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FULL LENGTH ARTICLE

A mathematical model of star formation in the Galaxy

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Abstract This paper is generally concerned with star formation in the Galaxy, especially blue stars. Blue stars are the most luminous, massive and the largest in radius. A simple mathematical model of the formation of the stars is established and put in computational algorithm. This algorithm enables us to know more about the formation of the star. Some real and artificial examples had been used to justify this model.

Introduction

Blue stars are the most luminous stars. They located in the upper left on the main sequence in the Hertsprung–Russel diagram. These stars are also the most massive of all the main sequence stars and the largest ones. Their large mass, as however, are not able to compensate for the high luminosity. This means that, although their supply of hydrogen is larger than stars like Sun, they will burn up this supply at such a high rate that their total lifetime is much shorter than for medium or massive stars.

Why is it important to look for these short-living stars? Since their total lifetime is so short, they will not have had the opportunity to move away very far from the place where they were born. Consequently, if we want to search for galactic regions in which star formation is possible, we have to look for places with luminous blue stars. This is not always possible in our own Galaxy where our view is blocked by interstellar clouds, but there are more than enough other galaxies. Luminous blue stars are found in the outer regions of spiral galaxies. The spiral components of these galaxies actually have a blue color because the luminosity of these blue stars is so high that it compensates for their small number. The spiral arms also have another important component, namely extended hydrogen and dust clouds, whose, presence leads to the suggestion that stars are born in these clouds by condensation and gravitational collapse. A number of pictures taken from dark clouds in the Galaxy clearly show spherical condensations, called *protostars*. Young stars often found in the immediate surroundings of these protostars.

A normal consequence of contraction of a gas cloud is the increasing of its temperature. During the first stages, however, the radiation is able to escape where the temperature and pressure in the cloud stay at low level. The cloud will therefore continue to collapse, eventually fragmenting into a number of smaller contracting clouds. By the time the central density becomes high enough to make the center opaque for infrared radiation, the gravitational field is strong enough to

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compensate for increasing pressure. The collapse is now inevitable and will only come to an end when the central temperature reaches several million Kelvin. This is the minimum temperature to start the nuclear fusion of hydrogen into helium, a process that releases an enormous amount of energy. This energy is transposed through the cloud and radiated away in space. The flow of energy also restores hydrostatic equilibrium. The cloud has now become a normal core hydrogen burning star (Bodifé and De Loore, 1985).

An important amount of kinetic energy is released during the contraction stage. It is necessary for the energy to be neutralized without heating the cloud to prevent any further contraction. CO molecules play an especially important role as cooling mechanisms in the dust clouds. On this context, a simple model for star formation can be illustrated as the following.

The model contains three active components: cool atomic clouds, cool molecular clouds, and active young stars. Each of these components may interact with the other components or with the rest of the Galaxy. The model is therefore an open system connected with two mass reservoirs outside the system. A schematic view of the system is presented in Fig. 1.

The main component of cool atomic clouds is neutral hydrogen, the most abundant chemical element in the Galaxy. Neutral hydrogen clouds are found in the spiral arms and disk of the Galaxy. Their typical radio emission at a wavelength of 21 cm enable.

The density varies over a large range, but direct star formation in these clouds seems not to occur. The cooling capacity of these clouds is not large enough to allow a sufficient condensation. This component is connected to an unlimited reservoir of new atomic gas outside the system.

Cool molecular clouds mainly consist of molecular hydrogen HII. The densities mast enormously and are generally much higher than in neutral clouds. The temperature in such cloud decreases as its density increases as a consequence of large cooling capacity of the CO molecules. Molecular clouds have smaller dimensions than neutral clouds. They are found in the spiral arms, often in the vicinity of OB associations, which are groups of young blue stars (Bodifé, 1986).

Young, active stars are mostly accompanied by hot ionized HII gas. These stars strongly affect the surrounding gas clouds and are responsible for shock waves in these clouds. In this way, new condensation regions may be formed in the molecular clouds of the system. The presence of young stars therefore has a positive effect on the stellar birth rate. The capacity of influencing the other components ends when these young stars evolve to neutron stars. Although these remnants are still physically present in the system, their masses have stopped playing an active role in the star formation process. We therefore say that this mass has left the active star formation system. The second reservoir is hence a waste reservoir containing the stellar remnants.

