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Original Article

Study some properties of reflexive and dihedral homology on operator algebras.

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Abstract:

The homology properties of Banach algebras have been a central topic in functional analysis, with foundational contributions by Johnson, Kadison, Sinclair, and Ringrose leading to classifications based on Hochschild (co)homology and the concept of amenability. The interplay between Hochschild, cyclic, and dihedral (co)homology has further enriched the study of Banach and operator algebras, with key developments in biflatness, bi-projectivity, and ideal amenability. Recent research has focused on the computation of dihedral homology for Banach algebras, utilizing projective tensor powers and Hochschild complexes. By extending classical homological tools, we introduce a framework for relative reflexive homology and analyze its properties within involutive Banach algebras. Furthermore, we construct free involutive resolutions and explore their role in the homological classification of Banach algebras. There are many applications in the general sciences. Our results establish fundamental connections between dihedral homology, cyclic homology, and reflexive homology, which offer new perspectives on the algebraic and functional structure of Banach algebras and operator algebras.

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1-Introduction

The study of Banach algebras and their (co)homology properties has been a central topic in functional analysis. In (Johnson et al.,1972) initiated the investigation of (co)homology in Banach algebras, demonstrating that these structures can be characterized by their Hochschild (co)homology groups. In particular, the vanishing of the Hochschild (co)homology group led to the classification of a significant subclass of Banach algebras, termed amenable algebras. This foundational work paved the way for further research into more complex structures, including operator algebras. In (Kadison, 1990; Ringrose, 1972; Sinclair & Smith, 2009) extended the framework of the Banach algebras to operator algebras, encompassing C^*-algebras and von Neumann algebras. A pivotal tool in this area is the Connes-Tsygan exact sequence, which connects cyclic (co)homology with simplicial (co)homology, offering a robust method for analyzing algebraic structures.

The relationship between the (co)homology group $H_n(A, A_*)$ of a Banach algebra A and its relative counterpart $H_n^B(A, A_*)$, where A_* denotes the dual Banach space of A, is of significant interest. This connection enables the exploration of dihedral (co)homology groups and their relative variants. In (Helemskii, 1989; Helemskii, 1992) advanced the study of Banach algebras through the development of Banach homology theory, introducing the concepts of biflat and bi-projective Banach algebras. For instance, a Banach algebra A is biflat if there exists a bounded A-bimodule morphism $\rho: A \to (A \otimes pA)^{**}$ such that $\pi^{**} \circ \rho$ represents the canonical embedding of A into A^{**}. Furthermore, Helemskii demonstrated that $G_1(G)$ is biflat when G is amenable and bi-projective when G is compact. (Kaniuth et al., 2008) later expanded on these ideas by defining a new notion of amenability for Banach algebras based on their character.

Operator algebras have also been the subject of extensive (co)homology studies. Simplicial (co)homology has been explored in works such as (Essmaili et al., 2011; Johnson et al.,1972; Kadison, 1990) while cyclic (co)homology has been examined in (Helemskii, 1992; Johnson et al.,1972; Sinclair & Smith, 2009). The dihedral homology of algebras with characteristic zero has been shown (Krasauskas et al., 1987; Loday, 1991) and others studying the dihedral cohomology of unital and involutive algebras over commutative rings. Alaa contributed important findings on the dihedral homology of operator algebras in (NorEldean, 2013). Subsequent studies by (Krasauskas et al., 1987) investigated the interplay between Hermitian

K-theory and dihedral homology. Further research into the (co)homology of operator algebras was conducted in (NorEldean & Gouda, 2013; NorEldean, 2014), with analyses of their triviality and non-appearing as shown by (NorEldean & Gouda, 2009).

This body of work highlights the richness of the (co)homology structures in Banach and operator algebras, offering a foundation for deeper exploration into their algebraic and functional properties.

2- Homology of Banach algebra

Recently, the simplicial, cyclic, and dihedral (co)homology groups for Banach algebras were computed. To proceed, we first recall some definitions and key facts required for this discussion.

For a Banach algebra A with a unit and involution, we define the unital Banach algebra as the Banach algebra with a unit element $e \quad s.h \ \|e\| = 1$. We denote $C_n(A)$ for $n = 0,1,\cdots$ as the projective tensor power of A taken $(n+1)$ times, written as $A^{\otimes(n+1)} = A \otimes \quad \ldots \otimes A$. The elements of these Banach spaces are referred to as n-dimensional chains.

