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Pairwise Generalized Semi Operation Continuous Mappings

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ABSTRACT

Our aim of this paper is to continue the study of operations on generalized semi open sets in bitopological spaces which was initiated in 2024 by Khedr. New types of pairwise continuous and pairwise irresolute mappings between two bitopological spaces by the concept of operations on generalized semi open sets are introduced with some of their properties and the relation between them are studied. Firstly, we introduce two concepts of pairwise generalized semi operation continuous mappings between two bitopological spaces with some of their properties and study the relation between them. Secondly, we introduce two concepts of pairwise generalized semi operation irresolute mappings between two bitopological spaces with some of their properties and study the relation between them. Finally, we introduce the notions of pairwise generalized semi operation closed mappings and pairwise generalized semi operation open mappings between two bitopological spaces and study some results and theorems.

1- INTRODUCTION

The concept of operations on topological spaces was introduced in 1979 by Kasahara [1]. Ogata [2] introduced the concept of operation-open sets and operation-continuous mappings. Many authors studied continuous and irresolute mappings using operations on topological spaces (see [3]).

Levine [4] in 1963 initiated the study of semi open sets in topological spaces. Arya and Nour [5] in 1990 initiated the study of generalized semi closed sets. Mappings and especially continuous and irresolute mappings stand among the most researched points in the whole of the mathematical sciences. Levine [4] and Crossley [6] introduced the notion of semi-continuous and irresolute mappings between topological spaces, respectively. After that many authors continued the study of other different forms of continuous mappings between topological spaces (see [7]).

The concept of a bitopological space was first introduced in 1963 by Kelly [8] and thereafter a large number of papers have been done to generalize the topological concepts to bitopological spaces. Bose [9] in 1981 introduced semi open sets and semi continuity in bitopological spaces. Khedr [10] extended the concept of generalized semi closed sets to bitopological settings. Tallafha [11] in 1999 studied continuous and pairwise continuous mappings between bitopological spaces. After that continuous and irresolute mappings between bitopological spaces were studied by many authors (see [12]). Operations on bitopological spaces were discussed in some manner [13] and [14]. A different technique to study operations on bitopological spaces is found in [15]. Khedr et al.[15] in 2010 studied continuous mappings between bitopological spaces using the concept of operations.

The aim of this paper is to continue the study of operations on generalized semi open sets in bitopological spaces which was initiated in 2024 by Khedr [16]. This paper is organized as follows: In Section 2, some definitions and results which are required to make this work self-contained are recalled. Section 3 is devoted to the study of pairwise generalized semi operation continuous and pairwise generalized semi operation irresolute mappings.

Throughout this paper $(\Upsilon, \tau_1, \tau_2)$ and $(Ƶ, \sigma_1, \sigma_2)$ (or briefly Υ and $Ƶ$) denote bitopological spaces. For a subset H of a bitopological space Υ, we shall denote the interior and the closure of H respect to τ_i (or σ_i) by $i - Int(\mathrm{H})$ and $i - Cl(\mathrm{H})$, respectively. Also, $i, j = 1,2, i \neq j$ and id denotes the identity map.

2- Preliminaries

In this section, we recall some concepts occurring in the papers [9], [10] and [16] which will be needed in the sequel.

Definition 2.1 [9]

A subset B of a bitopological space Υ is called ij −semi open (briefly $ij - so$) if there is $H \in \tau_i$ such that $\mathrm{H} \subset \mathrm{B} \subset j - Cl(\mathrm{H})$ or equivalently $\mathrm{B} \subset j - Cl(i - Int(\mathrm{B}))$.

$ij - SO(\Upsilon)$ denotes the family of all ij −so subsets of Υ. For $y \in \Upsilon$, $ij - SO(\Upsilon, y)$ denotes the family of all ij −so subsets of Υ containing y. The complement of an ij −so set is called $ij-$ semi closed (briefly $ij-sc$). The $ij-$semi closure of B, denoted by $ij-sCl$(B), is the intersection of all $ij-$sc subsets of ϒ containing B.

Definition 2.2 [10]

A subset B of a bitopological space ϒ is called $ij-$generalized semi closed (briefly $ij-gs-$closed) if $ji-sCl(B)\subset$ H whenever B $\subset$ H and H $\in \tau_i$.

If B is $ij-gs-$closed and $ji-gs-$closed, then it is called pairwise $gs-$closed. $ij-GSC$(ϒ) denotes the family of all $ij-$generalized semi closed subsets of ϒ. Every τ_j-closed subset of ϒ is $ij-$generalized semi closed. The complement of an $ij-gs-$closed set is called $ij-$generalized semi open (briefly $ij-gs-$open). $ij-GSO$(ϒ) denotes the family of all $ij-$generalized semi open subsets of ϒ.

