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The Marshall-Olkin Weibull-Lomax distribution: properties and applications

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The Marshall-Olkin Weibull-Lomax distribution: properties and applications

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ABSTRACT:

We are introducing a new flexible five parameters lifetime distribution called Marshall-Olkin Weibull-Lomax distribution. Some properties of the distribution such as the quantile function, moments, order statistics are derived. The unknown parameters of the new distribution are estimated using the maximum likelihood method. Monte Carlo simulation study is carried out to estimate the parameters and the performance of the estimates is judged via the average biases , mean squared error values and confidence interval. The usefulness of the proposed model is illustrated via real-life data set.

Key-words: : Marshal Olkin distribution, Weibull Lomax distribution, Hazard function, Likelihood estimation, Moments, Simulation, Application.

1. Introduction

In fact, there are hundreds of continuous univariate distributions. However, in recent years, applications from the environmental, financial, biomedical sciences, engineering among others, have further shown that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

The Lomax distribution which is also referred to as Pareto Type II distribution was initially proposed for modeling business failure Lomax [2]. Its application has been found in many other areas such as income, and size of city by Ahsanullah [1], Receiver Operating Characteristics (ROC) by Campbell and Ratnapaikhi [3], reliability and testing study Hassan and Al-Ghamdi,[4]; and estimation of the Stress-Strength Reliability by Salman and Hamad[5].

This paper aims to introduce a new generalization to the Lomax distribution extended and made more flexible to model data sets which ordinarily would not have provided adequate fit by means of addition of one or more parameters. Some of such extensions are Marshall-Olkin extended Lomax distribution by Ghitany et al. [6], McDonald Lomax by Lemonte and Cordeiro[7], Beta-Lomax by Rajab et al. [8], Gamma-Lomax by Cordeiro et al. [9], Logistic Lomax by Zubair et al.[11] Weibull and gamma distribution Bryson [17],,Half-Logistic Lomax by Anwar and Zahoor [14] and generalized Lomax by Maurya et al. [15].- Tahir et al.[10][17] introduced the gumbel Lomax distribution and weibull lomax distribution and studied its mathematical and statistical properties.

In recent years, various generalizations of the Marshall–Olkin (MO) distribution have been proposed to enhance flexibility in modeling lifetime data and hazard rate shapes.For instance,Elsherpieny et al.[20] introduced the Marshall–Olkin Lomax distribution (MOL), which accommodates increasing, decreasing, and bathtub-shaped hazard functions, making it suitable for reliability and insurance data. They applied it under median ranked set sampling and assessed risk measures such as Value at Risk (VaR) and Expected Shortfall (ES).Meanwhile, Boushra and El-Din [21] extended the Marshall–Olkin generalized extreme value distribution (MO-GEVD) under progressive Type-II censoring, employing genetic algorithms and Bayesian-Lindley approximation for parameter estimation. Their approach demonstrates strong performance even with incomplete data, which is common in survival analysis and Saiar et al [22]introduced a new distribution with Marshall Olkin MOEBXII distribution and analyzes the properties. Based on complete sample, and Bayesian estimators of the parameters

The cumulative distribution function (cdf) (for $x>0$) of the Weibull lomax(WL) distribution is given by

$$G(x,a,b,\alpha,\beta) = 1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\} \qquad (1)$$

where b is a scale parameter, a,a and b are shape parameters. The corresponding probability density function (pdf) is given by

$$g(x,a,b,\alpha,\beta) = \frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\} \qquad (2)$$

In this study, we propose a new extension of Lomax distribution called Mrshall olkin Weibull distribution (MOWL) distribution. The maximum likelihood (MLE) method is considered to estimate the parameters. We considered real-life data to illustrate the usefulness of the proposed model.

