Computational Journal of Mathematical and Statistical Sciences
4(2), 577–600
DOI:10.21608/cjmss.2025.395958.1208
https://cjmss.journals.ekb.eg/

Research article

A New Probability Chen Model: Properties, Risk Analysis and Distributions Validation for Testing using the Right Censored Real Data

Mohamed Ibrahim[1,*], **Abdullah H. Al-Nefaie**[1], **Ahmad M. AboAlkhair**[1], **M. Masoom Ali**[2], **Khaoula Aidi**[3], **and Haitham M. Yousof**[4]

[1] Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia; miahmed@kfu.edu.sa, aalnefaie@kfu.edu.sa, aaboalkhair@kfu.edu.sa

[2] Department of Mathematical Sciences, Ball State University, Muncie, IN, USA; mali@bsu.edu.

[3] Laboratory of probability and statistics LaPS, University Badji Mokhtar, Annaba, Algeria; khaoula.aidi@ensmm-annaba.dz.

[4] Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt; haitham.yousof@fcom.bu.edu.eg.

* **Correspondence:** miahmed@kfu.edu.

Abstract: This paper introduces and explores a new flexible probability distribution called the Burr-X generalized Chen (BXGZC) model, with a focus on its properties, applications in actuarial risk analysis, and validation using real right-censored data. The proposed model builds upon the Chen distribution, offering enhanced adaptability for modeling both positively and negatively skewed datasets commonly encountered in insurance and financial risk assessment. We examine several key risk indicators, such as Value-at-Risk (VaR), Tail-Value-at-Risk (TVaR), tail variance, tail mean-variance, and the mean excess loss function, and apply them under different estimation techniques including maximum likelihood, ordinary least squares, weighted least squares, and Cramervon Mises methods. These approaches are tested through simulation studies involving various sample sizes to evaluate their performance in capturing risk measures accurately. Additionally, we apply the BXGZC model to real-life insurance claims data to assess its practical utility in actuarial evaluation. To further validate the modelâ€™s fit, especially in the context of censored data, we employ a modified version of the Nikulin-Rao-Robson goodness-of-fit test. This test is particularly useful when dealing with survival or reliability data where censoring is present. The results demonstrate that the BXGZC model outperforms the standard Chen distribution in fitting a wide range of right-censored datasets across different domains such as medical research, engineering reliability, and insurance.

Keywords: Actuarial Risk; Censored Reliability Data; Value-at-Risk; Right-censored Data; Validation.
Mathematics Subject Classification: 60E05, 62E10, 62F162H05; 62N01; 65C20; 62N02; 62E10; 60E050.
Received: 19 June 2025; Revised: 11 August 2025; Accepted: 13 August 2025; Online: 19 August 2025.

1. Introduction

Every property/casualty claim procedure uses two independent random variables (RVs): the claim-size RV and the claim-count RV. The first two basic claim RVs can be combined to produce the aggregate-loss RV, which represents the total claim amount generated by the underlying claim procedure. This work presents the model called the BXGZC distribution for risk analysis and right-censored validity (see Lane [19] and Klugman et al. [18]). As part of a review of the business's risk exposure, risks are usually ranked according to their likelihood of occurring in the future multiplied by the potential loss if they did. The firm can differentiate between little and large losses by ranking the likelihood of likely losses in the future. Speculative risks frequently result in losses such as failures to comply with regulations, a decline in brand value, security flaws, and liability issues. Distributions based on probabilities can then provide an accurate depiction of the risk exposure and recently used for this actuarial purpose (see Shrahili et al. [31] and Mohamed [22]). The levels of exposure are functions frequently referred to as major risk indicators (see Klugman et al. [18]). Such main risk indicators provide risk managers and actuaries with information on the level of risk that the firm is exposed to. There are variety of RIs that can be taken into consideration and researched, including tailed-value-at-risk (TVAR) (also known as the conditional tail expectation (CTE)). We also study how the mean excess loss (MEL) function may be used to reduce actuarial and economic risks, (see Wirch [35], Tasche [33], Furman and Landsman [11] for the value-at-risk (VAR), conditional-VAR (CVAR), tail-Variance (TV)). We provide a simulation study to compare the effectiveness of the main risk indicators based on insurance data in order to satisfy the requirements of the actuarial analysis of risks. For risk analysis purpose, we analyse and model a new set of negatively skewed insurance claims data. Additionally, the risk exposure is an actuarial estimation of the potential loss that might develop in the future as a result of a specific action or occurrence.

In the framework of distributional validation and statistical hypothesis tests for the censored data, a modified Nikulin-Rao-Robson (NRR) statistic test (see Nikulin [23], Voinov et al. [34] and Rao and Robson [24] for more main details), which is based on the censored maximum likelihood estimators on initial non-grouped data, is considered under the BXGZC model. The modified NRR statistic is assessed under four right censored data sets and some results are highlighted. At the beginning of this introduction, it is worth mentioning to provide some simple details about the genesis, importance and uses of the new probability model. Let z be a non-negative RV with a generalized Chen (GzC) distribution (see Chaubey and Zhang [7]), then its corresponding cumulative distribution function (CDF) is given by

$$G_{\sigma_2,\sigma_3}(z) = (1 - \exp\{[1 - \exp(z^{\sigma_3})]\})^{\sigma_2}, \tag{1.1}$$

where $z > 0, \sigma_2 > 0$ and $\sigma_3 > 0$. Chaubey and Zhang [7] present two propositions studying probability density function (PDF) and hazard rate function (HRF). The first proposition shows that the PDF shapes are either "decreasing" or "unimodal". The second proposition concludes that the HRF shapes are either "increasing" or "bathtub". Chaubey and Zhang [7] also addressed the problem of estimation of parameters of the GzC distribution, focusing on the maximum likelihood estimation method. Due to Dey et al. [10], the shape of the PDF of the GzC distribution may be characterized as follows: for $\sigma_2 < 1, \sigma_3 < 1$, $g_{\sigma_2,\sigma_3}(z)$ is a decreasing density, for $\sigma_2 > 1, \sigma_3 > 1$, $g_{\sigma_2,\sigma_3}(z)$ is a unimodal density and for $\sigma_2 < 1, \sigma_3 > 1$ and $\sigma_2 > 1, \sigma_3 < 1$, $g_{\sigma_2,\sigma_3}(z)$ may be unimodal or decreasing density. Chaubey and Zhang [7] presented a proof that the failure behavior of the GzC distribution are, respectively, bathtub ($\sigma_2 < 1, \sigma_3 < 1$), increasing ($\sigma_2 > 1, \sigma_3 > 1$), increasing or bathtub ($\sigma_2 < 1, \sigma_3 > 1$ and $\sigma_2 > 1, \sigma_3 < 1$). For $\sigma_2 = 1$, the GzC distribution reduces to Chen (C) distribution (see Chen (2000)) with $G_{\sigma_3}(z) = 1 - \nabla_{\sigma_3}(z)$ where

$$\nabla_{\sigma_3}(z) = \exp\{[1 - \exp(z^{\sigma_3})]\}.$$

Dey et al. [10] addressed various mathematical properties and estimation methods for the GzC distribution. They described different estimation methods such as the method of maximum likelihood estimation (MLE), percentile estimation (PE), ordinary least square (OLSE) weighted least square estimation (WLSE), maximum product of spacings estimation (MPSE), Cramér-von Mises estimation (CVME). In this work, we shall use the Burr X (BX-G) family to derive a new version of the the BXGZC distribution. The CDF of the BX-G is defined as

$$F_{\sigma_1,\underline{\xi}}(z) = \left\{1 - \exp\left[-O^2_{\underline{\xi}}(z)\right]\right\}^{\sigma_1}, \tag{1.2}$$

where

$$O^2_{\underline{\xi}}(z) = \left(\frac{G_{\underline{\xi}}(z)}{\overline{G}_{\underline{\xi}}(z)}\right)^2.$$

Inserting (1.1) into (1.2), the CDF of the BXGZC distribution can be expressed as

$$F_{\underline{V}}(z) = \left\{1 - \exp\left[-O^{-2}_{\sigma_2,\sigma_3}(z)\right]\right\}^{\sigma_1}, \tag{1.3}$$

where

$$O^{-2}_{\sigma_2,\sigma_3}(z) = \left[(1 - \nabla_{\sigma_3}(z))^{-\sigma_2} - 1\right]^{-2}.$$

The corresponding CDF of the BXGZC can be derived as

$$f_{\underline{V}}(z) = 2\sigma_1\sigma_2\sigma_3 \frac{z^{\sigma_3-1}\exp(z^{\sigma_3})\nabla_{\sigma_3}(z)\exp\left[-O^{-2}_{\sigma_2,\sigma_3}(z)\right][1 - \nabla_{\sigma_3}(z)]^{2\sigma_2-1}}{\{1 - [1 - \nabla_{\sigma_3}(z)]^{\sigma_2}\}^3\left\{1 - \exp\left[-O^{-2}_{\sigma_2,\sigma_3}(z)\right]\right\}^{1-\sigma_1}}, \tag{1.4}$$

where $\underline{V} = (\sigma_1, \sigma_2, \sigma_3)$. Consider the power series

$$\left(1 - \frac{\zeta_1}{\zeta_2}\right)^{\zeta_3} = \sum_{l_1=0}^{+\infty} \frac{(-1)^{l_1}\,\Gamma(1+\zeta_3)}{l_1!\,\Gamma(1+\zeta_3-l_1)}\left(\frac{\zeta_1}{\zeta_2}\right)^{l_1} \Big|_{\left|\frac{\zeta_1}{\zeta_2}\right|<1,\ \zeta_3>0}. \tag{1.5}$$