Let $P-GSO(ϒ)=12-GSO(ϒ)\cup 21-GSO(ϒ)$.

Definition 2.3 [10]

Let B be a subset of a bitopological space ϒ. The $ij-$generalized semi closure of B, denoted by $ij-gsCl$(B), is the intersection of all $ij-gs-$closed subsets of ϒ containing B.

Definition 2.4 [16]

Let ϒ be a bitopological space. An operation κ on $P-GSO$(ϒ) is a mapping $\kappa: P-GSO(ϒ)\rightarrow P(ϒ)$ such that H $\subset$ H$^\kappa$ for every H $\in P-GSO$(ϒ), where H$^\kappa$ denotes the value of κ at H.

The operators H$^\kappa=$ H and H$^\kappa=j-Cl$(H) for H $\in ij-GSO$(ϒ) are operations on $P-GSO$(ϒ).

It is known that $\tau_1\cup\tau_2\subset P-GSO$(ϒ). Then if we restrict the κ operation to $\tau_1\cup\tau_2$, we obtain the γ operation in [15].

Definition 2.5 [16]

A subset B of a bitopological space ϒ is called $ij-gs\kappa-$open if for every $y\in B$, there is H $\in ij-GSO$(ϒ) with $y\in$ H such that H$^\kappa\subset B$.

If B is $ij-gs\kappa$ −open and $ji-gs\kappa$ −open, then it is called pairwise $gs\kappa$ −open. $ij-GS\kappa O(\Upsilon)$ denotes the family of all $ij-gs\kappa$ −open subsets of Υ. In any bitopological space Υ, we remark that Υ and Φ are $ij-gs\kappa$ −open sets. The complement of an $ij-gs\kappa$ −open set is $ij-gs\kappa$ −closed. If B is $ij-gs\kappa$ −closed and $ji-gs\kappa$ −closed, then it is called pairwise $gs\kappa$ −closed. $ij-GS\kappa C(\Upsilon)$ denotes the family of all $ij-gs\kappa$ −closed subsets of Υ.

If $H^{\kappa}=H$ for $H\in ij-GSO(\Upsilon)$, then the concept of an $ij-gs\kappa$ −open set coincide with the concept of an $ij-gs$ −open set [10] and if $H^{\kappa}=j-Cl(H)$ for $H\in ij-GSO(\Upsilon)$, then an $ij-gs\kappa$ −open set is called $ij-gs\theta$ −open.

Definition 2.6 [16]

Let Υ be a bitopological space. An operation κ on $P-GSO(\Upsilon)$ is called $ij-gs\kappa$ −open if for every $y\in\Upsilon$ and for every $H\in ij-GSO(\Upsilon)$ with $y\in H$, there is $S\in ij-GS\kappa O(\Upsilon)$ with $y\in S$ such that $S\subset H^{\kappa}$.

If κ is $ij-gs\kappa$ −open and $ji-gs\kappa$ −open, then it is called pairwise $gs\kappa$ −open.

If $H^{\kappa}=H$ for $H\in ij-GSO(\Upsilon)$, then an $ij-gs\kappa$ −open operation is called $ij-gs$ −open and if $H=j-Cl(H)$ for $H\in ij-GSO(\Upsilon)$, then an $ij-gs\kappa$ −open operation is called $ij-gs\theta$ −open.

Definition 2.7 [16]

Let B be a subset of a bitopological space Υ, $y\in\Upsilon$ and κ be an operation on $P-GSO(\Upsilon)$. Then y is said to be an $ij-\kappa gs$ closure point of B if $H^{\kappa}\cap B\neq\Phi$ for every $H\in ij-GSO(\Upsilon)$ with $y\in H$. The set of all $ij-\kappa gs$ closure points of B is called $ij-\kappa gs$ closure of B (briefly $ij-\kappa gsCl(B)$).

If $H^{\kappa}=H$ for $H\in ij-GSO(\Upsilon)$, then the concept of $ij-\kappa gsCl(B)$ coincide with the concept of $ij-gsCl(B)$ [10] and if $H^{\kappa}=j-Cl(H)$ for $H\in ij-GSO(\Upsilon)$, then $ij-\kappa gsCl(B)$ is called $ij-\theta gs$ closure of B (briefly $ij-\theta gsCl(B)$).

Definition 2.8 [10]

Let Υ and $Ƶ$ be bitopological spaces. A mapping $f:\Upsilon\rightarrow Ƶ$ is called ij −generalized semi-continuous (briefly $ij-gs$ −continuous) if $f^{-1}(W)\in ij-GSC(\Upsilon)$ for every $W^{c}\in\sigma_j$, equivalently $f^{-1}(W)\in ij-GSO(\Upsilon)$ for every $W\in\sigma_j$.

