Alfarama Journal of Basic & Applied Sciences

https://ajbas.journals.ekb.eg

ajbas@sci.psu.edu.eg

Faculty of Science Port Said University

http://sci.psu.edu.eg/en/

October 2025, Volume 6, Issue IV

DOI: 10.21608/ajbas.2025.403080.1267

ISSN 2682-275X

Submitted: 12/07/2025

Accepted: 24/08/2025

Pages: 431 - 442

Super Edge Bimagic and Trimagic Total Labeling for some Graphs

Mohamed R. Zeen El Deen[1], I. M. Hanafy[2], Sara G. Shehda[2], Nora. A. Omar[2,*]

[1] Mathematics and computer science Department, Faculty of Science, Suez University, Suez, Egypt

[2] Mathematics and computer science Department, Faculty of Science, Port-Said University, Port-Said, Egypt

***Corresponding author:** noraalaa19862013@gmail.com

ABSTRACT

A super edge trimagic total labeling (SETTL) of a graph Γ with α vertices and β edges is a bijection $\Phi: [V(\Gamma) \cup E(\Gamma)] \longrightarrow \{1, 2, 3, \cdots, \alpha+\beta\}$ such that for each edge $\vartheta\omega \in E(\Gamma)$, the value of the formula $[\Phi(\vartheta)+\Phi(\omega)+\Phi(\vartheta\omega)]$ is either K_1 or K_2 or K_3, with the additional condition that $\Phi: [V(\Gamma)] \longrightarrow \{1, 2, 3, \cdots, \alpha\}$. A super edge trimagic total graph is the one that allows a super edge trimagic total labeling. The idea of super bimagic total labelling of connected graphs gets further investigated in this study. First, we present the triangulated prism graph $T\Pi_r$ and demonstrate its bimagic total labelling by using the bimagic numbers $K_1 = 6r$ and $K_2 = 8r$, showing that this graph admits a super bimagic total labeling. Secondly, the idea of super edge trimagic total graph labeling was introduced. We found some complicated graphs with trimagic total numbers, including the triangulated wheel graph, the double vertex wheel graph, and the closed triangulated water wheel graph.

Keywords: Super edge trimagic total labeling, Triangulated prism, Double vertex wheel graph.

1. INTRODUCTION

All graphs considered in this study are finite, undirected, and simple connected graphs. Given a graph $\Gamma = (V(\Gamma), E(\Gamma))$, let $\alpha = |V(\Gamma)|$ represent the cardinality of its vertices, and $\beta = |E(\Gamma)|$ represent the cardinality of its edges. We follow [1,2] for general graph theoretical notion.

A graph labeling is a mapping that, based on certain restrictions, maps a graph's elements (vertices, edges, or both) to positive numbers. There are many different types of labeling, and all those labeling issues will share the three common factors.

[(i)] A set of numbers from which vertex or edge labels are chosen.

[(ii)] A rule that assigns a value to each edge or vertex.

[(iii)] A condition that these values must satisfy.

Graph labeling techniques can be applied to deal with problems in communication networks, improve communication speed in sensor networks, create fault-tolerant systems using facility graphs, design good radar type codes in coding theory, and address problems in mobile ad hoc networks [3].

Rosa proposed the concept of graph labelings in [4] at the beginning of the 1960s. Different graph labelling strategies have been researched after the publication of this study. See a dynamic survey of graph labelling, [5], for a thorough analysis of the topic.

The labeling of a graph is called total labeling if the domain of the mapping is the union of the vertices and the edges. Introduction of edge bimagic total labelling (EBTL) of graphs by J. Baskar Babujee [6] in 2004 described by a graph Γ with a bijection $\Phi: [V(\Gamma) \cup E(\Gamma)] \longrightarrow \{1,2,3,\ldots,\alpha+\beta\}$ so that the results of edge wights $\omega t_{\Phi}(\vartheta\omega) = [\Phi(\vartheta) + \Phi(\omega) + \Phi(\vartheta\omega)]$ for each edge $\vartheta\omega \in E(\Gamma)$ is either the constant represented by $k_1$or k_2.

2013 saw the introduction of edge trimagic total labelling (ETTL) of graphs by C. Jayasekaran et al. [7]. For a (α,β) graph Γ, an edge trimagic total labelling can be characterised as a bijection $\Phi: [V(\Gamma) \cup E(\Gamma)] \longrightarrow \{1, 2, 3,\cdots, \alpha+\beta\}$, so that the value of edge wights $\omega t_{\Phi}(\vartheta\omega) = [\Phi(\vartheta) + \Phi(\omega) + \Phi(\vartheta\omega)]$, for each edge $\vartheta\omega \in E(\Gamma)$, is equal to the distinct constant k_1 or k_2 or k_3. The authors of [8, 9, 10,11] demonstrated that certain graph classes have an edge trimagic total labeling.

Definition 1.1. A super edge trimagic total labeling (SETTL) of a graph Γ with α vertices and β edges is a bijection $\Phi: [V(\Gamma) \cup E(\Gamma)] \longrightarrow \{1, 2, 3,\cdots, \alpha+\beta\}$ such that for each edge $\vartheta\omega \in E(\Gamma)$, the value of the formula $[\Phi(\vartheta) + \Phi(\omega) + \Phi(\vartheta\omega)]$ is either K_1 or K_2 or K_3, with the additional condition that $\Phi: [V(\Gamma)] \longrightarrow \{1, 2, 3,\cdots, \alpha\}$. A super edge trimagic total graph is the one that allows a super edge trimagic total labeling.

As an illustration, C. Jayasekera et al. [12,13] proved that the generalized prism and web graph admit super edge trimagic total labeling. Furthermore, they found super edge trimagic total labeling of square of a cycle and gear graphs. In this paper, we try to push the super edge trimagic total labeling (SETTL) by finding more complicated graphs which possess super edge trimagic total labeling.

Definition 1.2. [14] For any two graphs $H = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $\Gamma = \{w_1, w_2, \ldots, w_k\}$. The Cartesian product, denoted by $H \boxtimes \Gamma$, is a new graph with vertex set $V(H \boxtimes \Gamma) = V(H) \times V(\Gamma)$ and the edge $\{(\lambda_1, w_1), (\lambda_2, w_2)\}$ is presented in the product whenever $\lambda_1\lambda_2 \in E(H)$ and $w_1 = w_2$ or symmetrically $w_1 w_2 \in E(\Gamma)$ and $\lambda_1 = \lambda_2$. The Cartesian product of a graph H with a path P_2 is called a prism over H. The ladder graph $L_2 = P_2 \boxtimes P_r$ is defined as a prism over a path P_r, while the circular ladder graph (prism) $\Pi_r = P_2 \boxtimes C_r$, see Fig 1.