Applying (1.5) to (1.4) we have

$$f_{\underline{V}}(z) = \exp(z^{\sigma_3})\nabla_{[1]}\sum_{l_1=0}^{+\infty}\frac{(-1)^{l_1}\,\Gamma(\sigma_1)}{l_1!\,\Gamma(\sigma_1-l_1)\exp\left[(l_1+1)\,O^{-2}_{\sigma_2,\sigma_3}(z)\right]}, \tag{1.6}$$

where

$$\nabla_{[1]} = \frac{2\sigma_1\sigma_2\sigma_3 z^{\sigma_3-1}\nabla_{\sigma_3}(z)(1 - \nabla_{\sigma_3}(z))^{\sigma_2}}{\{1 - [1 - \nabla_{\sigma_3}(z)]^{\sigma_2}\}^3(1 - \nabla_{\sigma_3}(z))^{1-\sigma_2}}.$$

Applying the power series to the term $\exp\left\{-(l_1+1)\,O^{-2}_{\sigma_2,\sigma_3}(z)\right\}$, equation (1.6) becomes

$$f_{\underline{V}}(z) = \exp\left(z^{\sigma_3}\right)\nabla_{[2]}\sum_{l_1,l_2=0}^{+\infty}\frac{(-1)^{l_1+l_2}\,(l_1+1)^{l_2}\,\Gamma(\sigma_1)}{l_1!\,l_2!\Gamma(\sigma_1-l_1)}\,\frac{\left[1-\nabla_{\sigma_3}(z)\right]^{(2l_2+1)\sigma_2}}{\left\{1-\left[1-\nabla_{\sigma_3}(z)\right]^{\sigma_2}\right\}^{2l_2+3}}, \tag{1.7}$$

where

$$\nabla_{[2]} = \frac{2\sigma_1\sigma_2\sigma_3 z^{\sigma_3-1}\nabla_{\sigma_3}(z)}{\left(1-\nabla_{\sigma_3}(z)\right)^{1-\sigma_2}}$$

Consider the series expansion

$$\left(1-\frac{\zeta_1}{\zeta_2}\right)^{-\zeta_3} = \sum_{l_3=0}^{+\infty}\frac{\Gamma(\zeta_3+l_3)}{l_3!\Gamma(\zeta_3)}\left(\frac{\zeta_1}{\zeta_2}\right)^{l_3}|_{\left|\frac{\zeta_1}{\zeta_2}\right|<1,\ \zeta_3>0}. \tag{1.8}$$

Applying the expansion in (1.8) to (1.7) for the term $\left[1-\left(1-\nabla_{\sigma_3}(z)\right)^{\sigma_2}\right]^{2l_2+3}$, equation (1.7) becomes

$$f_{\underline{V}}(z) = \sum_{l_2,l_3=0}^{+\infty}\varsigma_{l_2,l_3}\,\pi_\Omega(z)|\left(\Omega=(2l_2+1)\,\sigma_2+l_3+1\right), \tag{1.9}$$

where

$$\varsigma_{l_2,l_3} = \frac{2\sigma_1\,(-1)^{l_2}\,\Gamma(\sigma_1)\,\Gamma(2l_2+l_3+3)}{l_2!l_3!\Gamma(2l_2+3)\,\sigma_2^{\cdot}}\sum_{l_1=0}^{+\infty}\frac{(-1)^{l_1}\,(l_1+1)^{l_2}}{l_1!\,\Gamma(\sigma_1-l_1)},$$

and $\pi_\Omega(z) = \Omega g_{\sigma_3}(z)\left[G_{\sigma_3}(z)\right]^{\Omega-1}$. Equation (1.9) reveals that the density of Z can be expressed as a linear mixture of GzC densities. So, several mathematical properties of the new family can be obtained by knowing those of the GzC distribution. Similarly, the CDF of the BXGZC model can also be expressed as a mixture of GzC CDFs given by

$$F_{\underline{V}}(z) = \sum_{l_2,l_3=0}^{+\infty}\varsigma_{l_2,l_3}\,\Pi_\Omega(z)\,|\left(\Omega=(2l_2+1)\,\sigma_2+l_3+1\right) \tag{1.10}$$

where $\Pi_\Omega(z) = \left[G_{\sigma_3}(z)\right]^\Omega$ is the CDF of the GzC family with power parameter Ω.

The new NRR statistical test showed that using the new model as a stand-in for looking at two right-censored data sets is successful. In this context, we will discuss a few recent research findings that added to or changed the NRR. It is important to note that the browser for statistical literature on this topic (NRR goodness-of-fit test) will not find many new NRR goodness-of-fit extensions but only a few research that applied this test because the NRR goodness-of-fit test has specific requirements, strict procedures, and demands censored data. It is a well-known fact that it is challenging to collect new censored data to apply to and stress the importance of the new test. In the next few paragraphs, we will discuss a few recent research results that use this test on actual data that had been subject to right-wing censoring, along with a description of the findings from each study independently.

The rest of this work is organized as follows: Sections 2 presentes some mathematical and statistical properties and two related theorems. The main risk indicators under the BXGZC model are given in Sections 3. Risk analysis under artificial and real data is presented in Sections 4. Sections 5 introduces the right censored distributional validity. Sections 6 offers some concluding remarks.

2. Mathematical and statistical properties

Following Dey et al. [10], we can extract the following two theorems:

Theorem 1: Let z be a RV having the GzC distribution. Then using the transformation $t = [G_{\sigma_2,\sigma_3}(z)]^{\frac{1}{\sigma_2}}$, the r^{th} ordinary moment of Z is given by

$$\mu'_r = \mathbb{E}[Z^r] = \sigma_2\sigma_3 \sum_{\rho,\tau=0}^{+\infty} \sigma_2\left(\rho; \frac{r}{\sigma_3}\right)\sigma_2\left(\tau; \frac{r}{\sigma_3}+\rho\right)\frac{(-1)^{\frac{2r}{\sigma_3}+\rho}}{[\sigma_3(\sigma_2+\rho+\tau)+r]},$$

where $\sigma_2\left(\rho; \frac{r}{\sigma_3}\right)$ is the coefficient of $[\log(1-t)]^{\frac{2r}{\sigma_3}+\rho}$ in the expansion of

$$\left\{\sum_{j_1=1}^{+\infty}\frac{1}{j_1}[\log(1-t)]\right\}^{\frac{r}{\sigma_3}}$$

and $\sigma_2\left(\tau; \frac{r}{\sigma_3}+\rho\right)$ is the coefficient of $t^{\rho+\tau+\frac{r}{\sigma_3}}$ in the expansion of

$$\left(\sum_{j_2=1}^{+\infty}\frac{t^{j_2}}{j_2}\right)^{\frac{r}{\sigma_3}+\rho}$$

(see Ibrahim et al. [14] and Dey et al. [10] for more details).

Theorem 2: Let z be a RV having the GzC distribution. Then, the r^{th} conditional moment can be derived as

$$\begin{aligned}\mathbb{E}(Z^r) &= \sigma_2\sigma_3\sum_{\rho,\tau=0}^{+\infty}\sigma_2\left(\rho;\frac{r}{\sigma_3}\right)\sigma_2\left(\tau;\frac{r}{\sigma_3}+\rho\right)\\ &\quad\times\frac{(-1)^{\frac{2r}{\sigma_3}+\rho}(\nabla_{\sigma_3}(z))}{[\sigma_3(\sigma_2+\rho+\tau)+r]\{1-[1-\nabla_{\sigma_3}(z)]^{\sigma_2}\}}\end{aligned}$$

Based on Theorem 1, the r^{th} ordinary moment of the BXGZC distribution can then be expressed as

$$\mu'_{r,z} = \mathbb{E}[Z^r] = \Omega\sigma_3\sum_{\rho,\tau=0}^{\infty}\varsigma_{l_2,l_3}\,\Omega_\rho\left(\frac{r}{\sigma_3}\right)\Omega\left(\tau;\frac{r}{\sigma_3}+\rho\right)\frac{(-1)^{\frac{2r}{\sigma_3}+\rho}}{[\sigma_3(\Omega+\rho+\tau)+r]}. \tag{2.1}$$

The variance (V(z)), cumulants, n^{th} central moment, skewness (S(z)), kurtosis (K(z)) and Index of dispersion od the variance to mean ratio (ID(z)) measures can be calculated from the ordinary moments using well-known relationships. For the increasing failure rate models, it is also of interest to know what $\mathbb{E}(Z^r|z>z)$ is. It can be easily seen that

$$\mathbb{E}(Z^r|z>z) = \Omega\sigma_3\sum_{l_2,l_3,\rho,\tau=0}^{+\infty}\frac{\varsigma_{l_2,l_3}\,\Omega\left(\rho;\frac{r}{\sigma_3}\right)\Omega\left(\tau;\frac{r}{\sigma_3}+\rho\right)(-1)^{\frac{2r}{\sigma_3}+\rho}(\nabla_{\sigma_3}(z))}{[\sigma_3(\Omega+\rho+\tau)+r]\left[1-(1-\nabla_{\sigma_3}(z))^{\Omega}\right]}. \tag{2.2}$$

The mean residual life (MRL) is the expected remaining life, $z-z$, given that the item has survived to time z. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the MRL. Since the MRL function is the expected remaining life, z must be subtracted, yielding

$$M_{1,z} = \mathbb{E}\left(z - z|z > z\right) = \frac{1}{S_{\underline{V}}(z)}\left[int_{z}^{+\infty} z f_{\underline{V}}(z)\, dz\right] - z,$$

where $S_{\underline{V}}(z) = 1 - F_{\underline{V}}(z)$. Then using (2.2), we get

$$M_{1,z} = \Omega\sigma_3 \sum_{l_2,l_3,\rho,\tau=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\Omega\left(\rho; \frac{1}{\sigma_3}\right)\Omega\left(\tau; \frac{1}{\sigma_3} + \rho\right)(-1)^{\frac{2}{\sigma_3}+\rho}\left(\nabla_{\sigma_3}(z)\right)}{\left[\sigma_3\left(\sigma_2^{\cdot} + \rho + \tau\right) + 1\right]\left[1 - \left(1 - \nabla_{\sigma_3}(z)\right)^{\Omega}\right]} - z.$$

In a real life situation, where systems often are not monitored continuously, one might be interested in getting inference more about the history of the system, for example, when the individual components have failed.

