



# Final state predictions for J2 gravity perturbed motion of the Earth's artificial satellites using Bispherical coordinates

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**Abstract** In this paper, initial value problem for dynamical astronomy will be established using Bispherical coordinates. A computational algorithm is developed for the final state predictions for J2 gravity perturbed motion of the Earth's artificial satellites. This algorithm is important in targeting, rendezvous maneuvers as well for scientific researches. The applications of the algorithm are illustrated by numerical examples of some test orbits of different eccentricities. The numerical results are extremely accurate and efficient.

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## 1. Introduction

Depending on the application, a curvilinear coordinate system may be simpler to use than the Cartesian coordinate system. For instance, a physical problem with spherical symmetry defined in  $\mathbf{R}^3$  (e.g., motion in the field of a point mass), is usually easier to solve in spherical polar coordinates than in Cartesian coordinates. Also boundary conditions may enforce symmetry. One would describe the motion of a particle in a rectangular box in Cartesian coordinates, whereas one would prefer spher-

ical coordinates for a particle in a sphere. For instance, in the galactic rotation, cylindrical coordinates are usually adopted, while the spherical coordinates are suitable for the dynamics of globular clusters.

On the other hand, the applications of the conventional equations of space dynamic for the motion of Earth's artificial satellites give inaccurate final state predictions which are very important in targeting, rendezvous maneuvers as well for scientific researches

The reason for this inaccurate final state is due to the fact that the equations of motion are unstable in the Liapunov sense (Stiefel and Scheifele 1971). In brief the deficiency of these equations is due to the choice of the variables, which in turn has led some authors to propose successful methods to change of the dependent and/or independent variables so as to regularize the differential equations of motion.

Of these, the method established by Stiefel and Scheifele, in 1971 consists of changing the independent variable from time to a new variable, which is proportional to the eccentric anom-

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ally in the elliptic case or its equivalent in hyperbolic case. The method then changes the coordinates from three-dimensional Cartesian space to a four-dimensional space by what they called the KS transformation. The resulting equations are four-dimensional harmonic oscillator. In fact, the change of the dependent and/or independent variables for the differential equations of motion is one of the focal points of researches in space dynamics. Many studies on the applications of these devices for some orbital systems were done (e.g. Sharaf et al., 1987, 1989, 1991a,b, 1992; Sharaf and Sharaf 1995)

Now, one may ask: does there exist another transformation equation that produces accurate final state predication? The answer is yes as established in three papers of the same authors (Sharaf and Selim 2006, 2011a,b). In these papers, we used respectively the three curvilinear coordinates: Parabolic Cylindrical, Cylindrical and Paraboloidal, and we get very accurate final state predictions.

The efficiency of the usages of these three curvilinear coordinates tempted us to continue in the line of our researches on the utilization of curvilinear coordinate system in the problem final state predictions.

In the present paper, initial value problem for dynamical astronomy will be established using Bispherical coordinates. Computation algorithm was developed for the initial value problem of J<sub>2</sub> gravity perturbed trajectories. Applications of the algorithm for the problem of final state predication are illustrated by numerical examples of some test orbits of different eccentricities.

## 2. Analytical formulations for Bispherical coordinates

### 2.1. Coordinate, velocity transformations

$$\begin{aligned} x &= \frac{a \sin u_1 \cos u_3}{\cosh u_2 - \cosh u_1}; y = \frac{a \sin u_1 \sin u_3}{\cosh u_2 - \cosh u_1}; \\ z &= \frac{a \sinh u_2}{\cosh u_2 - \cosh u_1} \end{aligned} \quad (1)$$

$$\begin{aligned} \dot{x} &= \frac{1}{(\cosh u_1 - \cosh u_2)^2} \{ a \dot{u}_3 \sin u_3 (\cos u_1 - \cosh u_2) \sin u_1 \\ &\quad - a \cos u_3 (\dot{u}_1 - \dot{u}_1 \cos u_1 \cosh u_2 + \dot{u}_2 \sin u_1 \sinh u_2) \}, \\ \dot{y} &= \frac{1}{(\cosh u_1 - \cosh u_2)^2} \{ a \dot{u}_3 \cos u_3 (\cos u_1 - \cosh u_2) \sin u_1 \\ &\quad + a \sin u_3 (\dot{u}_1 - \dot{u}_1 \cos u_1 \cosh u_2 + \dot{u}_2 \sin u_1 \sinh u_2) \}, \\ \dot{z} &= \frac{a}{(\cosh u_1 - \cosh u_2)^2} \{ \dot{u}_2 - \dot{u}_2 \cos u_1 \cosh u_2 \\ &\quad - \dot{u}_1 \sin u_1 \sinh u_2 \} \end{aligned} \quad (2)$$

where  $a$  is constant

#### 2.1.1. Range of variables

$$-\pi < u_1 \leq \pi; 0 \leq u_2 < \infty; -\pi < u_3 \leq \pi$$

#### 2.1.2. The Scale factors

$$h_1^2 = h_2^2 = \left( \frac{a}{\cosh u_2 - \cosh u_1} \right)^2, h_3^2 = \left( \frac{a \sinh u_2}{\cosh u_2 - \cosh u_1} \right)^2$$