Illustration: The Cartesian product of the path P_2 and the cycle C_5 gives the prism Π_5 is depicted in Fig 1.

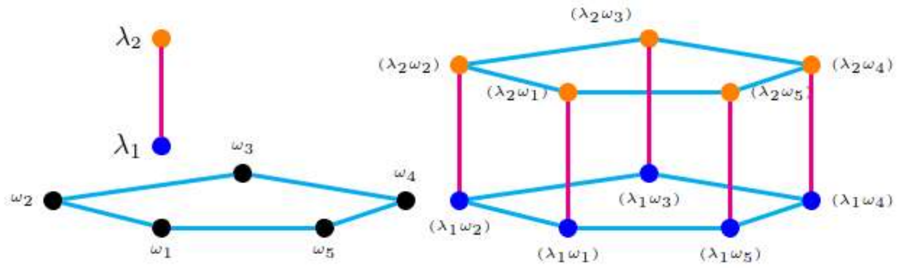


Figure 1: The Cartesian product of the path P_2 and the cycle C_5.

2. SUPER EDGE BIMAGIC TOTAL LABELING OF TRIANGULATED PRISM GRAPH $T\Pi_r$

Definition 2.1. The triangulated prism graph $T\Pi_r$ is a prism with additional edges $\omega_\kappa\lambda_{(\kappa+1)\text{mod } r}$, $\kappa \in [1, r]$, so the vertex set $V(T\Pi_r) = \{\lambda_\kappa, \omega_\kappa, \kappa \in [1, r]\}$ and edge set $E(T\Pi_r) = \{\lambda_\kappa\omega_\kappa, \lambda_\kappa\lambda_{(\kappa+1)\text{mod } \ell}, \omega_\kappa\omega_{(\kappa+1)\text{mod } r}, \omega_\kappa\lambda_{(\kappa+1)\text{mod } r}, \kappa \in [1, r]\}$. As a result, the graph $T\Pi_r$ has $\alpha = 2r$ vertices and $\beta = 4r$ edges.

Theorem 2.2. For any positive integer r, the triangulated prism graph $T\Pi_r$ is super edge bimagic total graph with bimagic numbers $K_1 = 6r$ and $K_2 = 8r$.

Proof: The suggested labelling transformation $\psi_1 : [(V \cup E)(\mathbf{T\Pi_r})] \longrightarrow \{1,2,3,\dots,6r\}$ is as follows:

$$\psi_1(\lambda_\kappa) = 2\kappa - 1, \qquad \kappa \in [1, r],$$

$$\psi_1(\omega_\kappa) = 2\kappa, \qquad \kappa \in [1, r],$$

$$\psi_1(\omega_\kappa\lambda_\kappa) = 6r - 4\kappa + 1, \qquad \kappa \in [1, r],$$

$$\psi_1(\omega_\kappa\omega_{(\kappa+1)\text{mod } r}) = \begin{cases} 6r - 4\kappa - 2 & \text{if} \quad \kappa \in [1, r-1]; \\ 6r - 2 & \text{if} \quad \kappa = r, \end{cases}$$

$$\psi_1(\lambda_\kappa\lambda_{(\kappa+1)\text{mod } r}) = \begin{cases} 6r - 4\kappa & \text{if} \quad \kappa \in [1, r-1]; \\ 6r & \text{if} \quad \kappa = r, \end{cases}$$

$$\psi_1(\omega_\kappa\lambda_{(\kappa+1)\text{mod } r}) = \begin{cases} 6r - 4\kappa - 1 & \text{if} \quad \kappa \in [1, r-1]; \\ 6r - 1 & \text{if} \quad \kappa = r, \end{cases}$$

To establish that the triangulated prism graph $T\Pi_r$ has bimagic constants, consider the edge $\omega_\kappa\lambda_\kappa$, $\kappa \in [1, r]$ then,

$$[\psi_1(\omega_\kappa) + \psi_1(\lambda_\kappa) + \psi_1(\omega_\kappa\lambda_\kappa)] = [(4\kappa - 1) + (6r - 4\kappa + 1)] = 6r = K_1.$$

For the edge $\lambda_\kappa\lambda_{\kappa+1}$, $\kappa \in [1, r-1]$ **then,**

$$[\psi_1(\lambda_\kappa) + \psi_1(\lambda_{\kappa+1}) + \psi_1(\lambda_\kappa\lambda_{\kappa+1})] = [(4\kappa) + (6r - 4\kappa)] = 6r = K_1.$$

For the edge $\lambda_\kappa\lambda_{(\kappa+1)mod\, r}$, $\kappa = r$ **then,**

$$[\psi_1(\lambda_\kappa) + \psi_1(\lambda_{(\kappa+1)mod\, r}) + \psi_1(\lambda_\kappa\lambda_{(\kappa+1)mod\, r})] = [(2r) + (6r)] = 8r = K_2$$

For the edge $\omega_\kappa\omega_{\kappa+1}$, $\kappa \in [1, r-1]$ **then,**

$$[\psi_1(\omega_\kappa)+\psi_1(\omega_{\kappa+1})+\psi_1(\omega_\kappa\omega_{\kappa+1})]=[(4\kappa+2)+(6r-4\kappa-2)]=6r=K_1$$

For the edge $\omega_\kappa\omega_{(\kappa+1)\bmod r}$, $\kappa=r$ **then,**

$$[\psi_1(\omega_\kappa)+\psi_1(\omega_{(\kappa+1)mod\, r})+\psi_1(\omega_\kappa\omega_{(\kappa+1)mod\, r})]=[(2r+2)+(6r-2)]=8r=K_2$$

For the edge $\omega_\kappa\lambda_{\kappa+1}$, $\kappa\in[1,r-1]$ **then,**

$$[\psi_1(\omega_\kappa)+\psi_1(\lambda_{\kappa+1})+\psi_1(\omega_\kappa\lambda_{\kappa+1})]=[(4\kappa+1)+(6r-4\kappa-1)]=6r=K_1$$