Equations of interaction between the components

The three mass components will be described by the variable S for the total mass of active stars, M for the total mass of molecular clouds, and A for the total mass of atomic clouds. It is assumed that the total mass of the system remains constant; thus, we assume that the amount of mass lost by stellar evolution is exactly replaced by fresh atomic clouds entering the star formation region from the rest of the Galaxy. If we call the total mass of the system T, we may write

$$T = A + M + S. \quad (1)$$

There are three kinds of interaction for the atomic cloud component A. First, there is a constant replenishment by new atomic clouds in an amount equal to the amount of mass leaving the active system by stellar evolution. The amount of new gas may therefore be considered as proprotional to the amount of stellar mass S. We will call the proportional constant of this process K_1. Secondly, the atomic component is increased as young, active stars lose mass by stellar wind. This process is also proportional to the number of stars and therefore to the

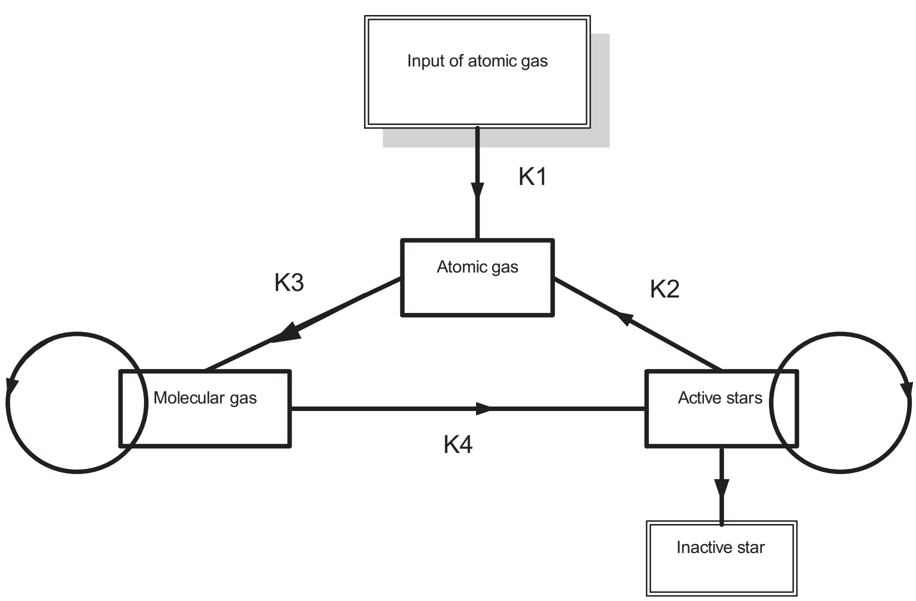

Fig. 1 A schematic view of the various components of the star formation model.

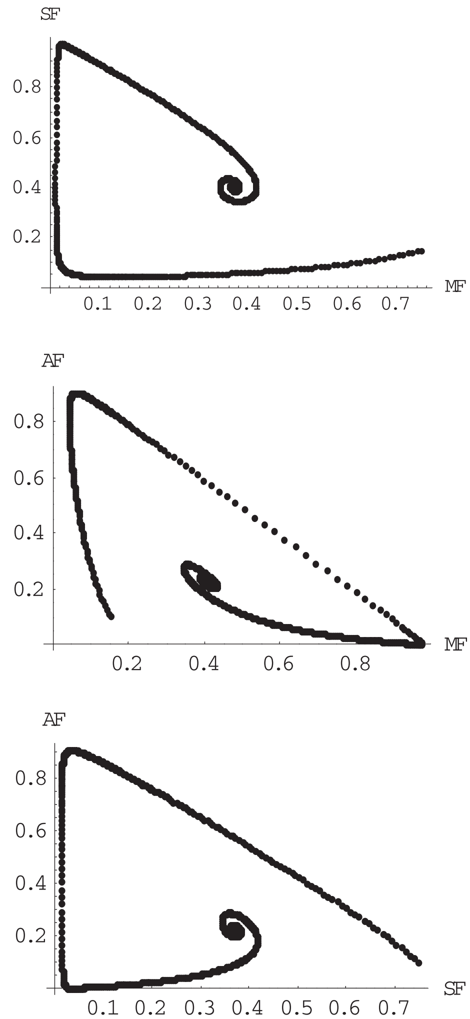

Fig. 2a Phase diagram of star formation model.