In the following, let $\kappa: P-GSO(Y) \to P(Y)$ and $\mu: P-GSO(Z) \to P(Z)$ be operations on $P-GSO(Y)$ and $P-GSO(Z)$, respectively.

3- Pairwise Generalized Semi Operation Continuous and Pairwise Generalized Semi Operation Irresolute Mappings

In this section, we introduce new types of pairwise continuous and pairwise irresolute mappings, namely, $ij-gs\kappa$ −continuous, $ij-gs(\kappa,\mu)$ −continuous, $ij-gs\kappa$ −irresolute and $ij-gs(\kappa,\mu)$ −irresolute. Also, we study some of their properties and the relation between them.

Definition 3.1

Let Y, Z be bitopological spaces and κ be an operation on $P-GSO(Y)$. A mapping f: Y → Z is called $ij-$ generalized semi operation continuous (briefly $ij-gs\kappa$ −continuous) if $f^{-1}(W) \in ij-GS\kappa C(Y)$ for every $W^c \in \sigma_j$, equivalently $f^{-1}(W) \in ij-GS\kappa O(Y)$ for every $W \in \sigma_j$.

If f is $ij-gs\kappa$ −continuous and $ji-gs\kappa$ −continuous, then it is called pairwise $gs\kappa$ −continuous.

If $H^\kappa = H$ for $H \in ij-GSO(Y)$, then the concept of an $ij-gs\kappa$ −continuous mapping coincide with the concept of an $ij-gs$ −continuous mapping [10].

Theorem 3.1

Let Y, Z be bitopological spaces, f: Y → Z be a mapping and κ be an operation on $P-GSO(Y)$. Then the following are equivalent:-

(i) f is $ij-gs\kappa$ −continuous.

(ii) For every $y \in Y$ and for every $W \in \sigma_j$ with $f(y) \in W$, there is $H \in ij-GSO(Y)$ with $y \in H$ such that $f(H^\kappa) \subset W$.

(iii) $f(ij-\kappa gsCl(B)) \subset ij-gsCl(f(B))$, for every subset B of Y.

(iv) $ij-\kappa gsCl(f^{-1}(D)) \subset f^{-1}(ij-gsCl(D))$, for every subset D of Z.

Proof.

$(i) \rightarrow (ii)$ Let f be $ij-gs\kappa$ $-$continuous, $y \in$ Y and $W \in \sigma_j$ with $f(y) \in W$, then $f^{-1}(W) \in ij-GS\kappa O(Y)$ and $y \in f^{-1}(W)$. Hence, there is $H \in ij-GSO(Y)$ with $y \in H$ such that $H^{\kappa} \subset f^{-1}(W)$. Thus $f(H^{\kappa}) \subset f(f^{-1}(W)) \subset W$.

$(ii) \rightarrow (iii)$ Let B be a subset of Y, $z \in f(ij-\kappa gsCl(B))$ and $W \in \sigma_j$ with $z \in W$, then there is $y \in Y$ such that $z = f(y)$, $y \in ij-\kappa gsCl(B)$ and $H \in ij-GSO(Y)$ with $y \in H$ such that $f(H^{\kappa}) \subset W$. Since $H \in ij-GSO(Y)$ with $y \in H$ and $y \in ij-\kappa gsCl(B)$, then $H^{\kappa} \cap B \neq \Phi$. Hence, $\Phi \neq f(H^{\kappa} \cap B) \subset f(H^{\kappa}) \cap f(B) \subset W \cap f(B)$. Therefore, we have $W \cap f(B) \neq \Phi$ and $W \in ij-GSO(Y)$ with $z \in W$ because every σ_j $-$open set is $ij-gs$ $-$open which implies that $z \in ij-gsCl(f(B))$ ([10] Lemma 3.14). Thus $f(ij-\kappa gsCl(B)) \subset ij-gsCl(f(B))$, for every subset B of Y.

$(iii) \rightarrow (iv)$ Let D be a subset of Z, then $f^{-1}(D)$ be a subset of Y. Hence by (iii), we have $f(ij-\kappa gsCl(f^{-1}(D))) \subset ij-gsCl\left(f(f^{-1}(D))\right) \subset ij-gsCl(D)$. Therefore, $f^{-1}(f(ij-\kappa gsCl(f^{-1}(D)))) \subset f^{-1}(ij-gsCl(D))$. Thus $ij-\kappa gsCl(f^{-1}(D)) \subset f^{-1}(ij-gsCl(D))$, for every subset D of Z.