For the edge $\phi_\kappa\lambda_{(\kappa+1)\bmod r}$, $\kappa=r$ then,

$$[\boldsymbol{\psi}(\phi_\kappa)+\boldsymbol{\psi}(\lambda_{(\kappa+1)\bmod r})+\boldsymbol{\psi}(\phi_\kappa\lambda_{(\kappa+1)\bmod r})]=[(2r+1)+(6r-1)]=8r=K_2$$

Similarly, it can be shown that the formula $[\boldsymbol{\psi_1}(\lambda)+\boldsymbol{\psi_1}(\omega)+\boldsymbol{\psi_1}(\lambda\omega)]$, for each edge $\lambda\omega\in E(T\Pi_r)$, provides either of the bimagic constants $K_1=6r$ **or** $K_2=8r$ **.** Therefore, a super edge bimagic total labelling for all r is allowed by triangulated prism graph $T\Pi_r$.

Example 2.3. In Fig. 2, we display the SEBTL of the graph $T\Pi_8$ with bimagic numbers $K_1=48$ and $K_2=64$ and the graph $T\Pi_9$ with bimagic numbers $K_1=54$ and $K_2=72$.

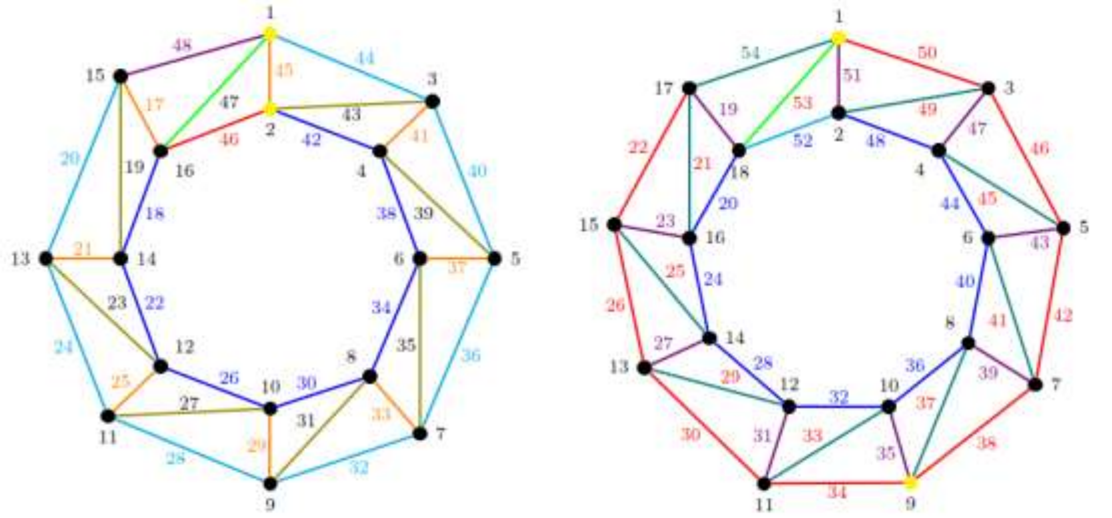


Figure 2: (a) $T\Pi_8$ with $K_1=48$ and $K_2=64$. **(b)** $T\Pi_9$ with $K_1=54$ and $K_2=72$.

3. SUPER EDGE TRIMAGIC TOTAL LABELING OF A TRIANGULATED WHEEL TW_r

Definition 3.1. The triangulated wheel graph TW_r , is a graph consists of a vertex set $V(TW_r)=\{\lambda_0\}\cup\{\lambda_\kappa,\omega_\kappa,\kappa\in[1,r]\}$and edge set $(TW_r)=\{\lambda_0\lambda_\kappa,\ \lambda_0\omega_\kappa,\ \lambda_\kappa\omega_\kappa,\ \lambda_\kappa\lambda_{(\kappa+1)\bmod r},\omega_\kappa\lambda_{(\kappa+1)\bmod r},\ \kappa\in[1,r]\}$. Thus, the graph TW_r has $\alpha=2r+1$ vertices and $\beta=5r$ edges.

Theorem 3.2. For any positive integer $\boldsymbol{r}$, the triangulated wheel graph $\mathbf{TW_r}$ is super edge trimagic total graph with trimagic numbers $\mathbf{K_1=7r+4}$, $\mathbf{K_2=4r+5}$ and $\mathbf{K_3=6r+4}$ when $\mathbf{r\equiv 1\bmod 2}, \mathbf{r\geq 3}$ while $\mathbf{K_3=6r+5}$ when $\mathbf{r\equiv 0\bmod 2},\ \mathbf{r\geq 4}$.

Proof: The suggested labelling transformation $\boldsymbol{\psi_2}:[(\mathbf{V}\cup\mathbf{E})(\mathbf{TW_r})]\longrightarrow\{\mathbf{1},\mathbf{2},\mathbf{3},\ldots,\mathbf{7r+1}\}$ is as follows**:**

$\boldsymbol{\psi_2(\lambda_0)=1},$

$\boldsymbol{\psi_2(\lambda_\kappa)=r+\kappa+1},$

$\boldsymbol{\psi_2(\omega_\kappa)=\kappa+1},$

$\boldsymbol{\psi_2(\lambda_0\omega_\kappa)=7r-\kappa+2}$, $\boldsymbol{\kappa\in[1,r]},$