3. Main risk indicators under the BXGZC model

The characterization of risk exposure that the probability-based distributions may offer is sufficient. One value, or at the very least a limited group of numbers, is frequently used to indicate the amount of risk exposure. These risk exposure statistics are obviously functions of a certain model and are frequently referred to as important main risk indicators. Such main risk indicators provide actuaries and risk managers with information on the degree to which a firm is exposed to specific types of risk. Numerous main risk indicators, including the VAR, the TVAR which also known as CVAR, the TV indicator, the Tail Mean–Variance (TMV) and the MELq function, among others, can be taken into account and examined. The VaR is a quantile of the distribution of aggregate losses in particular. Actuaries and risk managers frequently focus on estimating the likelihood of a negative result, which may be conveyed using the VaR indicator at a certain probability/confidence level. This indicator is frequently used to calculate the amount of capital needed to deal with such probable negative situations. The VAR of the BXGZC distribution at the $100q\%$ level, say VAR(z) or $\pi(q)$, is the $100q\%$ quantile (or percentile). Then, we can simply write

$$\mathrm{VAR}(z) = \Pr\left(X > Q(U)\right) = \begin{cases} 1\%|_{q=99\%} \\ 5\%|_{q=95\%} \\ \vdots \end{cases}, \tag{3.1}$$

where $Q(U) = F_{\underline{V}}^{-1}(z)$, for a one-year time when $q = 99\%$, the interpretation is that there is only a very small chance (1%) that the insurance company will be bankrupted by an adverse outcome over the next year. Generally speaking, if the distribution of gains (or losses) is limited to the normal distribution, it is acknowledged that the number VAR(z) meets all coherence requirements. The data sets for insurance such as the insurance claims and reinsurance revenues are typically skewed whether to the right or to the left , though. Using the normal distribution to describe the revenues from reinsurance and insurance claims is not suitable. The TVAR of Z at the $100q\%$ confidence level is the expected loss given that the loss exceeds the $100q\%$ of the distribution of Z, then the TVAR of Z can be expressed as

$$\mathrm{TVAR}(z) = \mathbb{E}\left(z|z > \pi(q)\right) = \frac{1}{1 - F_{\underline{V}}(\pi(q))}\int_{\pi(q)}^{\infty} z\, f_{\underline{V}}(z)\, dz = \frac{1}{1-q}\int_{\pi(q)}^{\infty} z\, f_{\underline{V}}(z)\, dz,$$

Then

$$\text{TVAR}(z) = \frac{\Omega\sigma_3}{1-q} \sum_{l_2,l_3,\rho,\tau=0}^{+\infty} C_1(\rho,\tau) \frac{\varsigma_{l_2,l_3}(-1)^{\frac{2}{\sigma_3}+\rho}(\nabla_{\sigma_3}(q))}{[\sigma_3(\Omega+\rho+\tau)+1]\left[1-(1-\nabla_{\sigma_3}(q))^{\Omega}\right]}, \tag{3.2}$$

where

$$C_1(\rho,\tau) = \Omega\left(\rho; \frac{1}{\sigma_3}\right)\Omega\left(\tau; \frac{1}{\sigma_3}+\rho\right).$$

The quantity TVAR(z), which gives further details about the tail of the BXGZC distribution, is therefore the average of all the VaR values mentioned above at the confidence level q. Moreover, the TVAR(z) can also be expressed as TVAR(z) = $e(z;q)$+VAR(z) , where $e(z;q)$ is the mean excess loss (MELq) function evaluated at the $100q\%^{th}$ quantile (see Acerbi and Tasche [2]; Tasche [33]; Wirch [35]. When the $e(z;q)$ value vanishes, then TVAR(z) =VAR(z) and for the very small values of $e(z;q)$, the value of TVAR(z) will be very close to VAR(z) .The TV risk indicator, which Furman and Landsman [11] developed, calculates the loss's deviation from the average along a tail. Explicit expressions for the TV risk indicator under the multivariate normal distribution were also developed by Furman and Landsman [11]. The TV risk indicator (TV(z)) can then be expressed as

$$\text{TV}(z) = \mathbb{E}\left(X^2|X>\pi(q)\right) - [\text{TVAR}(z)]^2, \tag{3.3}$$

where

$$\mathbb{E}\left(X^2|X>\pi(q)\right) = \Omega\sigma_3 \sum_{\rho,\tau=0}^{\infty} C_2(\rho,\tau) \frac{\varsigma_{l_2,l_3}(-1)^{\frac{4}{\sigma_3}+\rho}(\nabla_{\sigma_3}(q))}{\left[\sigma_3\left(\sigma_2^{\cdot}+\rho+\tau\right)+2\right]\left[1-(1-\nabla_{\sigma_3}(q))^{\Omega}\right]},$$

where

$$C_2(\rho,\tau) = \Omega_\rho\left(\frac{2}{\sigma_3}\right)\Omega_\tau\left(\frac{2}{\sigma_3}+\rho\right).$$

As a statistic for the best portfolio choice, Landsman [12] developed the TMV risk indicator, which is based on the TV risk indicator. Consequently, the TMV risk indicator may be written as

$$\text{TMV}(z) = \text{TVAR}(z) + \pi\text{TV}(z)|_{0<\pi<1}. \tag{3.4}$$

Then, for any continuous RV, TMV(z) >TV(z) and, for $\pi = 1$, TMV(z) =TVAR(z). In view of the theoretical complexities and the fact that the quantile function is not known in a certain closed form, we will use the methods that provide numerical solutions. To make numerical processes easier, pre-made programmes like "R" and "MATHCAD" will be used. Numerous factors have contributed to the recent rise in popularity of numerical methods. The presence of several mathematically sophisticated distributions and models, as well as the availability of ready-made statistical programmes, are the two most significant. The complexity of models is no longer the main issue facing researchers in the fields of statistical analysis and mathematical modelling, as statistical programmes and packages have significantly helped to simplify these complexities by offering numerical solutions. This is a fact that has come to be accepted and cannot be ignored. Numerical approaches were used in this paper's risk analysis and evaluation process, as well as in the issue of distributional validation under the NRR and its new matching version.

4. Risk analysis

4.1. *Artificial analysis using different methods*

In this section, we consider the following estimation methods: maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares estimation (WLSE) and Cramer-von Mises (CVM) for calculating the main risk indicators. These quantities are estimated using $N = 1,000$ with different sample sizes ($n = 20, 50, 100$) and three confidence levels (CLs) ($q = (50\%, 60\%, 70\%, 80\%, 90\%, 99\%)$). All results are reported in Table 1 (n=50), Table 2 (n=50), Table 3 (n=100), from which we conclude: VAR(z), TVAR(z) and TMV(z) increase when q increases for all estimation methods.

1-$\text{VAR}(z)_{\text{WLS}} < \text{VAR}(z)_{\text{CVM}} < \text{VAR}(z)_{\text{MLE}} < \text{VAR}(z)_{\text{OLSE}}$ for most q.

2-$\text{TVAR}(z)_{\text{WLS}} < \text{TVAR}(z)_{\text{CVM}} < \text{TVAR}(z)_{\text{MLE}} < \text{TVAR}(z)_{\text{OLSE}}$ for most q.

4.2. *Real data analysis under insurance claims*

The historical growth of claims through time for each appropriate exposure (or origin) period is frequently shown in the historical insurance actual data in the form of a triangle presentation. The year the insurance policy was purchased or the time period during which the loss occurred may be regarded as the exposure period. It is obvious that the genesis period need not be annual. For instance, it may be monthly or quarterly origin periods. The development time of an origin period is known as the ”claim age” or ”claim lag.” Data from separate insurance is frequently combined to represent uniform company lines, division levels, or risks. We examine the insurance claims payment triangle from a U.K. Motor Non-Comprehensive account in this article as a practical illustration. We choose a convenient origin period of 2007 to 2013. The insurance claims payment data frame displays the claims data in the manner in which a database would normally keep it. The origin year, which ranges from 2007 to 2013, the development year, and the incremental payments are all included in the first column. It’s important to note that this data on insurance claims was initially examined using a probability-based distribution. The capability of the insurance firm to handle such occurrences is of importance to actuaries, regulators, investors, and rating agencies. This work proposes certain main risk indicators quantities for the left-skewed insurance claims data under the EEC distribution, including VAR, TVAR, TV, and TMV (see Artzner [6]). One of the finest techniques for heavy-tailed distributions is based on the t-Hill approach, an upper order statistic modification of the t-estimator.

Table 4 (first part) lists the main risk indicators under the insurance calims data and MLE method for the BXGZC model where $\underline{\widehat{V}} = (0.242, 598.318, 0.085)$. Table 4 (second part) gives the main risk indicators under the insurance calims data and OLSE method for the BXGZC model where $\underline{\widehat{V}} = (0.926, 48.459, 0.062)$. Table 4 (third part) shows the main risk indicators under the insurance calims data and WLSE method for the BXGZC model where $\underline{\widehat{V}} = (1.351, 34.820, 0.0603)$. Table 4 (fourth part) presents the main risk indicators under the insurance calims data and CVM method for the BXGZC model where $\underline{\widehat{V}} = (1.169, 39.769, 0.0619)$.

Table 1. main risk indicators under artificial data for n=20.