The cumulative distribution function (cdf) of Marshall and Olkin (MO) family is defined by

$$F(x) = \frac{G(x)}{1-(1-\theta)(1-G(x))} \qquad x\in R \qquad (3)$$

with probability density function

$$f(x)=\frac{\theta g(x)}{[1-(1-\theta)(1-GX)]^2} \qquad x\in R \qquad (4)$$

Where G(x) and g(x) are cdf and pdf of the baseline distribution. Using this approach an additional shape parameter (θ) is added which is responsible for the skewness, kurtosis and tail weights.

2 . The MOWL Distribution

We define the five parameter MOWL distribution by inserting (1) into (3) for obtaining the cdf of the MOWL distribution as

$$G(x,\theta,a,b,\alpha,\beta)=\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)(1-(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right)\right\}\right\}} \qquad (5)$$

The corresponding pdf is

$$g(x,\theta,a,b,\alpha,\beta)=\frac{\frac{\theta ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left[1-(1-\theta)(1-(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right)\right\}\right\}\right]^{2}} \qquad (6)$$

where $x>0$, and $a, b, \theta, \alpha, \beta$ are positive parameters.

3.Mathematical Properties of MOWL Distribution

The mathematical properties of MOWL distribution including shapes of the pdf and hrf, quintile function (qf), random number generator, ordinary moments, are investigated in this section. The MOWL distribution can be applied in survival analysis, hydrology, and economics.

Established algebraic expansions to determine some structural properties of the MOWL distribution can be more efficient than computing those directly by numerical integration of its density function.

From equation (5) we can obtain survival function of the MOWL distribution as follows:

$$= \frac{\theta \exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}+\theta \exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)} \qquad (7)$$

where $x > 0$, and $a, b, \theta, \alpha, \beta > 0$.

Then, the hazard function is given by

$$h(x,a,b,\theta,\alpha,\beta) = \frac{g(x)}{1-G(x)}$$

$$= \frac{\frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}}{\left(\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}+\theta \exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)} \qquad (8)$$

Plots of the probability density of MOWL distribution for the several values of the parameters are shown in Figure 1.

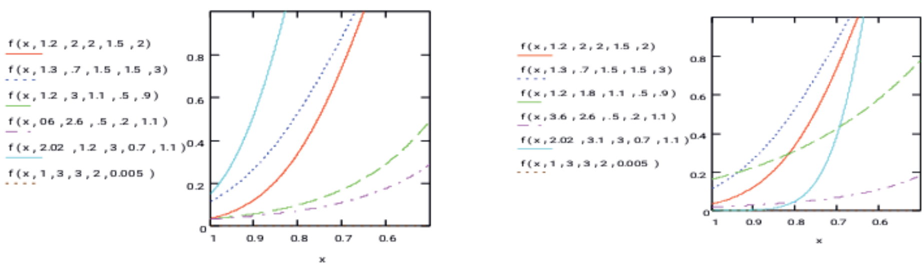


Figure 1. Plot for density function of the MOWL distribution for some selected parameter values.

Figure 1 shows that the plots of pdf for MOWL distribution for several values of parameters can take various forms depending on the parameter values. Both monotonically increasing, skewed shapes appear to be possible. It is evident that the MOWL distribution is very flexible.

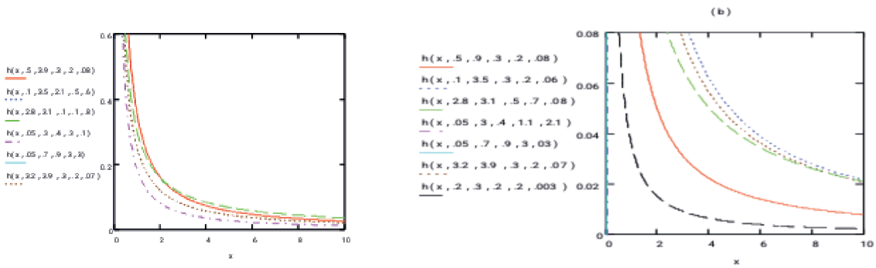


Figure 2. Plot for hazard function of the MOHL distribution for some selected parameter values

(a)shows different values of α, b and θ with a =2 and b=0.08.and (b) shows different values of α, b and θ with a =1.1 and b=2.2.