$\boldsymbol{\psi_2(\lambda_0\lambda_\kappa)=6r-\kappa+2}$, $\boldsymbol{\kappa\in[1,r]},$

$$\psi_2(\lambda_\kappa\lambda_{(\kappa+1)\bmod r}) = \begin{cases} 4r-2\kappa+2 & \text{if} \quad \kappa\in\left[1,\frac{r}{2}\right], r \text{ is even} \\ 4r-2\kappa+1 & \text{if} \quad \kappa\in\left[1,\frac{r-1}{2}\right], \ r \text{ is odd}; \\ 5r-2\kappa+1 & \text{for} \quad \kappa\in\left[\frac{r}{2}+1, r-1\right], \quad r \text{ is even} \\ & \text{or} \quad \kappa\in\left[\frac{r+1}{2}, r-1\right], \quad r \text{ is odd}; \\ 4r+1 & \text{if} \quad \kappa = r; \end{cases}$$

$$\psi_2(\omega_\kappa\lambda_\kappa) = \begin{cases} 3r-2\kappa+3 & \text{for} \quad \kappa\in\left[1,\frac{r}{2}\right], \quad \ell \text{ is even}; \\ & \text{or} \quad \kappa\in\left[1,\frac{r-1}{2}\right], \quad \ell \text{ is odd}; \\ 6r-2\kappa+2 & \text{for} \quad \kappa\in\left[\frac{r}{2}+1, r\right], \quad \ell \text{ is even}; \\ & \text{or} \quad \kappa\in\left[\frac{r+1}{2}, r\right], \quad \ell \text{ is odd}; \end{cases}$$

$$\psi_2(\omega_\kappa\lambda_{(\kappa+1)\bmod r}) = \begin{cases} 3r-2\kappa+2 & \text{for} \quad \kappa\in\left[1,\frac{r}{2}\right], \ r \text{ is even} \\ & \text{or} \quad \kappa\in\left[1,\frac{r-1}{2}\right], \ r \text{ is odd}; \\ 6r-2\kappa+1 & \text{for} \quad \kappa\in\left[\frac{r}{2}+1, r-1\right], \ r \text{ is even} \\ & \text{or} \quad \kappa\in\left[\frac{r+1}{2}, r-1\right], \ r \text{ is odd}; \\ 5r+1 & \text{if} \quad \kappa = r, \quad r \text{ is even}; \\ 2r+2 & \text{if} \quad \kappa = r, \quad r \text{ is odd}. \end{cases}$$

To establish that the triangulated wheel graph TW_r *has trimagic total constants, consider the edge* $\lambda_0\omega_\kappa$, $\kappa \in [1, r]$ then **,**

$$[\psi_2(\lambda_0) + \psi_2(\omega_\kappa) + \psi_2(\lambda_0\omega_\kappa)] = [(\kappa+2) + (7r-\kappa+2)] = 7r+4 = K_1$$

For the edge $\boldsymbol{\omega_\kappa\lambda_\kappa}$**,** $\boldsymbol{\kappa \in \left[1, \frac{r-1}{2}\right]}$**,** and r is odd, then

$$[\psi_2(\omega_\kappa) + \psi_2(\lambda_\kappa) + \psi_2(\omega_\kappa\lambda_\kappa)] = [(r+2\kappa+2) + (3r-2\kappa+3)] = 4r+5 = K_2$$

For the edge $\boldsymbol{\lambda_\kappa\lambda_{\kappa+1}}$, $\boldsymbol{\kappa \in \left[1, \frac{r-1}{2}\right]}$, and r is odd, then

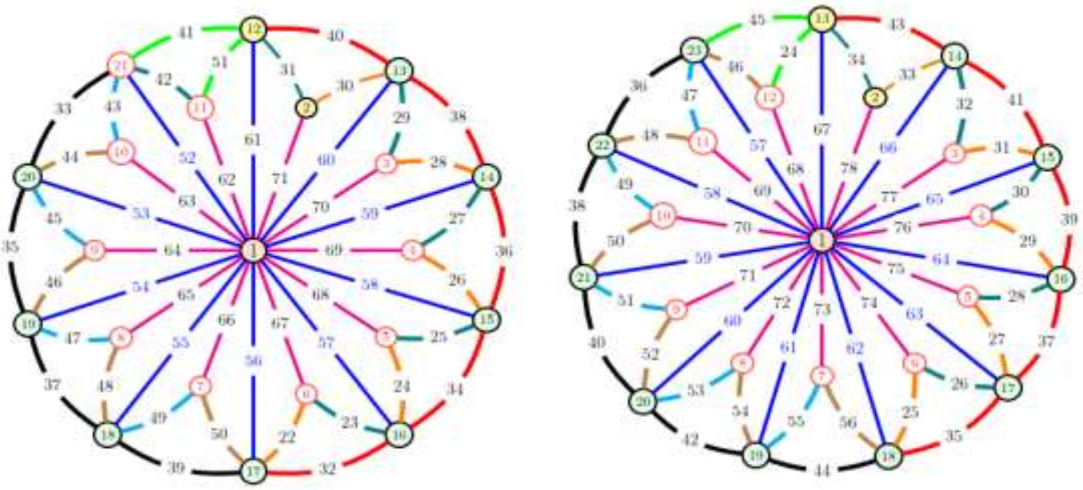
$$[\psi_2(\lambda_\kappa) + \psi_2(\lambda_{\kappa+1}) + \psi_2(\lambda_\kappa\lambda_{\kappa+1})] = [(2r+2\kappa+3) + (4r-2\kappa+1)] = 6r+4 = K_3$$

For the edge $\boldsymbol{\lambda_\kappa\lambda_{\kappa+1}}$, $\boldsymbol{\kappa \in \left[1, \frac{r}{2}\right]}$, and r is even, then

$$[\psi_2(\lambda_\kappa) + \psi_2(\lambda_{\kappa+1}) + \psi_2(\lambda_\kappa\lambda_{\kappa+1})] = [(2r+2\kappa+3) + (4r-2\kappa+2)] = 6r+5 = K_3$$

Similarly, it can be shown that the expression $[\boldsymbol{\psi_2}(\lambda) + \boldsymbol{\psi_2}(\omega) + \boldsymbol{\psi_2}(\lambda\omega)]$, for each edge $\lambda\omega \in E(\mathbf{TW}_\ell)$, yields either of the magic constants $\mathbf{K_1 = 7r+4}$, $\mathbf{K_2 = 4r+5}$ and $\boldsymbol{K_3 = 6r+4}$ or $\boldsymbol{K_3 = 6r+5}$. Therefore, a super edge trimagic total labeling for all r is allowed by the triangulated wheel graph TW_r.

Example 3.3. In Fig. 3, we display the SETHL of the triangulated wheel TW_{10} with trimagic total numbers $K_1 = 74$, $K_2 = 45$ and $K_3 = 65$ and the triangulated wheel TW_{11} with trimagic total numbers $K_1 = 81$, $K_2 = 49$ and $K_3 = 70$



((b

Figure 3: (a) TW_{10} with $K_1 = 74$, $K_2 = 45$ and $K_3 = 65$ **(b)** TW_{11} with $K_1 = 81$, $K_2 = 49$ and $K_3 = 70$.