Method	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	VAR(z)	TVAR(z)	TV(z)	TMV(z)	MELq(z)
MLE	2.113	1.980	0.100					
50%				0.157092	0.4329251	0.0881785	0.4770143	0.2758331
60%				0.2131667	0.4951766	0.0907816	0.5405673	0.2820098
70%				0.2882234	0.57735	0.0938764	0.6242882	0.2891266
80%				0.3985333	0.6963983	0.0977948	0.7452957	0.297865
90%				0.5971916	0.9072721	0.1034496	0.9589969	0.3100805
95%				0.8060192	1.1256475	0.1079804	1.1796377	0.3196282
99%				1.3186969	1.6531813	0.115138	1.7107503	0.3344844
OLSE	2.085	1.988	0.100					
50%				0.1576542	0.4364144	0.0901364	0.4814826	0.2787602
60%				0.2142574	0.4993355	0.0928086	0.5457398	0.2850781
70%				0.2900813	0.5824122	0.0959789	0.6304016	0.2923309
80%				0.4015843	0.7027889	0.0999833	0.7527806	0.3012047
90%				0.6024707	0.9160314	0.1057452	0.968904	0.3135607
95%				0.8136649	1.1368474	0.1103476	1.1920212	0.3231824
99%				1.3320705	1.6701568	0.1175811	1.7289473	0.3380863
WLSE	2.068	1.996	0.099					
50%				0.1555917	0.4373376	0.0931964	0.4839357	0.2817459
60%				0.2123289	0.5009935	0.0961682	0.5490776	0.2886647
70%				0.2886102	0.5852012	0.0996997	0.635051	0.296591
80%				0.4011945	0.707475	0.1041735	0.7595618	0.3062805
90%				0.60486	0.9246501	0.1106495	0.9799748	0.3197901
95%				0.8197635	1.1501218	0.1158725	1.2080581	0.3303583
99%				1.3494693	1.6964142	0.1242583	1.7585433	0.3469449
CVM	2.103	1.982	0.101					
50%				0.1593845	0.4358835	0.0878817	0.4798244	0.276499
60%				0.2158841	0.4982483	0.090339	0.5434178	0.2823642
70%				0.2913541	0.5804704	0.0932527	0.6270967	0.2891163
80%				0.4020269	0.6994189	0.096927	0.7478824	0.297392
90%				0.600803	0.909719	0.1021931	0.9608156	0.3089161
95%				0.8092097	1.1270754	0.1063694	1.1802601	0.3178657
99%				1.3192285	1.6508218	0.1128276	1.7072356	0.3315933

Table 2. main risk indicators under artificial data for n=50.

Method	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	VAR(z)	TVAR(z)	TV(z)	TMV(z)	MELq(z)
MLE	2.038	1.990	0.100					
50%				0.1544573	0.4325662	0.0901393	0.4776359	0.2781089
60%				0.2106891	0.4953719	0.0928857	0.5418147	0.2846828
70%				0.2861912	0.578372	0.0961339	0.6264389	0.2921808
80%				0.3974413	0.6987406	0.100224	0.7488526	0.3012993
90%				0.5982302	0.9121481	0.106089	0.9651926	0.3139179
95%				0.8095751	1.1332698	0.1107599	1.1886498	0.3236948
99%				1.3287942	1.6675677	0.1180806	1.726608	0.3387735
OLSE	2.033	1.994	0.100					
50%				0.1549363	0.4346197	0.091244	0.4802417	0.2796834
60%				0.2114468	0.4977862	0.0940387	0.5448055	0.2863394
70%				0.2873497	0.581276	0.0973434	0.6299477	0.2939264
80%				0.399226	0.7023741	0.1015041	0.7531261	0.303148
90%				0.6012092	0.9171128	0.1074703	0.970848	0.3159036
95%				0.8138639	1.1396486	0.1122227	1.1957599	0.3257846
99%				1.3364259	1.6774539	0.1196773	1.7372925	0.341028
WLSE	2.021	1.995	0.100					
50%				0.1540865	0.433995	0.0916003	0.4797951	0.2799084
60%				0.2105417	0.4972254	0.0944438	0.5444473	0.2866837
70%				0.286437	0.5808329	0.0978049	0.6297354	0.2943959
80%				0.3983925	0.702151	0.1020356	0.7531688	0.3037585
90%				0.6006834	0.9173793	0.1081023	0.9714305	0.3166959
95%				0.8138002	1.1405135	0.1129372	1.1969821	0.3267133
99%				1.3378274	1.6800049	0.120536	1.7402729	0.3421775
CVM	2.040	1.991	0.101					
50%				0.1556107	0.4343478	0.0902951	0.4794954	0.2787371
60%				0.2120757	0.4972817	0.0929993	0.5437814	0.285206
70%				0.2878291	0.580415	0.0961958	0.6285129	0.2925859
80%				0.3993579	0.7009179	0.1002173	0.7510265	0.30156
90%				0.6004607	0.9144319	0.1059744	0.9674191	0.3139712
95%				0.8119545	1.1355286	0.1105474	1.1908023	0.3235741
99%				1.3310292	1.6693611	0.1176748	1.7281985	0.3383319

Table 3. main risk indicators under artificial data for n=100.

Method	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	VAR(z)	TVAR(z)	TV(z)	TMV(z)	MELq(z)
MLE	2.030	1.994	0.100					
50%				0.1546226	0.4341807	0.0912138	0.4797876	0.2795581
60%				0.2110828	0.4973221	0.0940168	0.5443305	0.2862393
70%				0.2869347	0.5807871	0.097331	0.6294526	0.2938524
80%				0.3987584	0.7018612	0.1015035	0.7526129	0.3031028
90%				0.6006873	0.9165818	0.1074863	0.970325	0.3158946
95%				0.8133187	1.139121	0.1122525	1.1952472	0.3258023
99%				1.3359033	1.6769922	0.1197318	1.7368581	0.3410889
OLSE	2.022	1.997	0.100					
50%				0.1546194	0.4353147	0.0920863	0.4813578	0.2806952
60%				0.211246	0.4987212	0.0949394	0.5461909	0.2874752
70%				0.2873642	0.5825572	0.0983117	0.6317131	0.2951931
80%				0.3996373	0.7042001	0.1025562	0.7554782	0.3045627
90%				0.6024798	0.9199893	0.1086418	0.9743101	0.3175095
95%				0.8161566	1.1436897	0.1134907	1.2004351	0.3275332
99%				1.3415038	1.6845062	0.1211078	1.74506	0.3430024
WLSE	2.020	1.996	0.100					
50%				0.1548396	0.4351772	0.0916922	0.4810233	0.2803376
60%				0.2114571	0.4984949	0.094503	0.5457464	0.2870378
70%				0.2875285	0.5821916	0.0978235	0.6311033	0.2946631
80%				0.3996791	0.7035954	0.1019992	0.754595	0.3039164
90%				0.6021836	0.9188755	0.1079779	0.9728644	0.3166919
95%				0.8153874	1.1419576	0.1127319	1.1983235	0.3265701
99%				1.3392256	1.6809976	0.1201685	1.7410818	0.341772
CVM	2.025	1.996	0.100					
50%				0.1549623	0.4353138	0.0917147	0.4811711	0.2803515
60%				0.2115836	0.4986344	0.0945293	0.545899	0.2870508
70%				0.2876555	0.5823352	0.0978557	0.6312631	0.2946798
80%				0.3998043	0.7037479	0.1020412	0.7547684	0.3039436
90%				0.6023112	0.9190551	0.1080383	0.9730742	0.3167439
95%				0.8155329	1.1421827	0.1128114	1.1985884	0.3266498
99%				1.339488	1.6814022	0.12029	1.7415472	0.3419142

Based on Table 4, the following results can be highlighted:

1. For all actuarial risk assessment approaches:

$$VAR(z|_{1-q=50\%}) < VAR(z|_{1-q=40\%}) < ... < VAR(z|_{1-q=10\%}) < VAR(z|_{1-q=1\%}).$$

2. For all actuarial risk assessment approaches:

$$TVAR(z|_{1-q=50\%}) < TVAR(z|_{1-q=40\%}) < ... < TVAR(z|_{1-q=10\%}) < TVAR(z|_{1-q=1\%}).$$

3. For all actuarial risk assessment approaches:

$$TV(z|_{1-q=50\%}) > TV(z|_{1-q=40\%}) > ... > TV(z|_{1-q=10\%}) > TV(z|_{1-q=1\%}).$$

4. For all actuarial risk assessment approaches:

$$TMV(z|_{1-q=50\%}) > TMV(z|_{1-q=40\%}) > ... > TMV(z|_{1-q=10\%}) > TMV(z|_{1-q=1\%}).$$

5. For all actuarial risk assessment approaches:

$$MELq(z|_{1-q=50\%}) > MELq(z|_{1-q=40\%}) > ... > MELq(z|_{1-q=10\%}) > MELq(z|_{1-q=1\%}).$$

6. Under the MLE technique: The VAR(z) is monotonically increasing starts with $2366.80069|_{1-q=50\%}$ and ends with $6501.06864|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 3848.80179 and ends with 6970.51489. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all $q = (50\%, 60\%, 70\%, 80\%, 90\%, 99\%)$.
7. Under the OLSE method: The VAR(z) is monotonically increasing starts with $2468.457|_{1-q=50\%}$ and ends with $8867.3465|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4397.12888 and ends with 10037.10797. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all $q = (50\%, 60\%, 70\%, 80\%, 90\%, 99\%)$.
8. Under the WLSE method: The VAR(z) is monotonically increasing starts with $2328.55844|_{1-q=50\%}$ and ends with $8245.58459|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4061.32709 and ends with 9414.17032. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all $q = (50\%, 60\%, 70\%, 80\%, 90\%, 99\%)$.
9. Under the CVM method: The VAR(z) is monotonically increasing starts with $2465.08067|_{1-q=50\%}$ and ends with $8743.84313|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4325.0162 and ends with 9947.3567. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all $q = (50\%, 60\%, 70\%, 80\%, 90\%, 99\%)$.