Figure 2 explains the behavior of hrf of the MOWL distribution for several parameter values. It shows that the distribution seems much flexible in explaining the death rate and existence rate for the lifetime of the certain product. This attractive flexibility makes the hrf of the MOWL useful and suitable for monotonically decreasing empirical hazard behaviors which are more likely to be observed in real life situations

3.1. Quantile Function

Quantile function is widespread, used in general statistics and often finds representations in terms of lookup tables for key percentiles. Now consider the quantile function of MOWL distribution. By inverting Eq. (5), we obtain the quantile function of X, say $Q(u) = G^{-1}(u)$ of X which is given by

$$u = \frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)(1-(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}$$

Upon some simplifications, it reduces to the following form

$$Q(u) = \left[\left(\left(\frac{1}{a}\right)\ln(\theta u-1)^{\frac{1}{b}}+1\right)^{\frac{1}{\alpha}}-1\right]\beta \qquad (9)$$

where u is obtained from a uniform distribution on the unit interval (0,1)

By putting q = 0.5 in Equation (9) we can get the median of X.

The skewness of the MOWL distribution can be derived by quantiles (seeKenney, 1939). The Bowleys' skewness (B) based on Eq. (9) is given by

$$B = \frac{Q(3/4)+2Q(1l2)+Q(1l4)}{Q(3l4)-Q(1/4)} \qquad (10)$$

3.2 Moments

If X is a continuous random variable with the MOWL distribution, then the $r th$ moment of X is given by

$$E(X^r) = \mu'_r = \int_0^\infty X^r \qquad dx \qquad (11)$$

$$= \int_0^\infty X^r \frac{\frac{\theta ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left[1-(1-\theta)(1-(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right)\right\}\right\}\right]^2}dx \qquad (12)$$

Let $\left(1+\frac{x}{\beta}\right)^{\alpha}-1 = y$ then

$$X=\beta(1+y)^{\frac{1}{\alpha}}-1 \qquad \text{and} \qquad dx=\frac{\beta dy}{\alpha\left(1+\frac{x}{\beta}\right)^{\alpha-1}}$$

By substituting in (12)

$$=\frac{\theta ab\alpha}{\beta}\int_0^\infty\left(\beta(1+y)^{\frac{1}{\alpha}}-1\right)^r\frac{\left(1+\frac{x}{\beta}\right)^{\alpha-1}(y)^{b-1}\exp\left\{-a\{(y)^b\}\right\} \qquad dy}{\left[1-(1-\theta)(1-(1-\exp\left\{-a(y)^b\right\}\right]^2} \qquad (13)$$

$$= \theta ab\int_0^\infty\left[(1+y)^{1/\alpha}-1\right]^r\frac{y^{b-1}\exp-a\ \ y^b}{(1-(1-\theta)\exp-a\ \ y^b)^2}dy \qquad (14)$$

Since r is an integer number then

Using Binomial Expansion in the part $(\beta(1+y)^{\frac{1}{\alpha}}-1)^r$ and substitute in(13) results

$$= \theta ab \sum_{i=0}^{r} C_i^r (-1)^i B^{r-i} \int_0^\infty (1+y)^{r-i/\alpha} \frac{y^{b-1} \exp -a \quad y^b}{(1-(1-\theta)\exp -a \quad y^b)^2} dy \tag{15}$$

Since $(1+y)^{r-i/\alpha}$ is areal number then using the expansion and replace in (15)

$$(1+y)^{r-i/\alpha} = \sum_{j=0}^{\infty} C_j^{\frac{r-i}{\alpha}} y^j$$

and replace in (15)

$$= \theta ab \sum_{i=0}^{r} C_i^r (-1)^i B^{r-i} \sum_{j=0}^{\infty} C_j^{r-i/\alpha} \int_0^\infty y^{j+b-1} \frac{\exp -a \quad y^b}{(1-(1-\theta)\exp -a \quad y^b)^2} dy \tag{16}$$