4. SUPER EDGE TRIMAGIC TOTAL LABELING OF A DOUBLE VERTEX WHEEL GRAPH $\boldsymbol{DW_r}$

Definition 4.1. [2] The double vertex wheel graph , denoted by $\boldsymbol{DW_r}$, is a graph consists of a vertex set $\mathbf{V}(\boldsymbol{DW_r}) = \{\boldsymbol{\nu_0}\} \cup \{\boldsymbol{\mu_0}\} \cup \{\boldsymbol{\lambda_\kappa},\ \boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]\}$ and an edge set $\mathbf{E}(\boldsymbol{DW_r}) = \{\boldsymbol{\nu_0\lambda_\kappa}, \boldsymbol{\mu_0\lambda_\kappa}, \boldsymbol{\lambda_\kappa\lambda_{(\kappa+1) \bmod r}}, \boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]\}$. Thus, the graph $\boldsymbol{DW_r}$ has $\boldsymbol{\alpha} = \mathbf{r} + \mathbf{2}$ vertices and $\boldsymbol{\beta} = \mathbf{3r}$ edges.

Theorem 4.2.

For any positive integer r, the double vertex wheel graph DW_r is super edge trimagic total graph with trimagic total numbers $K_1 = 4r + 5, K_2 = 3r + 4$ and $K_3 = 2r + 6$ when $r \equiv 1 \bmod 2, r \geq 3$ while $K_3 = 2r + 5$ when $r \equiv 0 \bmod 2$, $r \geq 4$.

Proof: The suggested labelling transformation $\boldsymbol{\psi_3} : [(\mathbf{V} \cup \mathbf{E})(\boldsymbol{DW_r})] \longrightarrow \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{4r+2}\}$ is as follows:

$\boldsymbol{\psi_3(\nu_0) = 1,}$

$\boldsymbol{\psi_3(\mu_0) = r + 2,}$

$\boldsymbol{\psi_3(\lambda_\kappa) = \kappa + 1,,} \qquad \boldsymbol{\kappa \in [1, r],}$

$\boldsymbol{\psi_3(\nu_0\lambda_\kappa) = 4r - \kappa + 3\,,} \qquad \boldsymbol{\kappa \in [1, r],}$

$\boldsymbol{\psi_3(\mu_0\lambda_\kappa) = 3r - \kappa + 2\,,} \qquad \boldsymbol{\kappa \in [1, r],}$

$$\psi_3(\lambda_\kappa\lambda_{(\kappa+1)\bmod r}) = \begin{cases} 2r-2\kappa+2 & if \quad \kappa\in\left[1,\frac{r}{2}-1\right],\ r\ is\ even \\ 2r-2\kappa+3 & \text{if} \quad \kappa\in\left[1,\frac{r-1}{2}\right],\ \text{r is odd}; \\ 3r-2\kappa+1 & \text{for} \quad \kappa\in\left[\frac{r}{2},r-1\right],\ \text{r is even}; \\ & \text{or} \quad \kappa\in\left[\frac{r+1}{2},r-1\right],\ \text{r is odd}; \\ 3r+2 & \text{if} \quad \kappa=r\ ; \end{cases}$$

To establish that the double vertex wheel DW_r, has trimagic total constant, consider the edge $\nu_0\lambda_\kappa$, $\kappa\in[1,r]$ then,

$$[\psi_3(\nu_0)+\psi_3(\lambda_\kappa)+\psi_3(\nu_0\lambda_\kappa)] = [(\kappa+2)+(4r-\kappa+3)] = 4r+5 = K_1$$

For the edge $\lambda_\kappa\lambda_{\kappa+1}$, $\kappa\in\left[1,\frac{r}{2}-1\right]$ and r is even, then,

$$[\psi_3(\lambda_\kappa)+\psi_3(\lambda_{\kappa+1})+\psi_3(\lambda_\kappa\lambda_{\kappa+1})] = [(2\kappa+3)+(2r-2\kappa+2)] = 2r+5 = K_3$$

For the edge $\lambda_\kappa\lambda_{\kappa+1}$, $\kappa\in\left[\frac{r}{2},r-1\right]$ and r is even, then,

$$[\psi_3(\lambda_\kappa)+\psi_3(\lambda_{\kappa+1})+\psi_3(\lambda_\kappa\lambda_{\kappa+1})] = [(2\kappa+3)+(3r-2\kappa+1)] = 3r+4 = K_2$$

For the edge $\lambda_\kappa\lambda_{\kappa+1}$, $\kappa\in\left[1,\frac{r-1}{2}\right]$ and r is odd, then,

$$[\psi_3(\lambda_\kappa)+\psi_3(\lambda_{\kappa+1})+\psi_3(\lambda_\kappa\lambda_{\kappa+1})] = [(2\kappa+3)+(2r-2\kappa+3)] = 2r+6 = K_3$$

By the same way, it can be shown that the expression $[\psi_3(\lambda)+\psi_3(\nu)+\psi_3(\lambda\nu)]$, for each edge $\lambda\nu\in E(DW_r)$, yields either of the magic constants $K_1=4r+5$, $K_2=3r+4$ **and** $K_3=2r+6$ or $K_3=2r+5$. Therefore, super edge trimagic total labeling for all r is allowed by the double vertex wheel graph DW_r.

Example 4.3.

In Fig. 4, we display the SETTL of the double vertex wheel DW_{10} with trimagic total numbers $K_1=45$, $K_2=34$ and $K_3=25$ and the double vertex wheel DW_{11} with trimagic numbers $K_1=49$, $K_2=37$ and $K_3=28$.