Define the operator $d_n\colon C_{n(A)} \to C_n(A), for\ n = 0,1,2, \ldots,$ as follows:

$$d_n(a_0, \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n}(-1)^i(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_{n+1}) + (-1)^{n+1}(a_{n+1}a_0 \otimes \ldots a_n) \qquad (1.1)$$

It is well known that $d_{n-1} \circ d_n = 0$ for all $n \in N$, which is equivalent to $Im(d_{n+1}) \subseteq ker\ (d_n)$. The elements of $Im(d_{n+1})$ are called n-boundaries, while the elements of $ker\ (d_n)$ are referred to as n-cycles. The chain complex $C(A) = (C.(A), d.)$ is thus a chain complex represented as:

$$C(A)\colon 0 \leftarrow C_0(A) \overset{d_0}{\leftarrow} \cdots \leftarrow C_n(A) \overset{d_n}{\leftarrow} C_{n+1}(A) \leftarrow \cdots \qquad (1.2)$$

This complex is said to be the Hochschild (simplicial) complex, and its homology is called Hochschild homology, denoted by $H_n = H_n(A, A) = \frac{\ker b_n}{Im\ b^{n+1}}$.

Define $t_n : C_n(A) \to C_n(A)$, for $n = 0, 1, \ldots$, be the operator given by:

$$t_n(a_0 \otimes a_0 \otimes \cdots \otimes a_n) = (-1)^n(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$$

with $t_0 = id$.

The space $\overline{C}_n(A)$ is the quotient of $C_n(A)$ by the closure of the linear span of elements of the form $x \mapsto t_n(x)$ for $n = 0, 1, \cdots$. According to (Helemskii, 1992), $Im(1 - t_n)$ is closed in

$C_n(A)$, and we have $\overline{C}_n(A) = \frac{C_n(A)}{Im(1-t_n)}$. This leads to a quotient complex $\overline{CC}.(A)$ of the complex $CC(A)$, and the homology of $\overline{CC}.(A)$, denoted $HC_n(A)$, is called the n-dimensional Banach cyclic homology group of A.

Define another operator $r_n : C_n(A) \to C_n(A)$, for $n = 0, 1, ...,$ by:

$$r_n(a_0 \otimes \cdots \otimes a_n) = (-1)^{\frac{n(n+1)}{2}} \in a_0^* \otimes a_n^* \otimes \cdots \otimes a_1^* \quad (1.3)$$

where $\in = \pm 1$, and $*$ denotes the involution on A. Note that $Im(1 - t_n)$ is closed in $C_n(A)$. The quotient complex $CR_n(A) = \frac{C_n(A)}{Im(1-t_n)+Im(1-r_n)}$ forms a subcomplex of $C_n(A)$. Its homology, denoted $HR_n(A)$, is known as the n-dimensional reflexive homology group of the unital Banach algebra A. When the involution $*$ is introduced, the resulting homology group is the dihedral homology group of the operator algebra A, denoted by $HD_n(A)$.

Let A and B be Banach algebras with units, and suppose both are involutive. Let $f: A \to B$ be a homomorphism between these involutive Banach algebras. We construct a free involutive resolution of B over f using the sequence $f: A \xrightarrow{i} R \xrightarrow{\pi} B$, where π is a quasi-isomorphism and ii is the inclusion map. Then, the relative reflexive homology is

$$HR_*\left(A \xrightarrow{f} B\right) = H_*(R/(A + [R, R] + Im(1 - r^{\varepsilon})) ,$$

where$[R, R]$ denotes the commutator of the Banach algebra R, and r^{ϵ} represents an involution on R. This framework is analyzed within the context of operator algebras.

Furthermore, if f is a homomorphism between involutive operator algebras A and B over K of characteristic zero, we treat R_f^B as a free resolution of B over f. $\forall\, r_1, r_2 \in R_f^B$, the graded commutator is defined as

$$[r_1, r_2] = r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1$$

in which $|r_i| = deg\, r_i$, $\forall\, i = 1,2$.