And using the negative Binomial expansion of as a special case that is

$\sum_{k=0}^{\infty} C_k^{-n} (x)^k \ (k+)$ ($exp - ka \ \ y^b$) and substituting in (16)we get

$$= \theta ab \sum_{i=0}^{r} C_i^r (-1)^i B^{r-i} \sum_{j=0}^{\infty} C_j^{r-i/\alpha} \sum_{k=0}^{\infty} C_k^{-n} (1-\theta)^k \ (k+1) \int_0^\infty y^{j+b-1} \exp -a \quad y^b \exp -ka \quad y^b dy \tag{17}$$

Finally Using gamma function and some other algebra simplification we get

$$\mu'_r = \theta \sum_{i=0}^{r} C_i^r (-1)^i B^{r-i} \sum_{j=0}^{\infty} C_j^{r-i/\alpha} \left(\frac{1}{a}\right)^{j/b} \sum_{k=0}^{\infty} (1-\theta)^k \left(\frac{1}{k+1}\right)^{\frac{j}{b}} \Gamma \frac{j}{b} + 1) \tag{18}$$

Setting r =1 in (18), we have the mean of X. we can get the variance by the relation

Var(x) = $Var(x) = E(X^2) - E^2(X)$

3.3. Order Statistics

Now, we obtain order statistics from the MOWL distribution. Let $x_{1r} \le x_{2r} \le x_{3r} \dots \le x_{rr}$ be the sample ordered from a population. The cdf of the rth order statistics, is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) [F_X(x)]^{r-1} [1-F_X(x)]^{n-r} \tag{19}$$

Substitute the values of (5) and (6) in equation (15), we will get the pdf of r th order statistics $X_{(r)}$ for MOWL distribution which is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{\frac{\theta ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left[1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)\right]^{2}}\times$$

$$\left[\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)}\right]^{r-1}\left[1-\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)}\right]^{n-r} \quad (20)$$

From equation (20), the density function of the largest order statistics $X_{(n)}$ is given by

$$f_{X(n)}(x)=\frac{n!}{(n-1)!0)!}\frac{\frac{\theta ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left[1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)\right]^{2}}\times$$

$$\left[\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)}\right]^{r-1}\left[1-\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)}\right]^{n-n} \quad (21)$$

and the pdf of the first order statistics $X_{(1)}$ is given by

$$f_{X(1)}(x)=n\left(\frac{\frac{\theta ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{\left[1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)\right]^{2}}\right)$$

$$\left[1-\frac{1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}}{1-(1-\theta)\left(1-\left(1-\exp\left\{-a\left\{\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right\}\right)\right)}\right]^{n-1} \quad (22)$$

3.4 Maximum Likelihood Estimation

In this section, we will use maximum likelihood method for estimating the parameters of MOWL distribution. Let $X1, X2, \ldots, Xn$ denote the random sample of size n from the MOWL distribution. Then the likelihood function is given by

$$L(x,a,b,\theta,\alpha,\beta) = \frac{\left(\frac{ab\alpha}{\beta\theta}\right)^n \prod_{i=1}^n \left(1+\frac{x}{\beta}\right)^{ab-1} \prod_{i=1}^n \left(1-\left(1+\frac{x}{\beta}\right)^{-a}\right)^{b-1}}{e^{-ab\sum_{i=1}^n \left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)}} \quad (23)$$

The log-likelihood function comes out to be

$$\log L(x,a,b,\theta,\alpha,\beta)=n\log\left(\frac{ab\alpha}{\beta\theta}\right)+(ab-1)\sum_{i=1}^n \ln\left(1+\frac{x}{\beta}\right)+(b-1)\sum_{i=1}^n \ln\left(1-\left(1+\frac{x}{\beta}\right)^{-a}\right)-ab\sum_{i=1}^n \ln\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right) \quad (24)$$