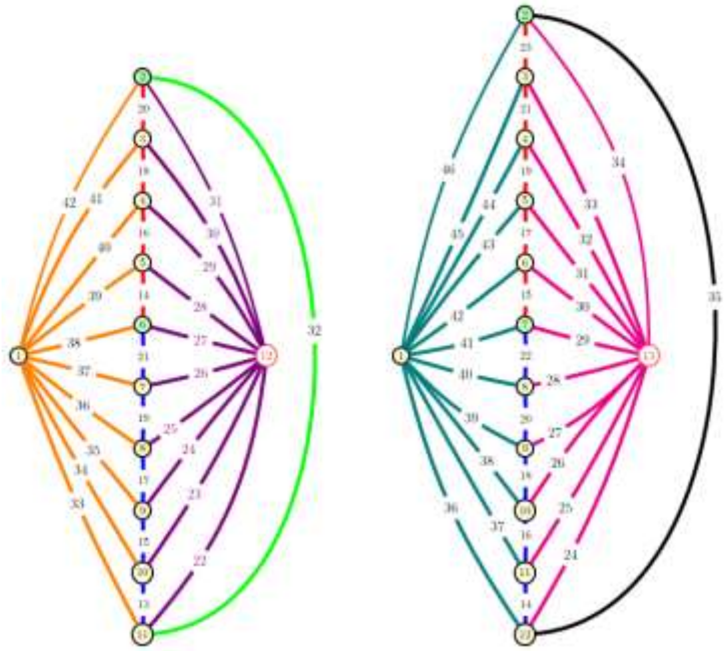


Figure 4: (a) DW_{10} with $K_1 = 45$, $K_2 = 34$ and $K_3 = 25$, **(b)** DW_{11} with $K_1 = 49$, $K_2 = 37$ and $K_3 = 28$.

5. SUPER EDGE TRIMAGIC TOTAL LABELING OF THE CLOSED TRIANGULATED WATER WHEEL GRAPH $CTWW_r$

Definition 5.1. [2] The closed triangulated water wheel graph $\mathbf{CTWW_r}$, is a graph consists of a vertex set $\mathbf{V(CTWW_r)} = \{\boldsymbol{\lambda_0}\} \cup \{\boldsymbol{\lambda_\kappa}, \boldsymbol{\nu_\kappa}, \boldsymbol{\omega_\kappa}, \ \boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]\}$ and edge set $\mathbf{E(CTWW_\ell)} = \{\boldsymbol{\lambda_0\lambda_\kappa}, \boldsymbol{\lambda_0\nu_\kappa}, \boldsymbol{\lambda_0\omega_\kappa}, \boldsymbol{\lambda_\kappa\omega_\kappa}, \boldsymbol{\nu_\kappa\omega_\kappa}, \boldsymbol{\nu_\kappa\lambda_{(\kappa+1)\bmod r}}, \ \boldsymbol{w_\kappa w_{(\kappa+1)\bmod r}}, \ \boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]\}$. As a result, the graph $\mathbf{CTWW_r}$ has $\boldsymbol{\alpha} = \mathbf{3r+1}$ vertices and $\boldsymbol{\beta} = \mathbf{7r}$ edges, see Fig. 5.

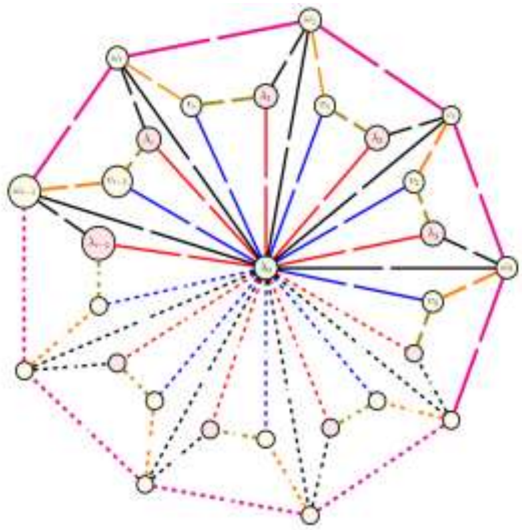


Figure 5: The graph $CTWW_9$ with standard labeling.

Theorem 5.2.

(1) when $r \equiv 1 \ mod\ 2$, $r \geq 3$, the closed triangulated water wheel $CTWW_r$ is edge bimagic total graph with bimagic numbers $K_1 = 10r + 4$ and $K_2 = 6r + 4$.
(2) when $r \equiv 0 \ mod\ 2$, $r \geq 4$, the closed triangulated water wheel $CTWW_\ell$ is edge trimagic total graph with trimagic numbers $K_1 = 10r + 4$, $K_2 = 6r + 4$ and $K_3 = 8r + 5$.

Proof: The suggested labelling transformation $\psi_4 : [(V \cup E)(CTWW_r)] \longrightarrow \{1, 2, 3, \dots, 10r + 1\}$ is as follows:

$\psi_4(\lambda_0) = 1,$

$\psi_4(\lambda_\kappa) = r + \kappa + 1,$

$\psi_4(\nu_\kappa) = \kappa + 1,$

$$\psi_4(\omega_\kappa) = \begin{cases} 2r + \kappa + 2 & \text{if} \quad \kappa \in [1, r-1]; \\ 2r + 2 & \text{if} \quad \kappa = r, \end{cases}$$

$\psi_4(\lambda_0\lambda_\kappa) = 9r - \kappa + 2, \quad \kappa \in [1, r],$

$\psi_4(\lambda_0\nu_\kappa) = 10r - \kappa + 2, \quad \kappa \in [1, r],$

$$\psi_4(\lambda_0\omega_\kappa) = \begin{cases} 8r - \kappa + 1 & \text{if} \quad \kappa \in [1, r-1]; \\ 8r + 1 & \text{if} \quad \kappa = r, \end{cases}$$

$$\psi_4(\nu_\kappa\omega_\kappa) = \begin{cases} 4r - 2\kappa + 1 & \text{for} \quad \kappa \in \left[1, \frac{r}{2} - 1\right], \ r \text{ is even}; \\ & \text{or} \quad \kappa \in \left[1, \frac{r-1}{2}\right], \ r \text{ is odd}; \\ 8r - 2\kappa + 1 & \text{for} \quad \kappa \in \left[\frac{r}{2}, r - 1\right], \ r \text{ is even}; \\ & \text{or} \quad \kappa \in \left[\frac{r+1}{2}, r - 1\right], \ r \text{ is odd}; \\ 5r + 2 & \text{if} \quad \kappa = r, \quad r \text{ is even}; \\ 7r + 1 & \text{if} \quad \kappa = r, \quad r \text{ is odd}; \end{cases}$$