$(iv) \rightarrow (i)$ Let $W^c \in \sigma_j$, then from (iv), $ij-\kappa gsCl(f^{-1}(W)) \subset f^{-1}(ij-gsCl(W))$. Since W is σ_j $-$closed, then W is $ij-gs$ $-$closed which implies that $ij-gsCl(W) = W$ ([10] Lemma 3.9). Therefore, $ij-\kappa gsCl(f^{-1}(W)) \subset f^{-1}(W)$. Since $f^{-1}(W) \subset ij-\kappa gsCl(f^{-1}(W))$ ([16] Theorem 4.3 (a)), then $ij-\kappa gsCl(f^{-1}(W)) = f^{-1}(W)$. Hence, $f^{-1}(W) \in ij-GS\kappa C(Y)$ ([16] Theorem 4.5). Thus ffis $ij-gs\kappa$ $-$continuous.

Corollary 3.1

In Theorem 3.1, if $H^{\kappa} = H$, for $H \in ij-GSO(Y)$, then the following are equivalent:-

(i) f is $ij-gs$ $-$continuous.

(ii) For every $y \in Y$ and for every $W \in \sigma_j$ with $f(y) \in W$, there is $H \in ij-GSO(Y)$ with $y \in H$ such that $f(H) \subset W$.

(iii) $f(ij-gsCl(B)) \subset ij-gsCl(f(B))$, for every subset B of Y.

(iv) $ij-gsCl(f^{-1}(D)) \subset f^{-1}(ij-gsCl(D))$, for every subset D of Z.

Corollary 3.2

In Theorem 3.1, if $H^{\kappa} = j - Cl(H)$, for $H \in ij - GSO(Y)$, then the following are equivalent:-

(i) f is $ij - gs\theta$ −continuous.

(ii) For every $y \in Y$ and for every $W \in \sigma_j$ with $f(y) \in W$, there is $H \in ij - GSO(Y)$ with $y \in H$ such that $f(H^{\theta}) \subset W$.

(iii) $f(ij - \theta gsCl(B)) \subset ij - gsCl(f(B))$, for every subset B of Y.

(iv) $ij - \theta gsCl(f^{-1}(D)) \subset f^{-1}(ij - gsCl(D))$, for every subset D of Z.

Definition 3.2

Let Y, Z be bitopological spaces and κ, μ be operations on $P - GSO(Y)$ and $P - GSO(Z)$, respectively. A mapping $f: Y \to Z$ is called $ij - gs(\kappa, \mu)$ −continuous at a point $y \in Y$ if for every $W \in \sigma_j$ with $f(y) \in W$, there is $H \in ij - GSO(Y)$ with $y \in H$ such that $f(H^{\kappa}) \subset W^{\mu}$. If f is $ij - gs(\kappa, \mu)$ −continuous at every point $y \in Y$, then it is called $ij - gs(\kappa, \mu)$ −continuous.

If f is $ij - gs(\kappa, \mu)$ −continuous and $ji - gs(\kappa, \mu)$ −continuous, then it is called pairwise $gs(\kappa, \mu)$ −continuous.

If $W^{\mu} = W$ for $W \in ij - GSO(Z)$, then the concept of an $ij - gs(\kappa, \mu)$ −continuous mapping coincide with the concept of an $ij - gs\kappa$ −continuous mapping.

If $H^{\kappa} = j - Cl(H)$ for $H \in ij - GSO(Y)$ and $W^{\mu} = j - Cl(W)$ for $W \in ij - GSO(Z)$, then an $ij - gs(\kappa, \mu)$ −continuous mapping is called $ij - gs\theta$ −continuous.

Remark 3.1

Every $ij - gs\kappa$ −continuous mapping is $ij - gs(\kappa, \mu)$ −continuous.

Theorem 3.2

Let Y, Z be bitopological spaces and κ, μ be operations on $P - GSO(Y)$ and $P - GSO(Z)$, respectively. If $f: Y \to Z$ is an $ij - gs(\kappa, \mu)$ −continuous mapping, then:-

(i) $f(ij - \kappa gsCl(B)) \subset ij - \mu gsCl(f(B))$, for every subset B of Y.

(ii) For every $ij - gs\mu$ −closed subset T of Z, $f^{-1}(T)$ is $ij - gs\kappa$ −closed subset of Y i.e., for every $\mathrm{S} \in ij - GS\mu O(\mathrm{Z})$, $\mathrm{f}^{-1}(\mathrm{S}) \in ij - GS\kappa O(\mathrm{Y})$.

(iii) $ij - \kappa gsCl\left(\mathrm{f}^{-1}(\mathrm{D})\right) \subset \mathrm{f}^{-1}(ij - \mu gsCl(\mathrm{D}))$, for every subset D of Z.

Proof.