Table 4. main risk indicators under insurance claims data

Method	VAR(z)	TVAR(z)	TV(z)	TMV(z)	MELq(z)
MLE					
50%	2366.80069	3848.80179	1201335.462	604516.53275	1482.001096
60%	2827.16262	4162.7392	1004479.704	506402.59109	1335.576577
70%	3345.63847	4523.50196	811251.6110	410149.30745	1177.863488
80%	3960.61743	4964.06764	618901.2187	314414.67701	1003.450212
90%	4787.12604	5582.14707	417447.5811	214305.93763	795.021027
95%	5425.62884	6082.04447	302440.8371	157302.46304	656.415627
99%	6501.06864	6970.51489	170469.0719	92205.05086	469.446252
OLSE					
50%	2468.457	4397.12888	2709322.0385	1359058.14812	1928.67188
60%	2989.91542	4815.57935	2505495.1618	1257563.16025	1825.66393
70%	3605.17011	5324.97984	2292224.1197	1151437.03971	1719.80974
80%	4394.59163	5997.60414	2055268.7449	1033631.97656	1603.0125
90%	5599.41252	7054.59324	1757463.3165	885786.25148	1455.18072
95%	6677.86559	8023.63854	1542231.5599	779139.41849	1345.77296
99%	8867.3465	10037.10797	1208939.1294	614506.67266	1169.76147
WLSE					
50%	2328.55844	4061.32709	2300413.8217	1154268.23794	1732.76865
60%	2786.71964	4438.56758	2159601.8532	1084239.49419	1651.84794
70%	3330.38852	4901.68738	2013367.4521	1011585.41341	1571.29886
80%	4035.648	5520.42809	1850954.7508	930997.8035	1484.78009
90%	5132.77458	6510.2026	1644322.0724	828671.2388	1377.42803
95%	6138.02961	7436.35862	1490625.9397	752749.32848	1298.32901
99%	8245.58459	9414.17032	1241308.0055	630068.17307	1168.58573
CVM					
50%	2465.08067	4325.0162	2596755.08221	1302702.5573	1859.93556
60%	2961.49325	4729.3575	2423361.55172	1216410.1334	1767.86428
70%	3548.94671	5224.0453	2242731.44223	1126589.7664	1675.09853
80%	4307.5111	5881.8935	2041989.96963	1026876.8783	1574.3824
90%	5478.62611	6926.9569	1787696.04351	900774.97867	1448.33081
95%	6541.97247	7897.1920	1600271.59533	808032.98965	1355.21951
99%	8743.84313	9947.3567	1302219.73987	661057.22665	1203.51358

5. Validation

5.1. Maximum likelihood estimation for censored data

In reliability studies and survival analysis, data are often censored. If $z_1, z_2, ..., z_n$ is a censored sample from the BXGZC distribution, each observation can be written as $z_i = min(z_i, C_i)$ for $i = 1, .., n$ where z_i are failure times and C_i censoring times. The likelihood function is

$$l_i(\Theta) = \Pi_{i=1}^{n} f_{\underline{V}}(z_i)^{\delta_i} S_{\underline{V}}(z_i)^{1-\delta_i}, \qquad \delta_i = 1_{X_i<C_i}.$$

The right censoring is assumed to be non informative, so the log-likelihood function can be written as:

$$L_i(\Theta) = \sum_{i=1}^{n} \delta_i \log f_{\underline{V}}(z_i) + \sum_{i=1}^{n} (1 - \delta_i) \log S_{\underline{V}}(z_i).$$

Let:

$$\begin{aligned} \nabla_{\sigma_3}(z) &= \exp\{[1 - \exp(z^{\sigma_3})]\}, \\ O^{-2}_{\sigma_2,\sigma_3}(z) &= \left[(1 - \nabla_{\sigma_3}(z))^{-\sigma_2} - 1\right]^{-2}, \end{aligned}$$

$$\varpi_i = 1 - e^{-O^{-2}_{\sigma_2,\sigma_3}(z)},$$

and

$$\varphi_i = 1 - \nabla_{\sigma_3}(z).$$

Then,

$$\begin{aligned} L_i(\Theta) = & \sum_{i=1}^{n} \delta_i \left[\begin{array}{c} \ln(2\sigma_1\sigma_2\sigma_3) - (\sigma_3 - 1)\ln z_i + z_i^{\sigma_3} + \nabla_{\sigma_3}(z) - O^{-2}_{\sigma_2,\sigma_3}(z) \\ +(2\sigma_2 - 1)\ln(\varphi_i) - 3\ln(1 - \varphi_i^{\sigma_2}) - (1 - \sigma_1)\ln(\varpi_i) \end{array} \right] \\ & + \sum_{i=1}^{n} (1 - \delta_i)\ln(1 - \varpi_i^{\sigma_1}) \end{aligned}$$

The maximum likelihood estimators $\widehat{\sigma_1}$,$\widehat{\sigma_2}$, and σ_3 of the unknown parameters σ_1, σ_2 and σ_3 are derived from the nonlinear following score equations:

$$\frac{\partial L}{\partial \sigma_1} = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\sigma_1} + \ln(\varpi_i)\right] - \sum_{i=1}^{n} (1 - \delta_i)\left[\frac{\varpi_i^{\sigma_1} \ln \varpi_i}{1 - \varpi_i^{\sigma_1}}\right] = 0,$$

$$\begin{aligned} \frac{\partial L}{\partial \sigma_2} = & \sum_{i=1}^{n} \delta_i \left[\begin{array}{c} \frac{1}{\sigma_2} - \frac{2\varphi_i^{-\sigma_2}\ln(\varphi_i)}{(\varphi_i^{\sigma_2}-1)^3} + 2\ln(\varphi_i) \\ +\frac{3\varphi_i^{\sigma_2}\ln(\varphi_i)}{1-\varphi_i^{\sigma_2}} - \frac{2(1-\sigma_1)\varphi_i^{-\sigma_2}\ln(\varphi_i)e^{-O^{-2}_{\sigma_2,\sigma_3}(z)}}{\varpi_i(\varphi_i^{\sigma_2}-1)^3} \end{array} \right] = 0, \\ & -2\sum_{i=1}^{n} (1 - \delta_i) \frac{\sigma_1\varphi_i^{-\sigma_2}\ln(\varphi_i)e^{-O^{-2}_{\sigma_2,\sigma_3}(z)}\varpi_i^{\sigma_1-1}}{(1 - \varpi_i^{\sigma_1})\left(\varphi_i^{\sigma_2} - 1\right)^3} \end{aligned}$$

and

$$\frac{\partial L}{\partial \sigma_3} = \sum_{i=1}^{n} \delta_i \left[\begin{array}{c} \frac{1}{\sigma_2} + \ln(z_i) + z_i^{\sigma_3} \ln(z_i) \\ -\frac{2\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z) \varphi_i^{-\sigma_2 - 1}}{\left(\varphi_i^{-\sigma_2} - 1\right)^3} + \frac{2(\sigma_2 - 1) z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z)}{\varphi_i} \\ -z^{\sigma_3} \ln(z_i) e^{-z_i^{\sigma_3}} + \frac{3\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \varphi_i^{\sigma_2 - 1} \nabla_{\sigma_3}(z)}{1 - \varphi_i^{\sigma_2}} \\ -\frac{2(1-\sigma_1)\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z) e^{-O^{-2}_{\sigma_2,\sigma_3}(z)} \varphi_i^{-\sigma_2 - 1}}{\left(\varphi_i^{-\sigma_2} - 1\right)^3 \varpi_i} \end{array} \right]$$

$$-2 \sum_{i=1}^{n} (1 - \delta_i) \frac{\sigma_1 \sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z) \, e^{-O^{-2}_{\sigma_2,\sigma_3}(z)} \varphi_i^{-\sigma_2 - 1} \varpi_i^{\sigma_1 - 1}}{\left(\varphi_i^{-\sigma_2} - 1\right)^3 (1 - \varpi_i^{\sigma_1})} = 0$$

The explicit form of $\widehat{\sigma}_1$, σ_2 and $\widehat{\sigma}_3$ cannot be obtained, so we use numerical methods.

5.2. *Test statistic for right censored data*

Let $z_1, z_2, ..., z_n$ be n i.i.d. random variables grouped into r classes I_j. To assess the adequacy of a parametric model F_0, where

$$H_0 : \quad P(z_i \leq z \mid H_0) = F_0(z; \Theta), z \geq 0, \quad \Theta = (\Theta_1, ..., \Theta_s)^T \in \Theta \subset R^s$$

when data are right censored and the parameter vector Θ is unknown, Bagdonavicius and Nikulin (2011) proposed a test statistic $T^2_{n,r-1,\alpha}$ based on the vector

$$Z_j = \frac{1}{\sqrt{n}}(U_j(z) - e_j(z)), \; j = 1, 2, ..., r \;\;, \text{ with } r > s.$$