Consider $C = [R_f^B, R_f^B]$, which is the linear span of the commutators $[r_1, r_2]$ for $r_1, r_2 \in R_f^B$. The complex is then defined by $\left(C = \left[R_f^B, R_f^B\right] + Im(1 - r^{\varepsilon})\right)$, where the involution r^{ϵ} acts on an element P as $r^{\varepsilon}(P) = \varepsilon(-1)^{|p|(|p|-1)/2} p^*$, with $*$ representing the involution on R_f^B, and ϵ being ± 1. Since $[Im\ (1 - r^{\varepsilon})]$ is a subcomplex of R_f^B, it follows that

$$\partial[r_1, r_2] = r_1 r_2 - (-1)^{|r_1||r_2|} r_1 r_2$$

Expanding the boundary operation, we obtain

$$\partial[r_1, r_2] = \partial r_1 r_2 + (-1)^{|r_1|} r_1 \partial r_2 - (-1)^{|r_1||r_2|}\left(\partial r_2 r_1 + (-1)^{|r_2|} r_2 \partial r_1\right)$$

Rearranging, this simplifies to

$$\partial[r_1, r_2] = \partial r_1 r_2 - (-1)^{|r_2|(|r_1|+1)} r_2 \partial r_1 - (-1)^{|r_1|}\left(r_1 \partial r_2 - (-1)^{|r_2|(|r_1|+1)} \partial r_2 r_1\right)$$
$$= [\partial r_1 r_2] + (-1)^{|r_1|}[r_1, \partial r_2]$$

where $|\partial r_i| = |r_1| - 1, \quad i = 1,2$. This confirms that $[R_f^B, R_f^B]$ is a subcomplex of R_f^B, and consequently, the chain complex associated with the K-module

$$\left([R_f^B, R_f^B] + Im(1 - r^\varepsilon)\right)$$

also forms a subcomplex within R_f^B.

Definition (2-1):

Suppose $f: A \to B$ is a homomorphism between F-Banach algebras A and B over K with characteristic zero. Given a free resolution R_f^B of the Banach algebra B over f, the relative reflexive homology is defined as:

$$HR_*\left(A \xrightarrow{f} B\right) = H_*\left(\frac{R_f^B}{A + [R, R] + Im(1 - r^\varepsilon)}\right)$$

where$[R, R]$ denotes the commutator of the Banach algebra R, and r^ϵ represents the involution acting on R_f^B.

Definition (2-2):

Let $A\langle t\rangle$ be F-Banach algebra, that generated by:

$$a_0 t a_1 t \dots t a_n, \qquad n \geq 0$$

is structured as a differential graded Banach algebra by ensuring that the morphism $A \to A\langle t\rangle$ of involutive differential graded Banach algebras. In this framework, the grading and differential properties are defined as follows: $\deg t = 1$, $\partial t = 0$, and $t^* = t$. This structure allows for a deeper understanding of the algebra's behavior, incorporating both the differential grading and the involution properties in a consistent manner.

Lemma (2-3):

States that $A\langle t\rangle$ is a splittable Banach algebra. The Banach algebras $B = 0$ over the homorphism $A \to 0$ are resolved by a free Banach algebra.

Proof:

We build the chain complex as follows

$$A \overset{\partial}{\leftarrow} AtA \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} At..tA \overset{\partial}{\leftarrow} \cdots,$$

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where K-module is formed by $At..tA$ (with n-times repetitions). The boundary operator ∂ is given by

$$\partial(a_0ta_1t..ta_{n-1}ta_n) = \sum_{i=0}^{n-1}(-1)^i a_0ta_1t..ta_i(\partial t)ta_{i+1}..ta_n$$

since, $\partial t = 0$, this simplifies to

$$\partial(a_0ta_1t..ta_{n-1}ta_n) = \sum_{i=0}^{n-1}(-1)^i a_0ta_1t..t(a_ia_{i+1})t..ta_n.$$

By comparing this differential with the operator $\delta_n: C_n(A) \to C_{n-1}(A)$, we observe that

$$\delta_n(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n.$$

As shown in (NorEldean & Gouda, 2009), the complex $(C_n(A), \delta_n)$ is splitable. This implies that $A\langle t\rangle$ also admits a decomposition, leading to $H_*(A\langle t\rangle) = 0$.

Thus, we conclude that $A\langle t\rangle$ serves as a free Banach algebra resolution of $B = 0$ over a homomorphism $A \to 0$.

Lemma (2-4):

The quotient $(A\langle t\rangle/[A, A\langle t\rangle])$ is the simplicial complex of a standard form.