amount of stellar mass. The proportional constant for this process is K_2. The third interaction is the transformation of atomic into molecular clouds. This process is clearly proportional to the amount of atomic gas A, but since the transformation becomes more and more effective with the cooling capacity of the cloud, and since this capacity increases with the square of the density of molecular content, we assume that the transformation of atomic into molecular gas is proportional to the square of the molecular mass. This third processes loss of atomic gas and therefore is written with a minus sign in the differential equation and a proportional constant K_3

$$\frac{dA}{dt} = K_1 S + K_2 S - K_3 AM^2. \tag{2}$$

The rate of the star formation may be considered as being proportional to the *n*th power of the density of molecular cloud. Values of n can be selected between 0.5 and 3.5 or between 1 and 2. It is assumed that the presence of other young stars is a necessary condition for star formation since they will perturb the molecular cloud and provoke condensations. In this way we may also state that the star formation rate is proportional to the number of active stars already present. Let us call the proportional constant K_4. This process increases the mass of stellar component. Two other process decrease it: stellar evolution, for which we may use K_1, and mass loss by stellar wind, for which we again use K_2. Both processes are proportional to the amount of stellar mass. Thus, the equation describing the variation of stellar mass in the system is

$$\frac{dS}{dt} = K_4 SM^n - K_1 S - K_2 S. \tag{3}$$

Finally, the variation of the total molecular mass is given by two processes already described: transformation of atomic into molecular gas, which increases for the variable M, and stellar formation, which decreases the amount of molecular mass. The equation for M will be

$$\frac{dM}{dt} = K_3 AM^2 - K_4 SM^n. \tag{4}$$

It is easy to see that every process in one of the three differential equations is a compensated by the process with an opposite sign in one of the other equations. This balance guarantees the conservation of the total amount of mass.

The three independent variables A, M and S in the previous section are linked by the fact that we consider the total mass T of the star formation system as constant. One may, for instance, replace S by T, A, M in Eqs. (2) and (4). The system then reduces to only two first-order equations. Furthermore, it is possible to transfer these two remaining equations by introducing new dimensional variables and a new dimensional time coordinate x.

$$a = \frac{A}{T} \tag{5.1}$$

$$m = \frac{M}{T} \tag{5.2}$$

$$s = \frac{S}{T} \tag{5.3}$$

$$x = (K_1 + K_2)t. \tag{5.4}$$

This implies that

$$a + m + s = 1, \tag{6}$$

at every moment. The parameters K_1, K_2, K_3, and K_4 are also transformed as a consequence of the dimensional variables and become two new parameters

$$k_1 = \frac{K_3 T^2}{K_1 + K_2}, \tag{7.1}$$

and

$$k_2 = \frac{K_4 T^n}{K_1 + K_2}. \tag{7.2}$$

when two out of the three variables are known, the third also is known, since the sum of the three is always one. The differential equations, after elimination of S and after introducing new variables, become

$$\frac{da}{dx} = 1 - a - m - k_1 m^2 a, \tag{8.1}$$

$$\frac{dm}{dx} = k_1 m^2 a + k_2 m^n (a - 1 + m). \tag{8.2}$$

These are the equations which can be solved. They contain three free parameters k_1, k_2 and n. The results may graphically be represented in two ways. It is, of course, possible to plot the three variables – the atomic, molecular, and stellar content – as functions of time. This is done for a number of models whose results will be discussed later. Another possible method often used in all applied sciences is to work with phase diagrams. Consider, for instance, a mass hanging on a spring. The mass will jump up and down, periodically oscillating around the state of rest. We may once again plot the position and the velocity as functions of the time in two separate drawings, but another way is to plot the position X on the horizontal axis and the corresponding velocity V on the vertical axis. We will then obtain a point that moving on circle in the XV plane. If the oscillations are damped, the radius will slowly decrease unit the circle shrinks to a point on the X-axis. This means that the position is constant and the velocity is zero. The mass has come to rest.

Diagrams in which two functions of time are plotted on the two axes are called *phase diagrams*, just as if the time had been eliminated from the two functions so that we have obtained one single relation containing the two variables. Phase diagrams are especially suited to study of periodic or oscillating systems.