$$\psi_4(\lambda_\kappa\omega_\kappa) = \begin{cases} 5r - 2\kappa + 2 & \text{if} \quad \kappa \in \left[1, \frac{r}{2} - 1\right], \ r \text{ is even}; \\ 7r - 2\kappa + 1 & \text{for} \quad \kappa \in \left[\frac{r}{2}, r - 1\right], \ r \text{ is even}; \\ & \text{or} \quad \kappa \in [1, r - 1], \ r \text{ is odd}; \\ 4r + 2 & \text{if} \quad \kappa = r, \quad r \text{ is even}; \\ 6r + 1 & \text{if} \quad \kappa = r, \quad r \text{ is odd}; \end{cases}$$

$$\psi_4(\nu_\kappa\lambda_{\kappa+1}) = \begin{cases} 7r - 2\kappa + 2 & \text{if} \quad \kappa \in [1, r - 1], \quad r \text{ is even}; \\ 5r - 2\kappa + 1 & \text{if} \quad \kappa \in [1, r - 1], \quad r \text{ is odd}; \\ 4r + 1 & \text{if} \quad \kappa = r, \end{cases}$$

$$\psi_4(\omega_\kappa\omega_{\kappa+1}) = \begin{cases} 4r + 2 & \text{if} \quad \kappa \in \left[1, \frac{r}{2} - 2\right], \ r \text{ is even}; \\ 6r - 2\kappa - 1 & \text{for} \quad \kappa \in \left[\frac{r}{2} - 1, r - 2\right], \ r \text{ is even}; \\ & \text{or} \quad \kappa \in [1, r - 2], \ r \text{ is odd}; \\ 3r + 2 & \text{if} \quad \kappa = r - 1, \ r \text{ is even}; \\ 5r + 1 & \text{if} \quad \kappa = r - 1, \ r \text{ is odd}; \\ 4r & \text{if} \quad \kappa = r, \quad r \text{ is even}; \\ 6r - 1 & \text{if} \quad \kappa = r, \quad r \text{ is odd}. \end{cases}$$

Case: $(\mathbf{1})$ When $\mathbf{r} \equiv \mathbf{1} \bmod \mathbf{2}, \mathbf{r} \geq \mathbf{3},$ to establish that the closed triangulated water wheel graph $\mathbf{CTWW_r}$ **has bimagic total constant the edge** $\boldsymbol{\lambda_0 \nu_\kappa}$, $\boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]$ **, then**

$$[\boldsymbol{\psi_4(\lambda_0) + \psi_4(\nu_\kappa) + \psi_4(\lambda_0 \nu \nu_\kappa)}] = [(\boldsymbol{\kappa} + \mathbf{2}) + (\mathbf{10r} - \boldsymbol{\kappa} + \mathbf{2})] = \mathbf{10r + 4 = K_1}.$$

For the edge $\boldsymbol{\nu_\kappa \omega_\kappa}$ **,** $\boldsymbol{\kappa} \in \left[\mathbf{1}, \frac{\mathbf{r-1}}{\mathbf{2}}\right]$**, then**

$$[\boldsymbol{\psi_4(\nu_\kappa) + \psi_4(\omega_\kappa) + \psi_4(\nu_\kappa \omega_\kappa)}] = [(\mathbf{2r + 2\boldsymbol{\kappa} + 3}) + (\mathbf{4r - 2\boldsymbol{\kappa} + 1})] = \mathbf{6r + 4 = K_2}.$$

By the same way, we can prove that, for each edge $\boldsymbol{\lambda\omega} \in \mathbf{E(CTWW_r)}$, the outcomes of the formula $[(\boldsymbol{\psi_4(\lambda) + \psi_4(\omega)) + \psi_4(\lambda\omega)}]$, provides either of the magic constant $\mathbf{K_1 = 10r + 4}$ **and** $\mathbf{K_2 = 6r + 4}$ **.**

Case: $(\mathbf{2})$ When $\mathbf{r} \equiv \mathbf{0} \bmod \mathbf{2}, \mathbf{r} \geq \mathbf{4}$, to establish that the closed triangulated water wheel graph $\mathbf{CTWW_r}$ **has trimagic total constants the edge** $\boldsymbol{\omega_\kappa \omega_{\kappa+1}}$, and $\boldsymbol{\kappa} \in \left[\mathbf{1}, \frac{\mathbf{r}}{\mathbf{2}} - \mathbf{2}\right]$**, then**

$$[\boldsymbol{\psi_4(\omega_\kappa) + \psi_4(\omega_\kappa) + \psi_4(\omega_\kappa \omega_{\kappa+1})}] = [(\mathbf{4r + 2\boldsymbol{\kappa} + 5}) + (\mathbf{4r - 2\boldsymbol{\kappa}})] = \mathbf{8r + 5 = K_3}$$

For the **edge** $\boldsymbol{\lambda_0 \nu_\kappa}$, and $\boldsymbol{\kappa} \in [\mathbf{1}, \mathbf{r}]$ **then**

$$[\boldsymbol{\psi_4(\lambda_0) + \psi_4(\nu_\kappa) + \psi_4(\lambda_0 \nu_\kappa)}] = [(\boldsymbol{\kappa} + \mathbf{2}) + (\mathbf{10r} - \boldsymbol{\kappa} + \mathbf{2})] = \mathbf{10r + 4 = K_1}$$

For the edge $\boldsymbol{\nu_\kappa \omega_\kappa}$, and $\boldsymbol{\kappa} \in \left[\mathbf{1}, \frac{\mathbf{r}}{\mathbf{2}} - \mathbf{1}\right]$ **then**

$$[\boldsymbol{\psi_4(\nu_\kappa) + \psi_4(\omega_\kappa) + \psi_4(\nu_\kappa \omega_\kappa)}] = [(\mathbf{2r + 2\boldsymbol{\kappa} + 3}) + (\mathbf{4r - 2\boldsymbol{\kappa} + 1})] = \mathbf{6r + 4 = K_2}$$

Similarly, it can be proved that, for each edge $\boldsymbol{\lambda\omega} \in \mathbf{E(CTWW_r)}$, the outcomes of the expression $[(\boldsymbol{\psi_4(\lambda) + \psi_4(\omega)) + \psi_4(\lambda\omega)}]$, yields either of the magic constant $\mathbf{K_1 = 10r + 4}, \mathbf{K_2 = 6r + 4}$ **and** $\mathbf{K_3 = 8r + 5}$ **. Therefore, a super edge trimagic total labelling for all** r **is allowed by a closed triangulated water wheel graph** $\mathbf{CTWW_r}$.