Therefore, the maximum likelihood estimators of the parameters a, b, œ, β, θ which maximize equation (24), must satisfy the following normal equations are given by

$$\frac{\partial}{\partial a}\log L(x,a,b,\theta,\alpha,\beta)=\frac{n}{a}+b\sum_{i=1}^n \ln\left(1+\frac{x}{\beta}\right)+(b-1)\sum_{i=1}^n \frac{\left(1+\frac{x}{\beta}\right)^{-a}}{\left(1-\left(1+\frac{x}{\beta}\right)^{-a}\right)}\ln\left(1+\frac{x}{\beta}\right)-b\sum_{i=1}^n \ln\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)=0 \quad (25)$$

$$\frac{\partial}{\partial b}\log L(x,a,b,\theta,\alpha,\beta)=\frac{n}{a}+a\sum_{i=1}^n \ln\left(1+\frac{x}{\beta}\right)+\sum_{i=1}^n \ln\left(1-\left(1+\frac{x}{\beta}\right)^{-a}\right)-a\sum_{i=1}^n \ln\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)=0 \quad (26)$$

$$\frac{\partial}{\partial \beta}\log L(x,a,b,\theta,\alpha,\beta)=\frac{-1}{\beta}-(ab-1)\sum_{i=1}^n \frac{\left(\frac{x}{\beta^2}\right)}{\left(1+\frac{x}{\beta}\right)}-a(b-1)\sum_{i=1}^n \frac{\left(\frac{x}{\beta^2}\right)\left(1+\frac{x}{\beta}\right)^{-a-1}}{\left(1-\left(1+\frac{x}{\beta}\right)^{-a}\right)}+ab\alpha\sum_{i=1}^n \left(\frac{\left(\frac{x}{\beta^2}\right)\left(1+\frac{x}{\beta}\right)^{\alpha-1}}{\left(1+\frac{x}{\beta}\right)^{\alpha}-1}\right)=0 \quad (27)$$

$$\frac{\partial}{\partial \theta}\log L(x,a,b,\theta,\alpha,\beta)=\frac{-n}{\theta}=0 \quad (28)$$

$$\frac{\partial}{\partial \alpha}\log L(x,a,b,\theta,\alpha,\beta)=\frac{n}{\alpha}+ab\sum_{i=1}^n \left(\frac{\left(1+\frac{x}{\beta}\right)^{a}}{\left(\left(1+\frac{x}{\beta}\right)^{a}-1\right)}\right)\log\left(1+\frac{x}{\beta}\right)=0 \quad (29)$$

Equations 25-29 can't be solved in closed form so that they can be solved numerically.

4. Simulation Study

Ion 0.1, are of sizes n = 20, 50, and 100. The process is repeated 1000 times using selected parameter values of the distribution .In each replication MLE estimation is performed, and the average estimates, biases, and mean square errors are obtained..Mathematica 7 is used for generating MOWL random variables and for solving the non-linear equations as well as for computing the minimization or maximization of the related functions.

Two quantities are computed in this simulation:

Average bias of the MLE of the parameters defined as

$$\text{Bias}=\sum_{n}^{i=1}\left(\tilde{\theta}-\theta\right)\Big/ n$$

the Root means square error (RMSE) of the MLE $\tilde{\theta}$ of the parameters defined as

$$\text{MSE}=\sqrt{\frac{1}{N}\sum_{n}^{i=1}\left(\tilde{\theta}-\theta\right)}$$