Mathematical procedure: star formation

Purpose

Computing a model of a star-forming region with negative and positive feedback mechanisms, where the model includes three components: cool HI, dust molecular gas, and young stars with their associated HII regions.

Input

1. n,
2. $k1$ (k_1),
3. $k2$ (k_2),
4. $m0$ (m_0),
5. $s0$ (s_0),
6. $a0$ (a_0).

Output

One of the two different regimes that simulate the three-components of a star-forming region:

1- evolution towards a stationary state, or

2- evolution towards a limit cycle.

List of the procedure

```
StarFormation [n_, k1_, k2_, m0_, s0_, a0_] := Module [{},
  sol = NDSolve [{a'[x] == 1 - a[x] - m[x] - k1 * m[x] ^2 * a[x],
  m'[x] == k1 * m[x] ^2 * a[x] + k2 * m[x] ^n * (a[x] - 1 + m[x]),
 a[0] == a0, m[0] == m0}, {a, m}, {x, 0, 2000}, MaxSteps -> 3000000 ];
  ww = Table [i, {i, 0, 100, 0.02}];
  ww = Table [i, {i, 0, 100, 0.02}];
  Nw = Length [ww];
  s[y_] = 1 - a[y] - m[y];
  q1 = N[a[x] /. sol /. x -> ww, 6];
  q2 = N[m[x] /. sol /. x -> ww, 6];
  q3 = N[s[x] /. sol /. x -> ww, 6];
  NN = 5;
  data1 = Table [{q3[[1, i]], q2[[1, i]]}, {i, 1, Nw / NN}];
  p1 = ListPlot [data1, AxesLabel -> {"MF ", "SF"},
    PlotStyle -> PointSize [0.017], DisplayFunction -> Identity ];
  A = Table [{i, q1[[1, i]]}, {i, 1, Nw / NN}];
  p2 = ListPlot [A, PlotLabel -> "Atomic fraction ",
    PlotStyle -> PointSize [0.017], DisplayFunction -> Identity ];
  data2 = Table [{q2[[1, i]], q1[[1, i]]}, {i, 1, Nw / NN}];
  B = Table [{i, q2[[1, i]]}, {i, 1, Nw / NN}];
  p3 = ListPlot [data2, AxesLabel -> {"MF ", "AF"},
    PlotStyle -> PointSize [0.017], DisplayFunction -> Identity ];
  p4 = ListPlot [B, PlotLabel -> "Molecular fraction ",
    PlotStyle -> PointSize [0.017], DisplayFunction -> Identity ];
  data3 = Table [{q3[[1, i]], q1[[1, i]]}, {i, 1, Nw / NN}];
  CC = Table [{i, q3[[1, i]]}, {i, 1, Nw / NN}];
  p5 = ListPlot [data3, AxesLabel -> {"SF ", "AF"},
    PlotStyle -> PointSize [0.017], PlotStyle -> PointSize [0.015],
    DisplayFunction -> Identity ];
  p6 = ListPlot [CC, PlotLabel -> "Stellar fraction ",
    PlotStyle -> PointSize [0.017], DisplayFunction -> Identity ];
  Show [GraphicsArray [{{p1, p2}}], DisplayFunction -> $DisplayFunction ];
   Show [GraphicsArray [{{p3, p4}}], DisplayFunction -> $DisplayFunction ];
  Show [GraphicsArray [{{p5, p6}}], DisplayFunction -> $DisplayFunction ]]
```

Numerical examples and conclusion

Evolution towards a stationary state

The module evolves towards a stationary state for certain combinations of the three free parameters. For fixed values of n and k_2, there are some critical values of k_1 above which the models evolve to such state. The model first starts with some periodic oscillations. These oscillations are then damped, and after a certain relaxation time, all variables (i.e., the atomic, molecular, and stellar content) reach constant values. This means that all the interactions between the three components are still present, but in such way their combined effects cancel