(i) Let f be $ij - gs(\kappa, \mu)$ −continuous, B be a subset of Y, $z \in \mathrm{f}(ij - \kappa gsCl(\mathrm{B}))$ and $\mathrm{W} \in \sigma_j$ with $z \in \mathrm{W}$, then there is $y \in \mathrm{Y}$ such that $z = f(y)$, $y \in ij - \kappa gsCl(\mathrm{B})$ and $\mathrm{H} \in ij - GSO(\mathrm{Y})$ with $y \in \mathrm{H}$ such that $\mathrm{f}(\mathrm{H}^{\kappa}) \subset \mathrm{W}^{\mu}$. Since $y \in ij - \kappa gsCl(\mathrm{B})$ and $\mathrm{H} \in ij - GSO(\mathrm{Y})$ with $y \in \mathrm{H}$, then $\mathrm{H}^{\kappa} \cap \mathrm{B} \neq \Phi$. Hence $\Phi \neq \mathrm{f}(\mathrm{H}^{\kappa} \cap \mathrm{B}) \subset \mathrm{f}(\mathrm{H}^{\kappa}) \cap \mathrm{f}(\mathrm{B}) \subset \mathrm{W}^{\mu} \cap \mathrm{f}(\mathrm{B})$. Therefore, $\mathrm{W}^{\mu} \cap \mathrm{f}(\mathrm{B}) \neq \Phi$ and $\mathrm{W} \in ij - GSO(\mathrm{Y})$ with $z \in \mathrm{W}$ because every σ_j −open set is $ij - gs$ −open which implies that $z \in ij - \mu gsCl(\mathrm{f}(\mathrm{B}))$. Thus $\mathrm{f}(ij - \kappa gsCl(\mathrm{B})) \subset ij - \mu gsCl(\mathrm{f}(\mathrm{B}))$, for every subset B of Y.

(ii) Let f be $ij - gs(\kappa, \mu)$ −continuous and T be an $ij - gs\mu -$ closed subset of Z, then $ij - \mu gsCl = \mathrm{T}$ ([16] Theorem 4.5) and $\mathrm{f}^{-1}(\mathrm{T})$ is a subset of Y. Since f is $ij - gs(\kappa, \mu)$ −continuous, then from (i), we have $\mathrm{f}\left(ij - \kappa gsCl\left(\mathrm{f}^{-1}(T)\right)\right) \subset ij - \mu gsCl\left(\mathrm{f}\left(\mathrm{f}^{-1}(T)\right)\right) \subset ij - \mu gsCl(T) = T$ ([16] Theorem 4.3 (d)). Hence, $\mathrm{f}^{-1}\left(\mathrm{f}\left(ij - \kappa gsCl\left(\mathrm{f}^{-1}(\mathrm{T})\right)\right)\right) \subset \mathrm{f}^{-1}(\mathrm{T})$. Therefore, $ij - \kappa gsCl(\mathrm{f}^{-1}(\mathrm{T})) \subset f^{-1}(\mathrm{T})$. Since $\mathrm{f}^{-1}(\mathrm{T}) \subset ij - \kappa gsCl(\mathrm{f}^{-1}(\mathrm{T}))$ ([12] Theorem 4.3 (a)), then $ij - \kappa gsCl(\mathrm{f}^{-1}(\mathrm{T})) = \mathrm{f}^{-1}(\mathrm{T})$. Thus $\mathrm{f}^{-1}(\mathrm{T})$ is $ij - gs\kappa$ −closed ([16] Theorem 4.5).

Now, let S be an $ij - gs\mu$ −open subset of Z, then S^{c} is an $ij - gs\mu$ −closed subset of Z. Hence, $\mathrm{f}^{-1}(\mathrm{S}^{c}) = (\mathrm{f}^{-1}(\mathrm{S}))^{c}$ is an $ij - gs\kappa$ −closed subset of Y. Thus $\mathrm{f}^{-1}(\mathrm{S})$ is an $ij - gs\kappa$ −open subset of Y.

(iii) Let f be $ij - gs(\kappa, \mu)$ −continuous and D be a subset of Z. So, $\mathrm{f}^{-1}(\mathrm{D})$ be a subset of Y and then from (i), we have $\mathrm{f}(ij - \kappa gsCl(\mathrm{f}^{-1}(\mathrm{D}))) \subset ij - \mu gsCl\left(\mathrm{f}\left(\mathrm{f}^{-1}(\mathrm{D})\right)\right)$ $\subset ij - \mu gsCl(\mathrm{D})$ ([16] Theorem 4.3 (d)). Hence, $\mathrm{f}^{-1}(\mathrm{f}(ij - \kappa gsCl(\mathrm{f}^{-1}(\mathrm{D})))) \subset \mathrm{f}^{-1}(ij - \mu gsCl(\mathrm{D}))$. Thus $ij - \kappa gsCl\left(\mathrm{f}^{-1}(\mathrm{D})\right) \subset \mathrm{f}^{-1}(ij - \mu gsCl(\mathrm{D}))$, for every subset D of Z.

Remark 3.2

In Theorem 3.2, suppose that μ be an $ij-gs\mu$ −open operation. Then $ij-gs(\kappa,\mu)$ −continuity of f, (i) and (ii) are equivalent to each other.