Proof:

Suppose that the complex $(A\langle t\rangle/[A, A\langle t\rangle])$, generated by elements of the form $a_0ta_1t \ldots ta_{n-1}t$. The relation

$$a_0ta_1t..ta_n = a_na_0ta_1t..ta_{n-1}t \qquad (mod\ [A, A\langle t\rangle])$$

indicates that these elements are cyclic permutations of each other. The boundary operator ∂ on this complex is defined as

$$\partial(a_0ta_1t..ta_{n-1}ta_n) = \sum_{i=0}^{n-1}(-1)^i a_0ta_1t..ta_i(\partial t)ta_{i+1}..ta_n + (-1)^n a_na_0ta_1t..a_{n-1}t.$$

Next, for differential in the standard Hochschild complex δ, we get the following chain complex:

$$A \overset{id}{\leftarrow} A \overset{\delta}{\leftarrow} A^{\otimes 2} \overset{\delta}{\leftarrow} \cdots \overset{\delta}{\leftarrow} A^{\otimes n} \overset{\delta}{\leftarrow} \cdots.$$

The space $(A\langle t\rangle/[A, A\langle t\rangle]_{n+1})$ can be identified with

$$A^{\otimes n+1}: a_0ta_1..ta_nt \to a_0\otimes a_1\otimes\cdots\otimes a_n.$$

Furthermore, the differential in $(A\langle t\rangle/[A, A\langle t\rangle])$ aligns precisely with δ. Consequently, $(A\langle t\rangle/[A, A\langle t\rangle])$ forms a standard simplicial complex.

Theorem (2-5):

Assume that A is an involution unital Banach algebra. Thus,

$HR_i\left(A \xrightarrow{f} B\right) = HR_i(A)$, where $HR_i(A)$ indicates the reflexive homology of the F-Banach algebras (with characteristic 0).

Proof:

Let the factor complex

$$(A\langle t\rangle/[A, A\langle t\rangle] + Im(1 - r^{\varepsilon})),$$

where the action of the boundary operator ∂ on the element $a_0ta_1t..ta_{n-1}t$ is given by

$$a_0ta_1t..ta_{n-1}t = (-1)^{n(n-1)/2}\varepsilon\, ta_n^*ta_{n-1}^* \dots ta_1^* \;=\; (-1)^{n(n-1)/2}\varepsilon\, ta_0^*ta_n^* \dots ta_1^*t,$$

here, $\varepsilon = \pm 1$, and the degrees are as follows:

- $\deg a_0ta_1t \dots ta_{n-1}t = n$,
- $\deg(a_n^*) = 0$,
- $\deg a_0ta_1t \dots ta_nt = n + 1$.

The dihedral homology of $A\langle t\rangle$ corresponds to the reflexive homology of the complex

$$(A\langle t\rangle/[A\langle t\rangle, A\langle t\rangle] + Im(1 - r^{\varepsilon})).$$

A homomorphism $CR_*(A \to 0) \to CR_{*-1}(A)$ is obtained by factoring $A\langle t\rangle$ firstly by the subcomplex $A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$ and then by the subcomplex $(A\langle t\rangle/[A\langle t\rangle, A\langle t\rangle] + Im(1 - r^{\varepsilon}))$. This results in an isomorphism of the reflexive homology groups $HR_*(A \to 0) \to HR_{*-1}(A)$. As a result, as needed, we have $HR_i(A \xrightarrow{f} B) = HR_i(A)$.

Theorem (2-6):

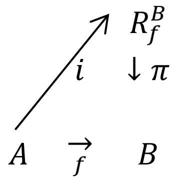
Let $f{:}\, A \to B$ be homomorphism of commutative Banach algebras over a field K (with $char(K) = 0$). The resolution selection is then independent of the relative reflexive homology $HR_i(A \xrightarrow{f} 0)$.