Example 5.3. In Fig. 6, we display the SEBTL of the closed triangulated water wheel graph $\boldsymbol{CTWW_9}$ with bimagic total numbers $\boldsymbol{K_1 = 94}$, and $\boldsymbol{K_2 = 58}$.

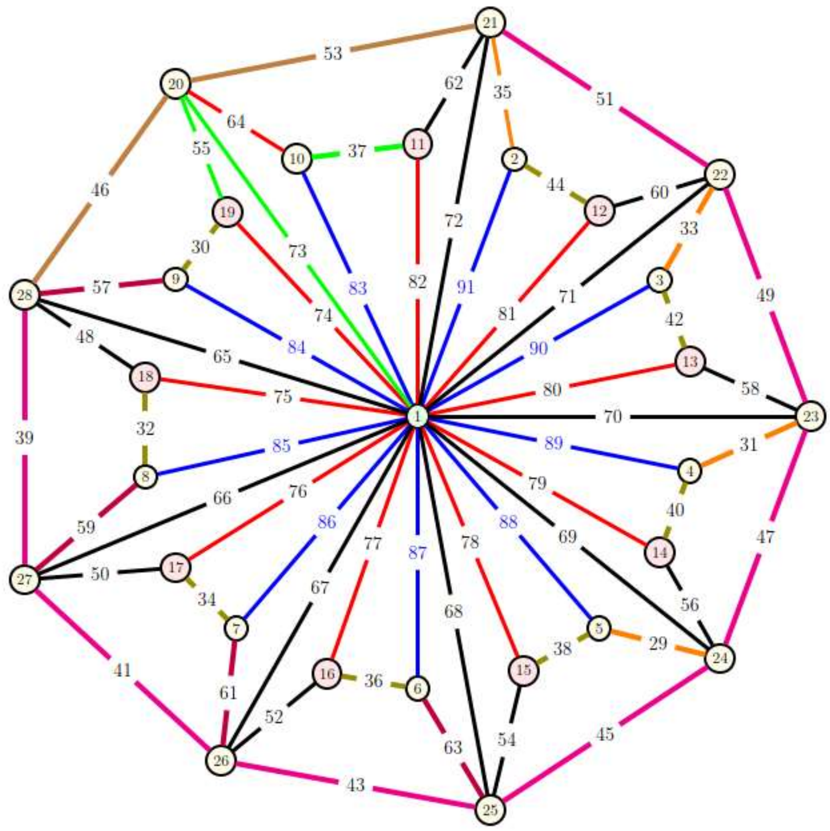


Figure 6: The graph $CTWW_9$ with bimagic total numbers $K_1 = 94$ and $K_2 = 58$.

Application of SEBTL and SETTL

Magic numbers can be used as secret numbers for any credit card or money safe in the manner described as follows: a graph is chosen from the graphs in this paper for each month of the year. The number of days in a month is represented by r, the number of vertices in that graph. No matter how odd or even the day is, the number of vertices r will change based on how the days of the month change. Therefore, we have secret numbers (trimagic total numbers) K_1, K_2 and K_3 for each day of the month that is different from the day before it. The following month, we picked a different graph from the collection, and r changes according to the day of the month. This leads to a special type of encryption and hacker-proof protection.

6. CONCLUSION

The novelty of the paper is to develop the concept of super bimagic total labelling of connected graphs. We demonstrate that the triangulated prism graph possesses super bimagic total labelling with bimagic total labelling using the bimagic numbers $\mathbf{K_1 = 6r}$ and $\mathbf{K_2 = 8r}$. Also, we introduce the concept of super edge trimagic total graph labelling. As a result, we discovered several complex networks with trimagic total numbers, including the triangulated wheel graph, the double vertex wheel graph, and the closed triangulated water wheel graph. The broader implication of introducing these labelings is to identifying new secret numbers for any issues.

7. REFERENCES

[1] J. Gross, and J. Yellen, Graph Theory and Its Applications; CRC Press: London, UK,1999.

[2] S. R. Kingan, Graphs and Networks, 2^{nd} edn. , Jhon Wiley and Sons. Inc., U.S. A. 2022.

[3] B.D. Acharya, S. Arumugam, and A. Rosa, Labeling of Discrete Structures and Applications; Narosa Publishing House: New Delhi, India 2008; pp. 1 –14.

[4] A. Rosa, On Certain Valuations of the Vertices of a Graph; Theory of Graphs (Internat. Symp, Rome, July 1966); Gordan and Breach: New York, NY, USA; Paris, France, 1967; pp. 349 –355.

[5] J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, (2022).

[6] J. B. Babujee, On edge bimagic labeling, Journal of Combinatorics Information and System Sciences, 1-4 (2004) 239-244.

[7] C. Jayasekaran, M. Regees and C. Davidraj, Edge trimagic labeling of some graphs, Intern. J. Combinatorial Graph Theory and Applications 6(2) (2013) 175-186.

[8] 6. C. Jayasekaran and J. Little Flower, On Edge Trimagic Labeling of Umbrella, Dumb bell and Circular Ladder Graphs, Annals of Pure and Applied Mathematics, 13(1) (2017) 73-87.

[9] C. Jayasekaran and J. Little Flower, Edge Trimagic Total Labeling of Mobius Ladder, Book and Dragon Graphs, Annals of Pure and Applied Mathematics, 13(2) (2017) 151-163.

[10] M. Regees and C. Jayasekaran, More Results On Edge Trimagic Labeling of Graphs, International Journal of Mathematical Archive, 4(11) (2013), 247-255.

[11] C. Jayasekaran, S. Robinson Chellathurai and J. Little Flower, Edge trimagic total labeling of cycle related graphs, International Journal of Pure and Applied Mathematics, 120(4) (2018) 511-519.

[12] M. Regees and C. Jayasekaran, Super Edge Trimagic Total Labeling of Generalized Prism and Web Graphs, Journal of Discrete Math. Sci. and Cryptography, 19 (2016) no.1, 81-92.

[13] M. Regees and C. Jayasekaran, Super Edge Trimagic Total Labeling of Square of a cycle and Gear Graphs, Mathematical Sciences International Research Journal, 2 (2015) 113- 115.

[14] R. Hammack, W. Imrich ,S. Klavzar, Handbook of Product Graph Theory 2nd edn., CRC Press . Taylor and Francis Group, 2011.