Proof.

From Theorem 3.2 and its proof, we have $ij-gs(\kappa,\mu)$ −continuity of f implies (i) and (i) implies (ii). Then it is sufficient to prove that (ii) implies $ij-gs(\kappa,\mu)$ −continuity of f. Let $W\in\sigma_j$ with $f(y)\in W f(y)\in W$and μ be an $ij-gs\mu$ −open operation, then $W\in ij-GSO(Z)$ with $f(y)\in W$. Hence, there is $S\in ij-GS\mu O(Z)$ with $f(y)\in S f(y)\in S$such that $S\subset W^{\mu}$. Thus from (ii), $f^{-1}(S)\in ij-GS\kappa O(Y)$ with $y\in f^{-1}(S)$. Therefore, there is $H\in ij-GSO(Y)$ with $y\in H$ such that $H^{\kappa}\subset f^{-1}(S)$. Hence, $f(H^{\kappa})\subset f(f^{-1}(S))\subset S\subset W^{\mu}$. Thus for every $W\in\sigma_j$ with $f(y)\in W$, there is $H\in ij-GSO(Y)$ with $y\in H$ such that $f(H^{\kappa})\subset W^{\mu}$ which proves that f is $ij-gs(\kappa,\mu)$ −continuous.

Definition 3.3

Let Y, Z be bitopological spaces and κ be an operation on $P-GSO(Y)$. A mapping $f: Y\to Z$ is called $ij-$ generalized semi operation irresolute (briefly $ij-gs\kappa$ −irresolute) if $f^{-1}(W)\in ij-GS\kappa C(Y)$ for every $W\in ij-GSC(Z)$, equivalently $f^{-1}(W)\in ij-GS\kappa O(Y)$ for every $W\in ij-GSO(Z)$.

If f is $ij-gs\kappa$ −irresolute and $ji-gs\kappa$ −irresolute, then it is called pairwise $gs\kappa-$ irresolute.

If $H^{\kappa}=H$ for $H\in ij-GSO(Y)$, then $ij-gs\kappa$ −irresolute mapping is called $ij-gs$ −irresolute.

Remark 3.3

Every an $ij-gs\kappa$ −irresolute mapping is $ij-gs\kappa$ −continuous.

Corollary 3.3

Every an $ij-gs$ −irresolute mapping is $ij-gs$ −continuous.

Theorem 3.3

Let Y, Z be bitopological spaces, $f: Y\to Z$ be a mapping and κ be an operation on $P-GSO(Y)$. Then the following are equivalent:-

(i) ƒ is $ij - gs\kappa$ −irresolute.

(ii) For every $y \in$ Ƴ and for every W $\in ij - GSO$(Ƶ) with $f(y) \in$ W, there is H $\in ij - GSO$(Ƴ) with $y \in$ H such that ƒ(H$^{\kappa}$) ⊂ W.

(iii) ƒ$(ij - \kappa gsCl(\mathrm{B})) \subset ij - gsCl$(ƒ(B)), for every subset B of Ƴ.

(iv) $ij - \kappa gsCl$(ƒ$^{-1}$(D)) ⊂ ƒ$^{-1}$($ij - gsCl$(B)), for every subset D of Ƶ.

Proof. Similar to that of Theorem 3.1.

Corollary 3.4

In Theorem 3.3, if H$^{\kappa}$ = H, for H $\in ij - GSO$(Ƴ), then the following are equivalent:-

(i) ƒ is $ij - gs$ −irresolute.

(ii) For every $y \in$ Ƴ and for every W $\in ij - GSO$(Ƶ) with $f(y) \in$ W, there is H $\in ij - GSO$(Ƴ) with $y \in$ H such that ƒ (H)⊂ W.

(iii) ƒ$(ij - gsCl(\mathrm{B})) \subset ij - gsCl$(ƒ(B)), for every subset B of Ƴ.

(iv) $ij - gsCl$(ƒ$^{-1}$(D)) ⊂ ƒ$^{-1}$($ij - gsCl$(D)), for every subset D of Ƶ.

Corollary 3.5

In Theorem 3.3, if H$^{\kappa} = j - Cl$(H), for H $\in ij - GSO$(Ƴ), then the following are equivalent:-

(i) ƒ is $ij - gs\theta$ −irresolute.

(ii) For every $y \in$ Ƴ and for every W $\in ij - GSO$(Ƶ) with $f(y) \in$ W, there is H $\in ij - GSO$(Ƴ) with $y \in$ H such that $f(\mathrm{H}^{\theta}) \subset$ W.

(iii) ƒ$(ij - \theta gsCl(\mathrm{B})) \subset ij - gsCl$(ƒ(B)), for every subset B of Ƴ.