This one represents the differences between observed and expected numbers of failures ($U_j(z)$ and $e_j(z)$) to fall into these grouping intervals $I_j = (\rho_{j-1}, \rho_j]$ with $\rho_0 = 0$, $\rho_r = \tau$, where τ is a finite time. The authors considered ρ_j as random data functions such as the r intervals chosen have equal expected numbers of failures $e_j(z)$. The test statistic $T^2_{n,r-1,\alpha}$ is defined by

$$T^2_{n,r-1,\alpha} = Z^T \widehat{\Sigma}^{-} Z = \sum_{j=1}^{r} \frac{1}{U_j(z)} (U_j(z) - e_j(z))^2 + Q$$

where $Z = (Z_1, ..., Z_k)^T$ and $\widehat{\Sigma}^{-}$ is a generalized inverse of the covariance matrix $\widehat{\Sigma}$ and

$$\begin{aligned} Q &= W^T \widehat{G}^{-} W \\ \widehat{A}_j &= U_j(z)/n, \\ U_j(z) &= \sum_{i: z_i \in I_j} \delta_i, \end{aligned}$$

$$\begin{aligned}
W &= (W_1, ..., W_s)^T, \\
\widehat{G} &= [\widehat{g}_{ll'}]_{sxs}, \\
\widehat{g}_{ll'} &= \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj}\widehat{C}_{l'j}\hat{A}_j^{-1}, \\
\widehat{C}_{lj} &= \frac{1}{n}\sum_{i:z_i\in I_j} \delta_i \frac{\partial}{\partial\Theta} \ln h(z_i, \hat{\Theta}), \\
\widehat{i}_{ll'} &= \frac{1}{q}\sum_{i=1}^{n} \delta_i \frac{\partial \ln h(z_i, \widehat{\Theta})}{\partial\Theta_l}\frac{\partial \ln h(z_i, \widehat{\Theta})}{\partial\Theta_{l'}}, \\
\widehat{W}_l &= \sum_{j=1}^{r} \widehat{C}_{lj}\widehat{A}_j^{-1}Z_j, \qquad l, l' = 1, ..., s,
\end{aligned}$$

where $\widehat{\Theta}$ is the maximum likelihood estimator of Θ on initial non-grouped data. Under the null hypothesis H_0 , the limit distribution of the statistic $T^2_{n,r-1,\alpha}$ is a chi-square with $r = rank(\Sigma)$ degrees of freedom. The description and applications of modified chi-square tests are discussed in Voinov et al. (2013). The interval limits ρ_j for grouping data into j classes I_j are considered as data functions and defined by

$$\hat{\rho}_{j,z} = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H\left(z_l, \hat{\Theta}\right)}{n-i+1}, \hat{\Theta}\right), \qquad \hat{\rho}_r = \max\left(z_{(n)}, \tau\right)$$

such as the expected failure times $e_j(z)$ to fall into these intervals are $e_j(z) = \frac{E_r}{r}$ for any j, with $E_r = \sum_{i=1}^{n} H\left(z_i, \hat{\Theta}\right)$. The distribution of this test statistic $T^2_{n,r-1,\alpha}$ is chi-square (see Voinov et al., 2013).

5.3. Criteria test for BXGZC

For testing the null hypothesis H_0 that data belong to the BXGZC model, we construct a modified chi-squared type goodness-of-fit test based on the statistic $T^2_{n,r-1,\alpha}$. Suppose that τ is a finite time, and observed data are grouped into $r > s$ sub-intervals $I_j = \left(\rho_{j-1}, \rho_j\right]$ of $[0, \tau]$. The limit intervals ρ_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures $e_j(z)$ are obtained as

$$E_j(z) = \frac{-j}{r-1}\sum_{i=1}^{n} \ln\left(1 - \left\{1 - \exp\left[-O^2_{\underline{\xi}}(z)\right]\right\}^{\sigma_1} \nabla\right), \quad j = 1, ..r-1$$

The components of the estimated matrix $\hat{W}$ are derived from the estimated matrix $\hat{C}$ which is given by:

$$\hat{C}_{1j}, z_i = \frac{1}{n}\sum_{i:z_i\in I_j}^{n} \delta_i \left[\frac{1}{\sigma_1} + \frac{\ln \varpi_i}{1 - \varpi_i^{\sigma_1}}\right]$$

$$\hat{C}_{2j}, z_i = \frac{1}{n}\sum_{i:z_i\in I_j}^{n} \delta_i \left[\begin{array}{c} \frac{1}{\sigma_2} - \frac{2\varphi_i^{-\sigma_2}\ln(\varphi_i)}{\left(\varphi_i^{\sigma_2}-1\right)^3} + 2\ln(\varphi_i) + \frac{3\varphi_i^{\sigma_2}\ln(\varphi_i)}{1-\varphi_i^{\sigma_2}} \\ + \frac{\varphi_i^{-\sigma_2}\ln(\varphi_i)e^{-O^{-2}_{\sigma_2,\sigma_3}(z)}\left(\varpi_i^{\sigma_1}+\sigma_1-1\right)}{\varpi_i(1-\varpi_i^{\sigma_1})\left(\varphi_i^{\sigma_2}-1\right)^3} \end{array}\right]$$

$$\hat{C}_{3j}, z_i = \frac{1}{n} \sum_{i:z_i \in I_j}^{n} \delta_i \left[\begin{array}{c} \frac{1}{\sigma_2} - z^{\sigma_3} \ln(z_i) e^{-z_i^{\sigma_3}} + z_i^{\sigma_3} \ln(z_i) \\ -\frac{2\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z) \varphi_i^{-\sigma_2 - 1}}{\left(\varphi_i^{-\sigma_2} - 1\right)^3} + \frac{2(\sigma_2 - 1) z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z)}{\varphi_i} \\ + \ln(z_i) + \frac{3\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \varphi_i^{\sigma_2 - 1} \nabla_{\sigma_3}(z)}{1 - \varphi_i^{\sigma_2}} \\ + \frac{2\sigma_2 z_i^{\sigma_3} \ln(z_i) e^{z_i^{\sigma_3}} \nabla_{\sigma_3}(z) e^{-O^{-2}_{\sigma_2, \sigma_3}(z)} \varphi_i^{-\sigma_2 - 1} (\varpi_i^{\sigma_1} + \sigma_1 - 1)}{\left(\varphi_i^{-\sigma_2} - 1\right)^3 \varpi_i (1 - \varpi_i^{\sigma_1})} \end{array} \right]$$

and

$$\hat{W}_l = \sum_{j=1}^{r} \hat{C}_{lj} A_j^{-1} Z_j, \quad l = 1, .., m \qquad j = 1, .., r$$

Therefore, the quadratic form of the test statistic can be obtained easily:

$$T^2_{n,r-1,\alpha}\left(\hat{\Theta}\right) = \sum_{j=1}^{r} \frac{\left(U_j(z) - e_j(z)\right)^2}{U_j(z)} + \hat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lj} \hat{C}_{l'j} \hat{A}_j^{-1}\right]^{-1} \hat{W}.$$

6. Validation via right censored data

6.1. Right censored lymphoma data

In this sebsection, we analyze the lymphoma data set consisting of times (in months) from diagnosis to death for 31 individuals with advanced non Hodgkin's lymphoma clinical symptoms, by using our model. This data has been analyzed by Gijbels and Gurler [13] by using exponential change point model. Among these 31 observations 11 of the times are censored, because the patients were alive at the last time of follow-up, where the data are given as: 2.5, 4.1, 4.6, 6.4, 6.7, 7.4, 7.6, 7.7, 7.8, 8.8, 13.3, 13.4, 18.3, 19.7, 21.9, 24.7, 27.5, 29.7, 30.1*, 32.9, 33.5, 35.4*, 37.7*, 40.9*, 42.6*, 45.4*, 48.5*, 48.9*, 60.4*, 64.4*, 66.4*. where * denotes a censored observation. We use the test statistic provided above to verify if these data are modeled by BXGZC distribution, and to that end,we first calculate the maximum likelihood estimators of the unknown parameters

$$\hat{\Theta} = (\sigma_1, \sigma_2, \sigma_3)^T = (1.6325, 1.9532, 1.0236)^T.$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in the following Table 5.

Table 5. values of $\widehat{\rho}_j, e_j(z), U_j(z), \hat{C}_{1j}, z_i, \hat{C}_{2j}, z_i, \hat{C}_{3j}, z_i$

$\hat{\rho}_{j,z}$	7.2	14.6	30	41.5	66.4
$U_j(z)$	5	7	6	6	7
$\hat{C}_{1j}, z_i$	0.9346	0.7367	0.8162	0.9934	1.0342
$\hat{C}_{2j}, z_i$	1.3426	1.2034	1.2963	1.4436	1.5133
$\hat{C}_{3j}, z_i$	0.8346	0.6746	0.7342	0.9347	1.0263
$e_j(z)$	2.862	2.862	2.862	2.862	2.862

Then we obtain the value of the test statistic $T^2_{n,r-1,\alpha}$: $T^2_{31,4,0.05} = X^2 + Q = 7.6329$. For significance level $\alpha = 0.05$, the critical value $\chi^2_5 = 11.0705$ is higher than the value of $T^2_{n,r-1,\alpha} = 7.6329$, so we can say that the proposed BXGZC model fit these data. Decision: for the right censored lymphoma data, $T^2_{31,4,0.05} = 7.6329 < \chi^2_{0.05}(6) = 11.0705$, therefore, we can accept the null hypothesis that the data of times to infection of kidney dialysis patients follows the BXGZC distribution.

6.2. *Right censored bone marrow transplant data*

The second data set , we consider the bone marrow transplant data (Klein and Moeschberger [17]) for patients suffering from acute lymphoblastic leukemia. This data consist of time (in days) to death or on study time after a allogenic bone marrow transplant for 38 patients. The bone marrow transplant is a standard treatment for acute leukemia. Recovery following bone marrow transplantation is a complex process. Immediately following transplantation, patients have depressed platelet counts and have higher hazard rate for the development of infections but as the time passes the hazard decreases. Data are given as: 1, 86, 107, 110, 122, 156, 162, 172, 243, 262, 262, 269, 276, 371, 417, 418, 466, 487, 526, 716, 781, 1111, 1182, 1199, 1279, 1377, 1433, 1496. Censored observations: 350, 1330, 194,226, 1167, 1462, 1602, 2081, 530, 996, 1330. We use the test statistic provided above to verify if these data are modeled by the BXGZC distribution, and to that end, we first calculate the maximum likelihood estimators of the unknown parameters

$$(\sigma_1, \sigma_2, \sigma_3)^T = (1.0342, 0.9238, 1.1342)^T .$$

Data are grouped into $r = 4$ intervals I_j. We give the necessary calculus in the following Table 6.