Table 1: Summary of the Means, Average bias and MSE from 1 000 simulations of

MOWL distribution for parameter values a=1.1 ,b=1.2; θ=1.3; α=1.5; μ 1.1

Sample size	MLE	MSE	Bias	U	L	Length
N=20	1.6126 1.0819 2.5669 1.4538 0.8533	0.3892 0.0597 2.0386 0.0719 0.1203	0.5125 -0.1180 1.2669 -0.0461 -0.2466	2.3095 1.5012 3.8575 1.9717 1.3314	0.9157 0.6628 1.2763 0.9359 0.3752	1.3938 0.8384 2.5811 1.0358 0.9562
N=50	1.6094 1.0695 2.6922 1.4231 0.8310	0.3079 0.0251 2.0641 0.0328 0.0972	-0.5094 -0.1304 -1.3922 -0.0768 -0.2689	2.0409 1.2458 3.3875 1.7444 1.1404	1.1779 0.8932 1.9968 1.1018 0.5216	0.8631 0.3526 1.3908 0.6424 0.6187
N=100	1.6109 1.0737 2.7150 1.4220 0.8333	0.2829 0.0185 2.0540 0.0193 0.0839	0.5109 -0.1263 1.4150 -0.0779 -0.2667	1.9006 1.1745 3.1609 1.6473 1.0553	1.3212 0.9728 2.2690 1.1967 0.6114	0.5794 0.2017 0.8919 0.4506 0.4439

Table 2: Summary of the Means, Average bias and MSE from 1 000 simulations of

MOWL distribution for parameter values a=1.1 ; b=1.2; θ=1.3; α=2.3; β =1.1

Sample size	MLE	MSE	Bias	U	L	Length
N=20	1.8892 1.0892 1.81862.0 129 1.2579	0.181800. 096740.35 6210.3718 10.1107	-0.2107 -0.1107 0.5186 0.5129 0.1579	2.61582 1.65899 2.3975 2.65911 1.8321	1.16275 0.5196 1.2397 1.3668 0.6837	1.4531 1.1394 1.1578 1.2923 1.1485
N=50	1.9147 1.0596 1.7942 2.0213 1.2658	0.7342 0.0619 0.2947 0.1332 0.0731	0.8147 -0.1403 0.4941 0.2787 0.1657	2.4351 1.4619 2.2345 2.4832 1.6846	1.3943 0.6574 1.3539 1.5594 0.8469	1.0408 0.8045 0.8806 0.9238 0.8376
N=100	1.9496 1.0381 1.8116 2.0211 1.2729	0.7521 0.0483 0.2907 0.1082 0.0539	0.8496 -0.1618 0.5116 -0.2781 0.1729	2.29045 1.32975 2.14538 2.36662 1.57617	1.60878 0.74649 1.47785 1.67718 0.9698	0.6816 0.5832 0.6675 0.6894 0.6063

Table3: Summary of the Means, Average bias and MSE from 1000 simulations of

MOWL distribution for parameter values a=3.1 ; b=1.2; θ=3.1; α=2.3; β=3.1

Sample size	MLE	MSE	Bias	U	L	Length
N=20	1.729941.1 92862.4261 91.742211. 10423	2.1969 0.1620 1.1239 0.4523 4.0938	-1.3700 -0.0071 -0.6738 -0.5577 -1.995	2.8384 1.9818 4.0304 2.4786 1.7565	0.6214 0.4038 0.8219 1.0057 0.4519	2.217 1.5779 3.2084 1.4728 1.3045
N=50	1.8717 1.0720 2.6947 1.7493 1.1218	2.2807 0.2435 2.1497 0.5580 4.1226	-1.2282 -0.1279 -0.4052 -0.5506 -1.9781	3.5938 2.0062 5.4566 2.7389 2.0187	0.1495 0.1378 -0.067 0.7598 0.2248	3.4443 1.8683 5.5236 1.9791 1.7939
N=100	1.9896 0.9981 2.9206 1.7864 1.1628	1.4593 0.1211 0.4585 0.3536 3.8236	-1.1103 -0.2018 -0.1793 -0.513 -1.9371	2.9223 1.5537 4.2005 2.3741 1.6849	1.0569 0.4425 1.6408 1.1987 0.6406	1.8654 1.1112 2.5596 1.1754 1.0442