(iv) $ij - \theta gsCl$(ƒ$^{-1}$(D)) ⊂ ƒ$^{-1}$($ij - gsCl$(D)), for every subset D of Ƶ.

Definition 3.4

Let Ƴ, Ƶ be bitopological spaces and κ, μ be operations on $P - GSO$(Ƴ) and $P - GSO$(Ƶ), respectively. A mapping ƒ: Ƴ → Ƶ is called $ij - gs(\kappa, \mu)$ −irresolute at a point $y \in$ Ƴ if for every W $\in ij - GSO$(Ƶ) with $f(y) \in$ W, there is H $\in ij - GSO$(Ƴ) with $y \in$ H such that $\mathfrak{f}(\mathrm{H}^{\kappa}) \subset \mathrm{W}^{\mu}$. If $\mathfrak{f}$fis $ij - gs(\kappa,\mu)$ −irresolute at every point $y \in \Upsilon$, then it is called $ij - gs(\kappa,\mu)$ −irresolute.

If $\mathfrak{f}$ is $ij - gs(\kappa,\mu)$ −irresolute and $ji - gs(\kappa,\mu)$ −irresolute, then it is called pairwise $gs(\kappa,\mu)$ −irresolute.

If $\mathrm{W}^{\mu} = \mathrm{W}$ for $\mathrm{W} \in ij - GSO(\mathrm{Z})$, then the concept of an $ij - gs(\kappa,\mu)$ −irresolute mapping coincide with the concept of an $ij - gs\kappa$ −irresolute mapping.

If $\mathrm{H}^{\kappa} = j - Cl(\mathrm{H})$ for $\mathrm{H} \in ij - GSO(\Upsilon)$ and $\mathrm{W}^{\mu} = j - Cl(\mathrm{W})$ for $\mathrm{W} \in ij - GSO(\mathrm{Z})$, then an $ij - gs(\kappa,\mu)$ −irresolute mapping is called $ij - gs\theta$ −irresolute.

Remark 3.4

(i) Every an $ij - gs\kappa$ −irresolute mapping is $ij - gs(\kappa,\mu)$ −irresolute.

(ii) Every an $ij - gs(\kappa,\mu)$ −irresolute mapping is $ij - gs(\kappa,\mu)$ −continuous.

Corollary 3.6

Every an $ij - gs\theta$ −irresolute mapping is $ij - gs\theta$ −continuous.

Theorem 3.4

Let Υ, Z be bitopological spaces and κ, μ be operations on $P - GSO(\Upsilon)$ and $P - GSO(\mathrm{Z})$, respectively. If $\mathfrak{f}: \Upsilon \rightarrow \mathrm{Z}$ is an $ij - gs(\kappa,\mu)$ −irresolute mapping, then:-

(i) $\mathfrak{f}\big(ij - \kappa gsCl(\mathrm{B})\big) \subset ij - \mu gsCl(\mathfrak{f}(\mathrm{B}))$, for every subset B of Υ.

(ii) For every $ij - gs\mu$ −closed subset T of Z, $\mathfrak{f}^{-1}(\mathrm{T})$ is $ij - gs\kappa$ −closed subset of Υ

i.e., for every $\mathrm{S} \in ij - GS\mu O(\mathrm{Z})$, $\mathfrak{f}^{-1}(\mathrm{S}) \in ij - GS\kappa O(\Upsilon)$.

(iii) $ij - \kappa gsCl\big(\mathfrak{f}^{-1}(\mathrm{D})\big) \subset \mathfrak{f}^{-1}(ij - \mu gsCl(\mathrm{D}))$, for every subset D of Z.

Proof. Similar to that of Theorem 3.2.

Remark 3.5

In Theorem 3.4, suppose that μ be an $ij - gs\mu$ −open operation. Then $ij - gs(\kappa,\mu)$ − irresoluteness of $\mathfrak{f}$, (i) and (ii) are equivalent to each other.

Proof. Similar to that of Remark 3.2.

Definition 3.5

Let Ƴ, Ƶ be bitopological spaces and κ, μ be operations on $P-GSO$(Ƴ) and $P-GSO$(Ƶ), respectively. A mapping ƒ: Ƴ → Ƶ is called:-

(i) $ij-gs(\kappa,\mu)$ −closed if for every $ij-gs\kappa$ −closed subset B of Ƴ, ƒ(B) is $ij-gs\mu$ −closed subset of Ƶ.

(ii) $ij-gs(\kappa,\mu)$ −open if for every $ij-gs\kappa$ −open subset B of Ƴ, ƒ(B) is $ij-gs\mu$ −open subset of Ƶ.

If ƒ is $ij-gs(\kappa,\mu)$ −closed and $ji-gs(\kappa,\mu)$ −closed, then it is called pairwise $gs(\kappa,\mu)$ −closed and if ƒ is $ij-gs(\kappa,\mu)$ −open and $ji-gs(\kappa,\mu)$ −open, then it is called pairwise $gs(\kappa,\mu)$ −open.