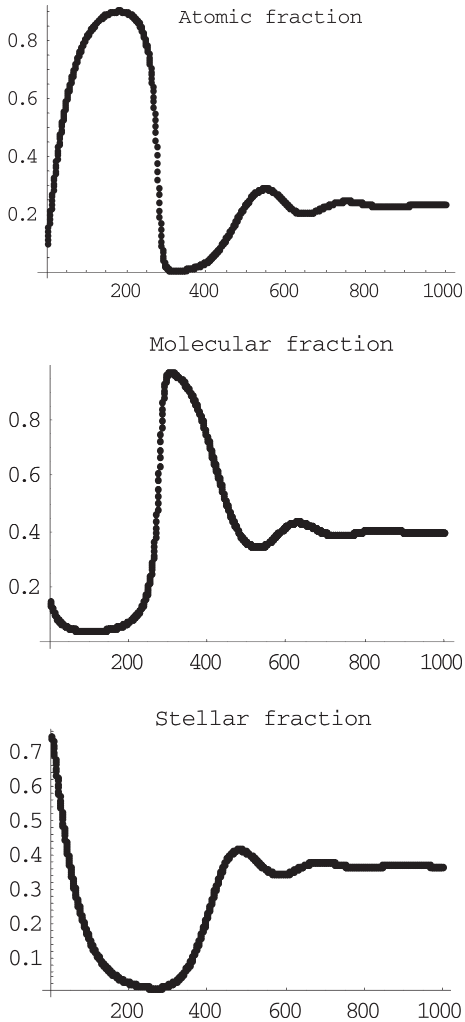

Fig. 2b Atomic, molecular, and stellar fractions as functions of time.

out. The decrease of the molecular fraction by star formation, for instance, is perfectly balanced by the increase caused by the transformation of atomic into molecular gas. The rate of star formation is also constant (Hellings, 1994).

For example consider the following values:

- Parameters
  $n = 1.0$, $k_1 = 10$, $k_2 = 10$.
- Initial conditions
  $m_0 = 0.15$, $s_0 = 0.15$, $a_0 = 0.15$.

The results are represented in Fig. 2a and 2b. We start with a galactic subsystem with high stellar mass fraction of 0.75. These stars all evolve together to an inactive state, meaning that their masses are regenerated in the system in the form of atomic gas. This is why the atomic content increases very sharply in the beginning (see Fig. 2a). All this atomic gas is then transformed completely into molecular gas, a process visible as the diagonal decrease on the second of Fig. 2a. Finally, the system evolves after some very quickly damped oscillations towards a stationary state with mass fraction of about 0.40 for the molecular clouds, 0.35 for the stellar content, and 0.25 for the atomic clouds. These final and stable fractions are represented by the constant levels in Fig. 2b.

Evolution towards a limit cycle

Another possible final state of the stellar formation model is the limit cycle. In this case, the model evolves into an oscillating but stable state. We present the two following examples:

Example (1):

Parameters
$n = 1.7$, $k_1 = 20$, $k_2 = 25$.
Initial conditions
$m_0 = 0.7$, $s_0 = 0.2$, $a_0 = 0.1$.

Example (2):

Parameters
$n = 1.5$, $k_1 = 8$, $k_2 = 15$.
Initial conditions
$m_0 = 0.3$, $s_0 = 0.3$, $a_0 = 0.4$.

The results are shown in Figs. 3 and 4, respectively. The period and amplitude of the oscillations are constants. In particular, this means for the stellar fraction that the birth rate is not constant in time. Star formation is concentrated at certain moments as is easily seen in the time dependency Figs. 3b and 4b. In Fig. 3 there are always active stars in the system, at least

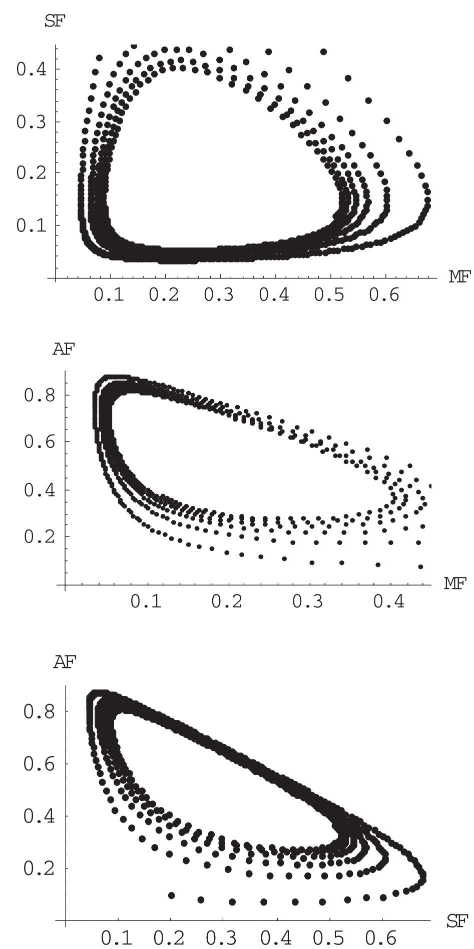

Fig. 3a Phase diagram of star formation model.