Table 4: Summary of the Means, Average bias and MSE from 1000 simulations of

MOWL distribution for parameter values a=1.1; b=1.2; θ=3.1; α=2.3; β =1.1

Sample Size	MLE	MSE	Bias	U	L	Length
N=20	1.5606 1.5206 1.8935 1.9560 1.2719	0.4229 0.1992 1.8623 0.2628 0.1504	0.4606 0.3205 -1.2064 -0.3439 0.1719	2.4604 2.1294 3.1437 2.7013 1.9533	0.6608 0.9118 0.6434 1.2107 0.5906	1.7996 1.2175 2.5003 1.4905 1.3627
N=50	1.4873 1.5339 1.8441 1.8782 1.2096	0.30393 0.1669 1.8886 0.2603 0.0812	0.3873 0.3339 -1.2558 -0.4217 0.1096	2.2562 1.9952 2.9377 2.4411 1.725	0.7185 1.0727 0.7505 1.3154 0.6942	1.5377 0.9224 2.1873 1.1256 1.0307
N=100	1.4467 1.5405 1.7999 1.8381 1.1766	0.2708 0.1596 2.0028 0.2846 0.0666	0.3467 0.3405 -1.3000 -0.4618 0.0767	2.2074 1.9498 2.8959 2.3616 1.6599	0.6859 1.1313 0.7039 1.3147 0.6934	1.5214 0.8185 2.1920 1.0469 0.9664

From tables1,2,3 and 4 it is noticed that

the estimates are approach to the value of the true parameters, the MSE is decreases as sample size increases, the length of the confidence interval of the parameters is decreasing as the sample size increasing in the most cases according to the values of the parameters.

5 Application of Real Data Analysis

This section is devoted to illustrating the potentiality of the MOWL distribution for the real data set. MOWL distribution is compared with the competitive model such as (WL) distribution.

The two data sets used are on aircraft windshield failure and service times applied by Murthy et al. [12] and used by Ramos et al.[15] to model exponentiated Lomax Poisson distribution. We consider the data on service time for a particular model windshield given below. The data consists of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h.0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183,2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

We estimate the unknown parameters of the models by MLE and the goodness-of-fit statistics are tested using the criteria Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC) and the Kolmogorov-Smirnov (K-S) test along with the corresponding p-value. In general, the smaller values of these statistics indicate the better fit for the data.

$$AIC = -2\, l(.) + 2p, \qquad BIC = -2l(.) + p \log(n),$$

$$CAIC = -2l(.) + \frac{2pn}{n-p-1}$$

where $l(.)$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size.

Table 5. MLEs and their standard errors (in parentheses) for service times of 63 Aircraft Windshield data

Distribution	a	b	[illegible]	A	β
WL	0.17 (0.296)	0.920 (0.427)	- -	3.9136 (3.8489)	3.0067 (8.276)
MOWL	3.31 (0.42)	0.262 (0.033)	144.2 (18.38)	50.66 (6.431)	46.23 (5.869)

(Tables 5) gives the MLEs and their corresponding standard errors (in parentheses)of the model parameters. The model selection is carried out in (table 6) using the AIC, BIC, and CAIC statistics defined by

$$AIC = -2\, l(.) + 2p, \qquad BIC = -2l(.) + p \log(n),$$

$$CAIC = -2l(.) + \frac{2pn}{n-p-1}$$

Where log l(.) denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size.