Proof:

A homomorphism of chain complexes is induced by the homomorphism f:

$$f_*{:}\, CR_*(A) \to CR_*(B)$$

Here, $CR_*(A)$ is a reflexive complex. Examine the following diagram:



where$i$ is the inclusion map, andR_f^B is a free resolution. Since

$$H_i(R_f^B) = \begin{cases} B, & i = 0 \\ 0, & i > 0 \end{cases}$$

the isomorphism between the homology of these complexes is produced by an isomorphism $\pi_*: CR_*(R_f^B) \to CR_*(B)$. We have:

$$HR_i\left(A \xrightarrow{f} B\right) \to HR_i\left(A \xrightarrow{g\circ f} C\right) \to HR_i\left(A \xrightarrow{g} C\right) \to HR_{i-1}\left(A \xrightarrow{f} B\right) \to \cdots$$

where$i_*: CR_*(A) \to CR_*(R_f^B)$ is an inclusion, and $M(i_*) \approx [CR_*(R_f^B)/CR_*(A)]$, where $M(i_*)$ is the cone of i.

The following diagram is commutative:

$$\begin{array}{ccc} & \nearrow & CR_*(R_f^B) \\ & i_* & \downarrow \pi_* \\ CR_*(A) & \xrightarrow[f_-*]{} & CR_*(B) \end{array}$$

This shows that $M(f_*) \approx [CR_*(R_f^B)/CR_*(A)]$. By using results from (NorEldean & Gouda, 2011), we obtain the following: $[CC_*(R_f^B)/CC_*(A)] \approx R_f^B/A + [R_f^B, R_f^B]]$,where CC_* denotes the Connes cyclic complex. By applying the spectral sequence$E_{ij}^2 = H_*(Z/2, H_*\ (R_f^B)) = HR_{i+j}(R_f^B)$, we derive:$CR_*(R_f^B)/CR_*(A)] \approx R_f^B/A + [R_f^B, R_f^B] + Im(1 - r^\varepsilon)$. Thus, we have, $M(f_*) \approx R_f^B/A + [R_f^B, R_f^B] + Im(1 - r^\varepsilon)$, which leads to the conclusion that,$HR_i(A \xrightarrow{f} B)$ is independent of the choice of R_f^B, as required.

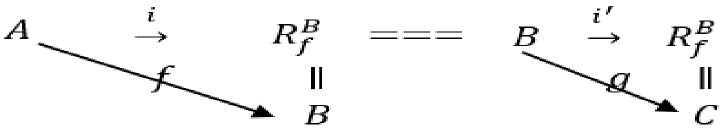
Theorem (2-7):

Consider the involutive Banach algebras $A, B$and C. Following that, the long exact sequence of relative reflexive homology is induced by the sequence $A \xrightarrow{f} B \xrightarrow{g} C$:

$$HR_i\left(A \xrightarrow{f} B\right) \to HR_i\left(A \xrightarrow{g\circ f} C\right) \to HR_i\left(B \xrightarrow{g} C\right) \to HR_{i-1}\left(A \xrightarrow{f} B\right) \to \cdots$$

Proof:

Any homomorphism $\boldsymbol{f}: \boldsymbol{B} \leftarrow \boldsymbol{A}$ of involutive algebras in any category is identical to an inclusion $\boldsymbol{i}: \boldsymbol{R_f^B} \leftarrow \boldsymbol{A}$, as demonstrated by Theorem (2-6). We have the following complex for a series of involutive Banach algebras $\boldsymbol{A} \xrightarrow{f} \boldsymbol{B} \xrightarrow{g} \boldsymbol{C}$:



where$g$and $f$are morphisms of the sequence.

Examine the mapping cone sequence:

$$0 \to M(i_*) \to M(i'_*) \to M(i_* \circ i'_*) \to 0.$$

The higher sequence is typically imprecise, and two morphisms will have zero composition. Canonically, however, the cone over the morphism $M(i_*) \to M(i'_*)$ is homotopy equal to $M(i_* \circ i'_*)$.

As a result, the exact sequence of relative reflexive homology that follows is obtained:

$$HR_i\left(A \xrightarrow{f} B\right) \to HR_i\left(A \xrightarrow{g \circ f} C\right) \to HR_i\left(B \xrightarrow{g} C\right) \to HR_{i-1}\left(A \xrightarrow{f} B\right) \to \cdots$$

3-Main results

Theorem (3-1):

Let $A$and A' is Banach algebras, there is an isomorphism for dihedral homology

$$\mathcal{H}D_n(A \times A') \cong \mathcal{H}D_n(A) \oplus \mathcal{H}D_n(A').$$

Proof:

Consider that $A \times A'$ is the direct product of the Banach algebras A and A', has natural projection and inclusion maps. These maps can be structured as:

$$\begin{array}{ccc} A \times A' & \overset{\psi}{\to} & A \\ \downarrow \rho & & \downarrow \phi \\ A' & \overset{\psi}{\to} & A \oplus A' \end{array},$$

where:

- $\psi: A \times A' \to A$ and $\rho: A \times A' \to A$ are projection maps, and
- $\phi: A \to A \oplus A'$and$\varphi: A' \to A \oplus A'$ are inclusion maps.