If $W^{\mu}=W$ for W ∈ $ij-GSO$(Ƶ), then an $ij-gs(\kappa,\mu)$ −closed (resp. $ij-gs(\kappa,\mu)$ −open) mapping is called $ij-gs\kappa$ −closed (resp. $ij-gs\kappa$ −open).

If $H^{\kappa}=H$ for H ∈ $ij-GSO$(Ƴ), then an $ij-gs(\kappa,\mu)$ −closed (resp. $ij-gs(\kappa,\mu)$ −open) mapping is called $ij-gs\mu$ −closed (resp. $ij-gs\mu$ −open).

If $H^{\kappa}=H$ for H ∈ $ij-GSO$(Ƴ) and $W^{\mu}=W$ for W ∈ $ij-GSO$(Ƶ), then an $ij-gs(\kappa,\mu)$ − closed (resp. $ij-gs(\kappa,\mu)$ −open) mapping is called $ij-gs$ −closed (resp. $ij-gs$ −open).

If $H^{\kappa}=j-Cl(H)$ for H ∈ $ij-GSO$(Ƴ) and $W^{\mu}=j-Cl(W)$ for W ∈ $ij-GSO$(Ƶ), then an $ij-gs(\kappa,\mu)$ −closed (resp. $ij-gs(\kappa,\mu)$ −open) mapping is called $ij-gs\theta$ −closed (resp. $ij-gs\theta$ −open).

Theorem 3.5

Let Ƴ, Ƶ be bitopological spaces and κ, μ be operations on $P-GSO$(Ƴ) and $P-GSO$(Ƶ), respectively. Then a bijective mapping ƒ: Ƴ → Ƶ is $ij-gs(\kappa,\mu)$ −open if and only if it is $ij-gs(\kappa,\mu)$ −closed.

Proof.

Let ƒ be a bijective $ij-gs(\kappa,\mu)$ −open mapping and T be an $ij-gs\kappa$ −closed subset of Ƴ, then there is S ∈ $ij-GS\kappa O$(Ƴ) such that $T=S^{c}$. Hence, $ƒ(T)=ƒ(S^{c})=(ƒ(S))^{c}$

because f is bijective. Since f is an $ij - gs(\kappa,\mu)$ −open mapping and $S \in ij - GS\kappa O(Y)$, then $f(S) \in ij - GS\mu O(Z)$. Hence, $(f(S))^c \in ij - GS\mu C(Z)$. Then $f(T) \in ij - GS\mu C(Z)$. Thus f is $ij - gs(\kappa,\mu)$ −closed. Similarly, we can prove the inverse direction.

Corollary 3.7

Let Y, Z be bitopological spaces and κ,μ be operations on $P - GSO(Y)$ and $P - GSO(Z)$, respectively. Then a bijective mapping $f: Y \to Z$ is $ij - gs\kappa$ −open (resp. $ij - gs\mu$ −open) if and only if it is $ij - gs\kappa$ −closed (resp. $ij - gs\mu$ −closed).

Corollary 3.8

Let Y, Z be bitopological spaces. Then a bijective mapping $f: Y \to Z$ is $ij - gs$ −open (resp. $ij - gs\theta$ −open) if and only if it is $ij - gs$ −closed (resp. $ij - gs\theta$ −closed).

Corollary 3.9

Let Y, Z be bitopological spaces and κ,μ be operations on $P - GSO(Y)$ and $P - GSO(Z)$, respectively. Then a bijective mapping $f: Y \to Z$ is pairwise $gs(\kappa,\mu)$ −open if and only if it is pairwise $gs(\kappa,\mu)$ −closed.

Remark 3.6

Let Y, Z be bitopological spaces and κ be an operation on $P - GSO(Y)$. If $f: Y \to Z$ is bijective and $f^{-1}: Z \to Y$ is $ij - gs\kappa$ −irresolute, then f is $ij - gs\kappa$ −closed.

Proof.

Let T be an $ij - gs\kappa$ −closed subset of Y. Since f is bijective and f^{-1} is $ij - gs\kappa$ − irresolute, then f(T) is $ij - gs$ −closed subset of Z. Thus f is $ij - gs\kappa$ −closed.

Corollary 3.10

Let Y, Z be bitopological spaces. If $f: Y \to Z$ is bijective and $f^{-1}: Z \to Y$ is $ij - gs$ − irresolute, then f is $ij - gs$ −closed.

Corollary 3.11

Let Y, Z be bitopological spaces and κ be an operation on $P - GSO(Y)$. If $f: Y \to Z$ is bijective and $f^{-1}: Z \to Y$ is pairwise $gs\kappa$ −irresolute, then f is pairwise $gs\kappa$ −closed.

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