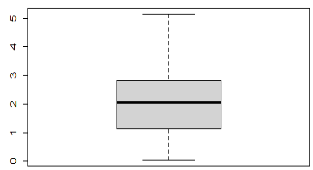
Table 6. The statistics $-l$(.), AIC, BIC, and CAIC for service times of 63 Aircraft Windshield data

Model	logL logL	AIC	BIC	CAIC	K-S Stat	(pvalue)
WL	-98.117	204.234	204.923	212.806	0.242	0.0375
MOWL	-91.605	199.21	174.319	172.099	0.215	0.083

the model with least goodness of fit statistics provides the best fit for the data set (Chen and Balakrishnan[18]. The values in Tables 6 and 7 indicate that the goodness fit statistics of the

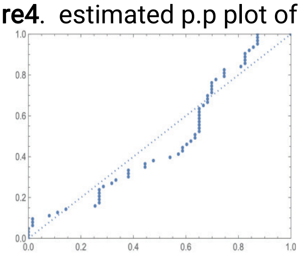
MOWL are the least among the competing model. These values also indicate that the MOWL model provides the best fit compared to the competing model.

Figure 3.The box plot of the data set

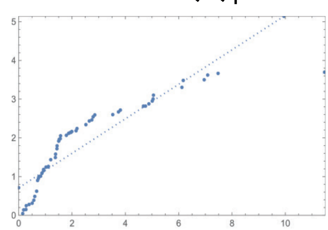


The box plot appears a right skewed for the data represented

Figure4. estimated p.p plot of data set

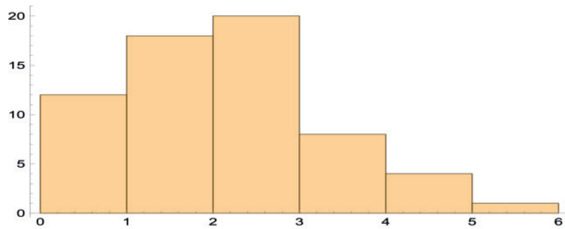


estimated Q-Q plot i



The P-P plot and Q-Q plot indicate that the MOWL distribution fits the data good.

Figure5.the histogram of the data set



The histogram of the data shows that the data are unimodal

6- Concluding remarks

In this paper, we propose a five-parameter Marshall Olkin Weibull-Lomax (MOWL) distribution. We study some structural properties of the (MOWL) distribution including an expansion for the density function and explicit expressions for the moments, quantile function. Further, the explicit expressions for order statistics are also derived. The (ML) method is employed for estimating the model parameters. We fit the (MOWL) model to a real-life data set to show the usefulness of the proposed distribution. The new model consistently provides a better fit than the (WL) distributions. We hope that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

A five-parametric MOWL distribution is introduced using Marshall Olkin family distribution. The newly proposed distribution is studied thoroughly, and various properties are derived. Parameters were estimated theoretically through the maximum likelihood method and numerically via Newton Raphson's iterative method for various sample sizes. Lastly, the proposed distribution is applied to a real data set and compared with other distribution in literature. The results showed that the newly proposed MOWL distribution performs better than the distribution compared.

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توزيع مارشال-أولكين-ويبل-لوماكس: خصائصه وتطبيقاته

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الملخص:

نُقدّم توزيعًا جديدًا مرنًا بخمسة معاملات، يُسمى توزيع مارشال-أولكين-ويبل-لوماكس .

تم اشتقاق بعض خصائص هذا التوزيع، مثل دالة الكميات، والعزوم، وإحصاءات الترتيب. قُدّرت المعاملات المجهولة للتوزيع الجديد باستخدام طريقة الامكان الأعظم . أُجريت دراسة محاكاة مونت كارلو لتقدير المعاملات، وتمّ تقييم أداء التقديرات من خلال متوسط الانحيازات، ومتوسط مربعات الخطأ، وفترة الثقة. تم توضيح فائدة النموذج المقترح من خلال مجموعة بيانات واقعية.]

الكلمات المفتاحية : توزيع مارشال أولكين، توزيع ويبل لوماكس، دالة الخطر ، تقدير الاحتمالية ، العزوم ، المحاكاة ، التطبيق.