Table 6. values of $\widehat{\rho}_j, e_j(z), U_j(z), \hat{C}_{1j}, z_i, \hat{C}_{2j}, z_i, \hat{C}_{3j}, z_i$

$\hat{\rho}_{j,z}$	197	402	1125	2081
$U_j(z)$	9	8	10	11
$\hat{C}_{1j}, z_i$	0.9734	0.8376	0.9436	0.9696
$\hat{C}_{2j}, z_i$	0.9816	0.8933	0.9212	0.9196
$\hat{C}_{3j}, z_i$	0.7347	0.6198	0.7417	0.7538
$e_j(z)$	3.6592	3.6592	3.6592	3.6592

Then we obtain the value of the test statistic $T^2_{n,r-1,\alpha}$: $T^2_{38,3,0.05} = X^2 + Q = 6.9326$. For significance level $\alpha = 0.05$, the critical value $\chi^2_4 = 9,4877$ is higher than the value of $T^2_{n,r-1,\alpha} = 6.9326$, so we can say that the proposed model BXGZC fit these data. Decision: For the right censored bone marrow transplant data, $T^2_{38,3,0.05} = 6.932 < \chi^2_{0.05}(4) = 9.4877$, therefore, we can accept the null hypothesis that the bone marrow transplant data follows the BXGZC distribution.

6.3. *Right censored reliability data*

For the third data set, we apply the results obtained from this study to real data established from reliability (Crowder et al. [8]). In an experiment to obtain information on the strength of a certain type of braided cord after the weather, the forces of 48 pieces of cord having resisted for a determined time were studied. The right censored force values observed are given below: 26.8*, 29.6*, 33.4*, 35*, 36.3, 40*, 41.7, 41.9*, 52.3, 52.3, 52.4, 52.6, 53.6, 42.5*, 57.3, 52.7, 53.1, 50.8, 51.9, 52.1, 53.6, 53.9, 53.9, 54.1, 54.6, 54.8, 54.8, 55.1, 55.4, 55.9, 56, 56.1, 56.5, 57.7, 57.8, 58.1, 58.9, 43.9, 49.9, 50.1, 56.9, 57.1, 57.1, 59, 59.1, 59.6, 60.4, 60.7. We use the test statistic provided above to verify if these data are modeled by the BXGZC distribution, and to that end, we first calculate the maximum likelihood estimators of the unknown parameters

$$(\sigma_1, \sigma_2, \sigma_3)^T = (1.326, 2.061, 1.4523)^T .$$

Data are grouped into $r = 5$ intervals I_j. We give the necessary calculus in Table 7.

Table 7. values of $\widehat{\rho}_j, e_j(z), U_j(z), \hat{C}_{1j}, z_i, \hat{C}_{2j}, z_i, \hat{C}_{3j}, z_i$

$\hat{\rho}_{j,z}$	42.30	52.02	53.76	56.7	60.7
$U_j(z)$	8	6	9	12	13
$\hat{C}_{1j}, z_i$	1.2346	1.3476	1.2019	0.9834	0.7934
$\hat{C}_{2j}, z_i$	1.3637	1.2133	1.3737	1.1136	0.9739
$\hat{C}_{3j}, z_i$	0.8534	0.9312	0.8994	0.7647	0.6345
$e_j(z)$	4.5316	4.5316	4.5316	4.5316	4.5316

Then we obtain the value of the test statistic $T^2_{n,r-1,\alpha}$: $T^2_{31,4,0.05} = X^2 + Q = 9.5326$. For significance level $\alpha = 0.05$, the critical value $\chi^2_5 = 11.0705$ is higher than the value of $T^2_{n,r-1,\alpha} = 9.5326$, so we can say that the proposed model BXGZC fit these data. Decision: For the right censored reliability data, $T^2_{31,4,0.05} = 9.5326 < \chi^2_{0.05}(5) = 11.0705$, therefore, we can accept the null hypothesis that thestrength of certain type of braided cord data follows the BXGZC distribution.

6.4. *Right censored survival data*

For the fourth data set, Woolson [9] has reported survival data on 26 psychiatric inpatients admitted to the university of Iowa hospitals during the years 1935-1948. This sample is part of a larger study of psychiatric Inpatients discussed by Woolson [9]. Data for each patient consists of age at rst admission to the hospital, sex, number of years of follow-up (years from admission to death or censoring) and patient status at the followup time. The data is given $1, 1, 2, 11, 14, 22, 22, 24, 25, 26, 28, 30^*, 30^*, 31^*, 31^*, 32, 33^*, 33^*, 34^*, 35, 35^*, 35^*, 36^*, 37^*, 39^*, 40$. (* indicates the censorship). We use the test statistic provided above to verify if these data are modeled by the BXGZC distribution, and to that end, we first calculate the maximum likelihood estimators of the unknown parameters

$$(\sigma_1, \sigma_2, \sigma_3)^T = (0.9532, 1.0315, 0.8239)^T .$$

Data are grouped into $r = 4$ intervals I_j. We give the necessary calculus in Table 8.

Table 8. values of $\widehat{\rho}_j, e_j(z), U_j(z), \hat{C}_{1j}, z_i, \hat{C}_{2j}, z_i, \hat{C}_{3j}, z_i$

$\hat{\rho}_{j,z}$	23.5	31.6	34.8	40
$U_j(z)$	7	8	4	7
$\hat{C}_{1j}, z_i$	0.9361	0.9712	0.9396	0.7346
$\hat{C}_{2j}, z_i$	0.8326	0.8263	0.8575	0.8633
$\hat{C}_{3j}, z_i$	1.0134	1.0492	1.1346	1.0034
$e_j(z)$	2.0314	2.0314	2.0314	2.0314

Then we obtain the value of the test statistic $T^2_{n,r-1,\alpha}$: $T^2_{26,3,0.05} = X^2 + Q = 7.2301$. For significance level $\alpha = 0.05$, the critical value $\chi^2_4 = 9.4877$ is higher than the value of $T^2_{n,r-1,\alpha} = 7.2301$, so we can say that the proposed model BXGZC fit these data. Decision: For the right censored survival data, $T^2_{26,3,0.05} = 7.2301 < \chi^2_{0.05}(4) = 9.4877$, therefore, we can accept the null hypothesis that thestrength of certain type of braided cord data follows the BXGZC distribution.

7. Conclusion

In this paper, we introduced and studied a novel probability distribution for risk analysis and censored validity. Several characterizations are provided. Indicators of financial risk include value-at-risk, tail-value-at-risk, tail variance, tail Mean-Variance, and mean excess loss function. These indicators are considered by the Cramer-von Mises method, ordi-nary least squares, weighted least squares, and maximum likelihood estimation. These four techniques were used in a sim-ulation study and an application to insurance payment claims data for the actuarial evaluation. The well-known Niku-lin-Rao-Robson statistics are taken into consideration for distributional validation under the whole set of data. Four com-plete real data sets and a simulation study are used to evaluate the Nikulin-Rao-Robson test statistic. An updated version of the Nikulin-Rao-Robson statistics are taken into consideration for censored distributional validation. Four censored real data sets and a thorough simulation analysis are used to evaluate the novel Nikulin-Rao-Robson test statistic. Under the artificial analysis, we have the following results:

1. $\mathrm{VAR}(z)_{\mathrm{WLS}} < \mathrm{VAR}(z)_{\mathrm{CVM}} < \mathrm{VAR}(z)_{\mathrm{MLE}} < \mathrm{VAR}(z)_{\mathrm{OLSE}}$ for most q.
2. $\mathrm{TVAR}(z)_{\mathrm{WLS}} < \mathrm{TVAR}(z)_{\mathrm{CVM}} < \mathrm{TVAR}(z)_{\mathrm{MLE}} < \mathrm{TVAR}(z)_{\mathrm{OLSE}}$ for most q.
   Based on Table 4, the following results can be highlighted:
3. For all actuarial risk assessment approaches:
$$VAR(z|_{1-q=50\%}) < ... < VAR(z|_{1-q=1\%}).$$
4. For all actuarial risk assessment approaches:
$$TVAR(z|_{1-q=50\%}) < ... < TVAR(z|_{1-q=1\%}).$$
5. For all actuarial risk assessment approaches:
$$TV(z|_{1-q=50\%}) > ... > TV(z|_{1-q=1\%}).$$
6. For all actuarial risk assessment approaches:
$$TMV(z|_{1-q=50\%}) > ... > TMV(z|_{1-q=1\%}).$$
7. For all actuarial risk assessment approaches:
$$MELq(z|_{1-q=50\%}) > ... > MELq(z|_{1-q=1\%}).$$
8. Under the MLE technique: The VAR(z) is monotonically increasing starts with $2366.80069|_{1-q=50\%}$ and ends with $6501.06864|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 3848.80179 and ends with 6970.51489. It is worth noting that the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all q.

9. Under the OLSE method: The VAR(z) is monotonically increasing starts with $2468.457|_{1-q=50\%}$ and ends with $8867.3465|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4397.12888 and ends with 10037.10797. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all q.
10. Under the WLSE method: The VAR(z) is monotonically increasing starts with $2328.55844|_{1-q=50\%}$ and ends with $8245.58459|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4061.32709 and ends with 9414.17032. It is worth noting that the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all q.
11. Under the CVM method: The VAR(z) is monotonically increasing starts with $2465.08067|_{1-q=50\%}$ and ends with $8743.84313|_{1-q=1\%}$, the TVAR(z) in monotonically incresing starts with 4325.0162 and ends with 9947.3567. However the TV(z), the TMV(z) and the MELq(z) are monotonically decreasing for all q.

In the context of the distributional validity and statistical hypothesis tests for the censored data, a modified NRR statistic, which is based on the censored maximum likelihood estimators on initial non-grouped data, is of considered under the BXGZC model. The modified NRR statistic is assessed under four right censored data sets and the following results can be highlighted:

- For the right censored lymphoma data, $T^2_{31,4,0.05} = 7.6329 < \chi^2_{0.05}(6) = 11.0705$, therefore, we can accept the null hypothesis that the data of times to infection of kidney dialysis patients follows the BXGZC distribution.
- For the right censored bone marrow transplant data, $T^2_{38,3,0.05} = 6.932 < \chi^2_{0.05}(4) = 9.4877$, therefore, we can accept the null hypothesis that the bone marrow transplant data follows the BXGZC distribution.
- For the right censored reliability data, $T^2_{31,4,0.05} = 9.5326 < \chi^2_{0.05}(5) = 11.0705$, therefore, we can accept the null hypothesis that thestrength of certain type of braided cord data follows the BXGZC distribution.
- For the right censored survival data, $T^2_{26,3,0.05} = 7.2301 < \chi^2_{0.05}(4) = 9.4877$, therefore, we can accept the null hypothesis that thestrength of certain type of braided cord data follows the BXGZC distribution.

