.1007, 410.5452, 412.9031 and 410.9972) respectively. Which indicates that the data of this group are the best and more fit for distribution.

Table 5. AIC-CAIC-BIC and QHIC

Data set	AIC	CAIC	BIC	QHIC
X1	411.2115	411.6560	414.0139	412.1080
X2	410.1007	410.5452	412.9031	410.9972
X	816.2016	816.4121	820.3902	817.8400

5. Conclusion

This study presents the maximum likelihood estimates for the unknown parameters of mixed Weibull distributions under a joint type-I censoring scheme, together with their approximate variance-covariance matrix and asymmetric confidence intervals. We conduct simulations to evaluate the performance of the MLEs and demonstrate the applicability of the proposed model in a practical scenario using a re-analyzed real data set. All proposed estimations closely approximate the appropriate parameter value. Furthermore, the corresponding standard errors of the proposed estimates are nearly nil, and the D statistic of the Kolmogorov-Smirnov test is less than the p-value. Consequently, the null hypothesis of no effect is not rejected, indicating that the proposed model is proportionate to the data. Consequently, the data adheres to a normal distribution, as illustrated in the figure.

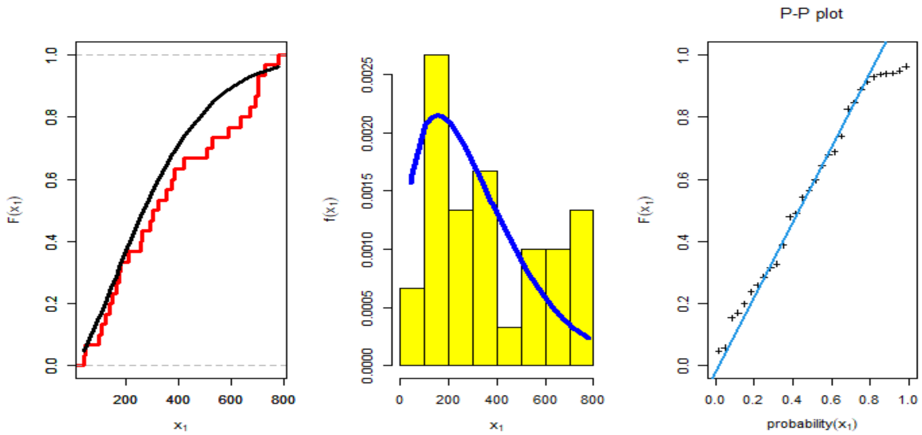

Figure 1: Empirical distribution and cdf – Estimated pdf with histogram – p-p plot for x_1

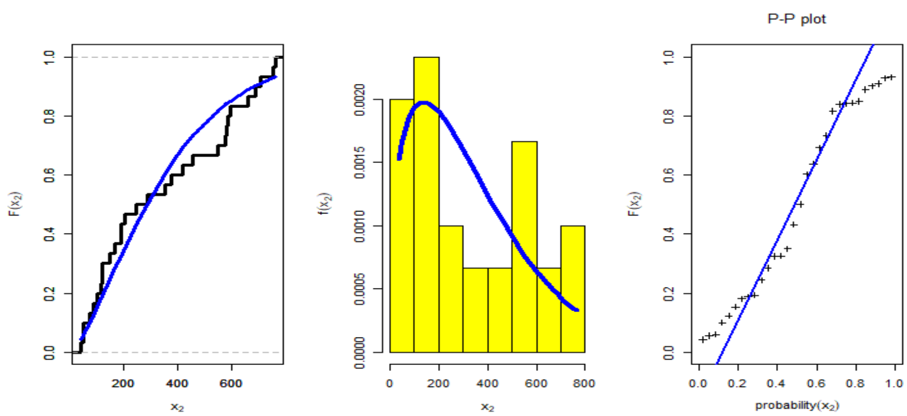

Figure 2: Empirical distribution and cdf – Estimated pdf with histogram – p-p plot for x_2

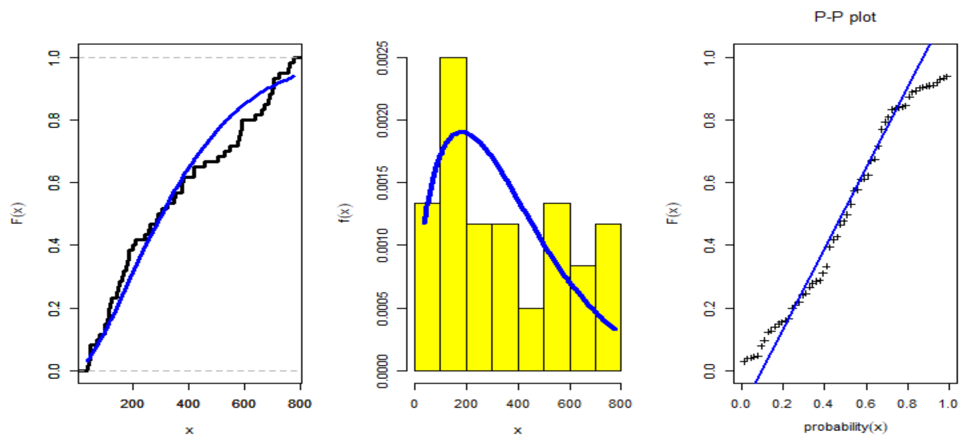

Figure 3: Empirical distribution and cdf – Estimated pdf with histogram – p-p plot for X

There is no difference between the data distribution and the normalized distribution

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