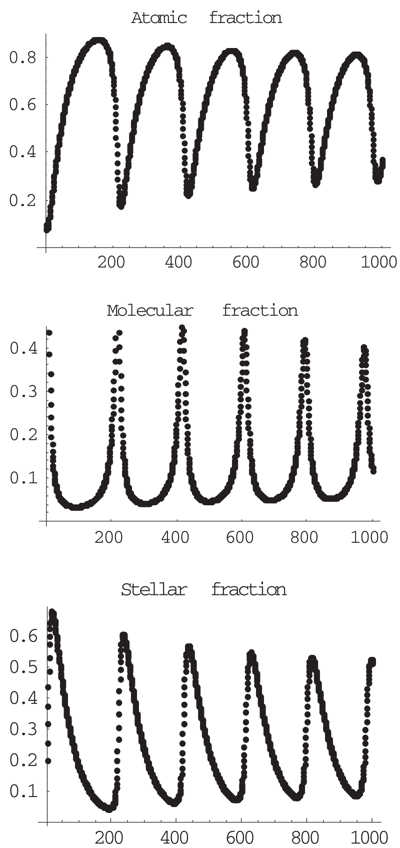

Fig. 3b Atomic, molecular, and stellar fractions as functions of time.

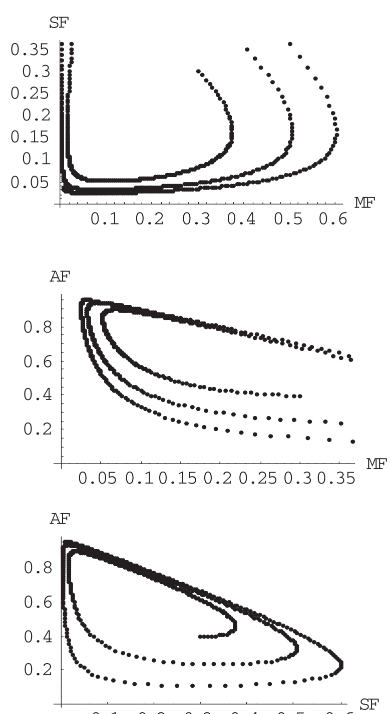

Fig. 4a Phase diagram of star formation model.

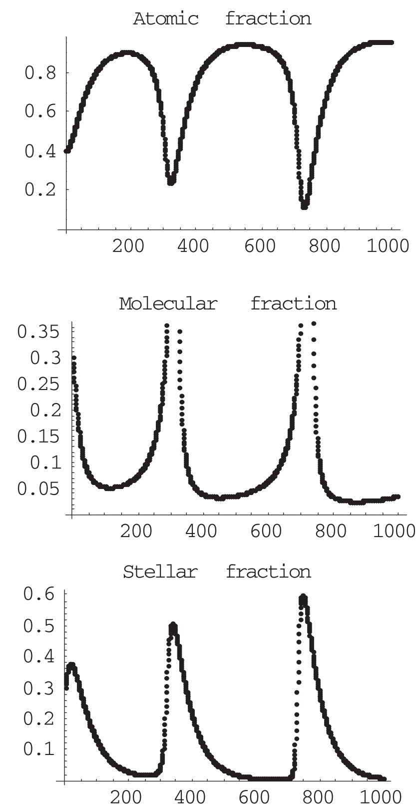

Fig. 4b Atomic, molecular, and stellar fractions as functions of time.

about 5%. The oscillations of Fig. 4 are more violent and this model is characterized by large periods in which the major part of the mass is atomic gas, and nearly all the rest molecular. Almost no active stars are present. Then there is a quite sudden transformation of almost all the atomic gas into molecular gas, immediately followed by a burst of the star formation. Then all these stars evolve more or less, leaving the active star formation system. Their mass is replaced by fresh atomic gas, which becomes again the most important component.

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