In the context of the distributional validity and statistical hypothesis testing for the censored data, a modified NIRR statistic is of consideration under the BXGZC model. This statistic is based on the censored maximum likelihood estimators on the original non-grouped data, and it is of interest because it is derived from the data. When evaluated using four different right-censored data sets, the modified NIRR statistic yields the following results, which can be highlighted. As a future potential work, we may consider other Chen extensions for making a useful compression. We may also consider other distractions for validations and risk analysis. Other modified test statistics could be considered for validation. Develop a new test for the right censored validation. Use some new risk indicators for risk analysis. Develop some new risk indicators for the risk analysis Other related papers can be used for some potential works as presented in Abonongo et al. [1]. For other useful works see Shaheed [28], Mohammad [20], Mohammad [21], Shaheed [29] and Shaheed [30]. Other future works could be developed based on Alzeley et al. [5], Tashkandy et al. [32], Jameel et al. [15], Salih and Abdullah [25], Salih and Hmood [26] and Salih and Hmood [27], and Alotaibil et al. [4].

Funding Statement: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU252849].

Acknowledgments: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU252849].

Conflict of Interest: The authors declare no conflict of interest.

Data availability Statement: Data is available from the corresponding authors upon reasonable request.

References

1. Abonongo, J., Abonongo, A. I. L., Aljadani, A., Mansour, M. M., & Yousof, H. M. (2025). Accelerated failure model with empirical analysis and application to colon cancer data: Testing and validation. *Alexandria Engineering Journal*, 113, 391-408.

2. Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance*, 26(7), 1487-1503.

3. AlKhayyat, S. L., Yousof, H. M., Goual, H., Hamida, T., Hamed, M. S., Hiba, A., & Ibrahim, M. (2025). Rao-Robson-Nikulin Goodness-of-fit Test Statistic for Censored and Uncensored Real Data with Classical and Bayesian Estimation. *Statistics, Optimization & Information Computing*, 13(6), 2205-2225.

4. Alotaibil, N., Al-Moisheer, A. S., Elbatal, I., & Alyami, S. A. (2024). Bivariate step-stress accelerated life test for a new three-parameter model under progressive censored schemes with application in medical. *AIMS Mathematics*, 9(2), 3521-3558.

5. Alzeley, O., Almetwally, E. M., Gemeay, A. M., Alshanbari, H. M., Hafez, E. H., & Abu-Moussa, M. H. (2021). Statisti-cal inference under censored data for the new exponential-X Fréchet distribution: Simulation and application to Leukemia data. *Computational Intelligence and Neuroscience*, 2021(1), 2167670.

6. Artzner, P. (1999). Application of coherent risk measures to capital requirements in insurance. *North American Actuarial Journal*, 3(2), 11-25.

7. Chaubey, Y. P., & Zhang, R. (2015). An extension of Chen's family of survival distributions with bathtub shape or increasing hazard rate function. *Communications in Statistics-Theory and Methods*, 44(19), 4049-4064.

8. Crowder M. J., Kimber A. C., Smith R.L and Sweeting, T. J. (1991). Statistical analysis of reliability data, CHAPMAN &HALL/CRC.Y

9. Woolson, R. F. (1981). Rank tests and a one-sample logrank test for comparing observed survival data to a standard population. Biometrics, 687-696.

10. Dey, S., Kumar, D., Ramos, P. L. and Louzada, F. (2017). Exponentiated Chen distribution: Properties and estimation. *Communications in Statistics-Simulation and Computation*, 46(10), 8118-8139.

11. Furman, E. and Landsman, Z. (2006). Tail Variance premium with applications for elliptical portfolio of risks. ASTIN Bulletin: T*he Journal of the IAA*, 36(2), 433-462.

12. Landsman, Z. (2010). On the tail mean–variance optimal portfolio selection. Insurance: Mathematics and Economics, 46(3), 547-553.

13. Gijbels, I. and Gürler, Ü. (2003). Estimation of a change point in a hazard function based on censored data. *Lifetime Data Analysis*, 9(4), 395-411.

14. Ibrahim, M., Aidi, K., Ali, M. M. and Yousof, H. M. (2023). A Novel Test Statistic for Right Censored Validity under a new Chen extension with Applications in Reliability and Medicine. *Annals of Data Science*, 10(5):1285–1299.

15. Jameel, S. O., Salih, A. M., Jaleel, R. A., & Zahra, M. M. (2022). On The Neutrosophic Formula of Some Matrix Equations Derived from Data Mining Theory and Control Systems. *International Journal of Neutrosophic Science* (IJNS), 19(1).

16. Kazemi, M. R., Jafari, A. A., Tahmasebi, S., Alizadeh, M., & Hamedani, G. G. (2022). Exponentiated Extended Chen Distribution: Regression Model and Estimations. *Statistics, Optimization & Information Computing*, 10(3), 710-724.

17. Klein J. P. and Moeschberger M. L. (2003). Survival Analysis: Techniques for Censored and Truncated Data, Springer, New York.

18. Klugman, S. A., Panjer, H. H. and Willmot, G. E. (2012). Loss models: from data to decisions (Vol. 715). John Wiley & Sons.

19. Lane, M. N. (2000). Pricing risk transfer transactions1. ASTIN Bulletin: *The Journal of the IAA*, 30(2), 259-293.

20. Mohammad, G. S. (2023). A New Weighted Topp-Leone Family of Distributions. *Statistics, Optimization & Infor-mation Computing*, 11(3).

21. Mohammad, G. S. (2024). A new mixture of exponential and Weibull distributions: properties, estimation and relibilty modelling. *São Paulo Journal of Mathematical Sciences*, 18(1), 438-458.

22. Mohamed, H. S., Cordeiro, G. M., Minkah, R., Yousof, H. M., & Ibrahim, M. (2024). A size-of-loss model for the negatively skewed insurance claims data: applications, risk analysis using different methods and statistical forecasting. *Journal of Applied Statistics*, 51(2), 348-369.

23. Nikulin, M. S. (1974). Chi-square test for continuous distributions with shift and scale parameters. *Theory of Probability & Its Applications*, 18(3), 559-568.

24. Rao, K. C., & Robson, B. S. (1974). A chi-squabe statistic for goodies-of-fit tests within the exponential family. *Communications in Statistics-Theory and Methods*, 3(12), 1139-1153.

25. Salih A.M., & Abdullah M.M. (2024). Comparison between classical and Bayesian estimation with joint Jeffrey's prior to Weibull distribution parameters in the presence of large sample conditions. Statistics in Transition new series, 25(4), pp. 191-202 https://doi.org/10.59139/stattrans-2024-010

26. Salih, A. M., & Hmood, M. Y. (2020). Analyzing big data sets by using different panelized regression methods with application: surveys of multidimensional poverty in Iraq. Periodicals of Engineering and Natural Sciences (PEN), 8(2), 991-999.

27. Salih, A. M., & Hmood, M. Y. (2021). Big data analysis by using one covariate at a time multiple testing (OCMT) method: Early school dropout in Iraq. International Journal of Nonlinear Analysis and Applications, 12(2), 931-938.
28. Shaheed, G. (2022). Novel Weighted G family of Probability Distributions with Properties, Modelling and Different Methods of Estimation. *Statistics, Optimization & Information Computing*, 10(4), 1143-1161.
29. Shaheed, G. (2022). A new two-parameter modified half-logistic distribution: properties and applications. *Statistics, Optimization & Information Computing*, 10(2), 589-60.
30. Shaheed, G. (2025). A Weighted Exponentiated class of Distributions: Properties with Applications for Modelling Reliability Data. *Statistics, Optimization & Information Computing*, 13(3), 1144-1161.
31. Shrahili, M., Elbatal, I., & M. Yousof, H. (2021). Asymmetric Density for Risk Claim-Size Data: Prediction and Bimodal Data Applications. *Symmetry*, 13(12), 2357.
32. Tashkandy, Y. A., Almetwally, E. M., Ragab, R., Gemeay, A. M., Abd El-Raouf, M. M., Khosa, S. K., ... & Bakr, M. E. (2023). Statistical inferences for the extended inverse Weibull distribution under progressive type-II censored sample with applications. *Alexandria Engineering Journal*, 65, 493-502.
33. Tasche, D. (2002). Expected Shortfall and Beyond, *Journal of Banking and Finance*, 26, 1519-1533.
34. Voinov, V., Nikulin, M., and Balakrishnan, N. (2013). Chi-Squared Goodness of Fit Tests with Applications, Academic Press, Elsevier.
35. Wirch, J. L. (1999). Raising value at risk. *North American Actuarial Journal*, 3(2), 106-115.
36. Yousof, H. M., Ali, E. I. A., Aidi, K., Butt, N. S., Saber, M. M., Al-Nefaie, A. H., Aljadani, A., Mansour, M. M., Hamed, M. S., & Ibrahim, M. (2025). The Statistical Distributional Validation under a Novel Generalized Gamma Distribution with Value-at-Risk Analysis for the Historical Claims, Censored and Uncensored Real-life Applications. *Pakistan Journal of Statistics and Operation Research*, 21(1), 51-69. https://doi.org/10.18187/pjsor.v21i1.4534
37. Yousof, H.M.; Emam, W., Tashkandy, Y.; Ali, M.M.; Minkah, R., Ibrahim, M. (2023). A Novel Model for Quantitative Risk Assessment under Claim-Size Data with Bimodal and Symmetric Data Modeling. *Mathematics*, 11, 1284. https://doi.org/10.3390/math11061284