From this structure, we note that $ker\ (\psi) = A'$ and $ker\ (\rho) = A$. These kernels satisfy the F_*-excision property, allowing the use of exact sequences.

Using the structure of A, A', and $A \times A'$, we consider the short exact sequence:

$$0 \to A \to A \oplus A' \to A' \to 0.$$

Applying the dihedral homology functors HD_*, this sequence induces a Mayer–Vietoris long exact sequence:

$$\cdots \to HD_{n+1}(A \oplus A') \to HD_n(A \times A') \to HD_n(A) \oplus HD_n(A') \to HD_n(A \oplus A') \to \cdots.$$

From the properties of dihedral homology:

1. $HD_n(A \oplus A') \cong HD_n(A) \oplus HD_n(A')$.

   This follows from the additive structure of $A \oplus A'$.

2. The natural projections ψ\psi and ρ\rho induce isomorphisms:

$$\psi_*: HD_n(A \times A') \to HD_n(A), \qquad \rho_*: HD_n(A \times A') \to HD_n(A').$$

we obtain:

$$HD_n(A \times A') \cong HD_n(A) \oplus HD_n(A').$$

The above process can be visualized using the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \cdots & \to & HD_{n+1}(A \oplus A') & \to & 0 & \to & \cdots \\ & & \downarrow \phi & & \downarrow \psi & & \\ \cdots & \to & HD_n(A) \oplus HD_n(A') & \to & HD_n(A \times A' & \to & \cdots \end{array}$$

This diagram shows that all maps commute, and the isomorphism holds.

Theorem (3-2):

Let A, B, C and D is Banach algebras, and the commutative diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow f, \\ C & \to & D \end{array}$$

then there is a long Mayer – Vietoris sequence of dihedral homology as a form

$$\cdots \to HD_n(A) \to HD_n(B) \oplus HD_n(C) \to HD_n(D) \to HD_{n-1}(A) \to \cdots$$

Proof:

Given the commutative diagram:

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow f \\ C & \to & D \end{array}$$

the maps $A \to B$, $A \to C$, $B \to D$, and $C \to D$ induce maps on the dihedral homology groups:

$$HD_n(A) \to HD_n(B),\ HD_n(A) \to HD_n(C), HD_n(B) \to HD_n(D), HD_n(C) \to HD_n(D).$$

Construct a short exact sequence of chain complexes C_n:

$$0 \to C_n(A) \to C_n(B) \oplus C_n(C) \to C_n(D) \to 0.$$

Here:

- $C_n(A), C_n(B), C_n(C)$, and $C_n(D)$ denote the chain complexes associated with A, B, C, and D.
- The middle map is defined by: $(b, c) \mapsto b - c$.

The above short exact sequence induces a long exact sequence in homology:

$$\cdots \to H_n(C(A)) \to H_n(C(B)) \oplus H_n(C(C)) \to H_n(C(D)) \to H_{n-1}(C(A)) \to \cdots..$$

Replacing H_n with HD_n, we obtain the desired long Mayer–Vietoris sequence for dihedral homology:

$$\cdots \to HD_n(A) \to HD_n(B) \oplus HD_n(C) \to HD_n(D) \to HD_{n-1}(A) \to \cdots.$$

4-Conclusion

In this study, we examined the homology properties of Banach algebras, particularly their Hochschild, cyclic, and dihedral homology. We established a homological framework for analyzing Banach algebras with involution, introducing the concept of reflexive homology and its relative analogs. Through explicit constructions, we demonstrated the role of free resolutions in defining and computing these homology groups. Our results confirmed that the relative dihedral homology remains invariant under homomorphisms and independent of resolution choices, reinforcing its stability as a homological invariant. These contributions enhance the understanding of Banach algebra structures and pave the way for further studies on the (co)homology of operator algebras and their applications in functional analysis.

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Contributions

All parts contained in the research were carried out by the authors through hard work and a review of the various references and contributions in the field of mathematics. Authors have read and approved the final manuscript.

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