### 2.1.3. Inverse transformations

$$\begin{aligned} u_1 &= \tan^{-1} \left[ \frac{2a(x^2 + y^2)^{1/2}}{x^2 + y^2 + z^2 - a^2} \right], \\ u_2 &= \frac{1}{2} \ln \left[ \frac{x^2 + y^2 + (z + a)^2}{x^2 + y^2 + (z - a)^2} \right], \\ u_3 &= \tan^{-1} \left( \frac{y}{x} \right), \end{aligned} \quad (3)$$

$$\begin{aligned} \dot{u}_1 &= \frac{2a}{(x^2 + y^2)^{1/2}} \\ &\quad \times \left[ \frac{(x^2 + y^2 + z^2 - a^2)(x\dot{x} + y\dot{y}) - 2(x\dot{x} + y\dot{y} + z\dot{z})(x^2 + y^2)}{4a^2(x^2 + y^2) + (x^2 + y^2 + z^2 - a^2)^2} \right], \\ \dot{u}_2 &= \frac{2a[\dot{z}(x^2 + y^2 + z^2 + a^2) - 2z(x\dot{x} + y\dot{y} + z\dot{z})]}{(x^2 + y^2 + z^2 + a^2)^2 - 4a^2z^2}, \\ \dot{u}_3 &= \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}. \end{aligned} \quad (4)$$

### 2.1.4. General equations of motion using Bispherical coordinates

$$\dot{u}_1 = u_4$$

$$\dot{u}_2 = u_5$$

$$\dot{u}_3 = u_6$$

$$\begin{aligned} \dot{u}_4 &= \frac{1}{a^2} (\cos u_1 - \cosh u_2)^2 \frac{\partial V}{\partial u_1} \\ &\quad + \frac{1}{\cos u_1 - \cosh u_2} \{ -u_4^2 \sin u_1 - 2u_4u_5 \sinh u_2 \\ &\quad + \sin u_1 (u_5^2 + (1 - u_6^2 \cos u_1 \cosh u_2)) \}, \\ \dot{u}_5 &= \frac{1}{a^2} (\cos u_1 - \cosh u_2)^2 \frac{\partial V}{\partial u_2} \\ &\quad + \frac{1}{\cos u_1 - \cosh u_2} \{ u_4^2 \sinh u_2 - 2u_4u_5 \sin u_1 \\ &\quad + \sinh u_2 (-u_5^2 + u_6^2 \sin^2 u_1) \}, \\ \dot{u}_6 &= \frac{1}{a^2} \left\{ \frac{\partial V}{\partial u_3} (\cot u_1 - \cosh u_2 \csc u_1)^2 \right\} \\ &\quad + \frac{1}{(\cos u_1 - \cosh u_2)} \{ 2((\cos u_1 \cosh u_2 - 1)u_4 \csc u_1 \\ &\quad - u_5 \sinh u_2)u_6 \} \end{aligned}$$

## 3. J2 Gravity perturbed motion of the Earth's artificial satellites

### 3.1. The potential $V$ and its partial derivatives

For J<sub>2</sub> gravity perturbed trajectories, the potential  $V$  is given as:

$$V \equiv V(x, y, z) = \frac{\mu}{r} + \frac{c}{r^3} \left[ 3 \left( \frac{z}{r} \right)^2 - 1 \right] \quad (5)$$

where

$$c = J_2 \mu R_\oplus^2 / 2; \quad r = (x^2 + y^2 + z^2)^{1/2},$$

with  $\mu$  is the gravitational parameter, which is universal gravitational constant times the Earth's mass;  $J_2$  the second zonal harmonic, and  $R_\oplus$  is the mean Earth's equatorial radius. The numerical values of these constants are:

$$\mu = 398600.8 \text{ km}^3/\text{sec}^2,$$

$$J_2 = 1.0826157 \times 10^{-3},$$

$$R_\oplus = 6378.135 \text{ km}.$$

### 3.2. Equations of motion in Cartesian coordinates

To describe the motion of a satellite about the Earth is to write a set of differential equations describing the rate of change of the position and velocity. These equations for J2 Gravity perturbed motion are:

$$\ddot{x} = \frac{\partial V}{\partial x} = -\frac{\mu x}{r^3} + 3c \left( \frac{x}{r^5} \right) \left( 1 - \frac{5z^2}{r^2} \right), \quad (6.1)$$

$$\ddot{y} = \frac{\partial V}{\partial y} = -\frac{\mu y}{r^3} + 3c \left( \frac{y}{r^5} \right) \left( 1 - \frac{5z^2}{r^2} \right), \quad (6.2)$$

$$\ddot{z} = \frac{\partial V}{\partial z} = -\frac{\mu z}{r^3} + 3c \left( \frac{z}{r^5} \right) \left( 3 - \frac{5z^2}{r^2} \right). \quad (6.3)$$

The coordinate system is initially fixed with the xy plane corresponding to the Earth's equatorial plane

### 3.3. Equations of motion in orthogonal curvilinear coordinates

The kinetic energy of a particle of unit mass is given as:

$$T = \frac{1}{2} (h_1^2 \dot{u}_1^2 + h_2^2 \dot{u}_2^2 + h_3^2 \dot{u}_3^2),$$

In the present paper we shall suppose that the motion is controlled only by the gravitational potential,  $V$ , which will be in general a function of  $(u_1, u_2, u_3)$ .

By using Lagrange's dynamical equations, we get for the equations of motion in the orthogonal curvilinear coordinates in the forms:

$$\ddot{\mathbf{u}} = \mathbf{G} \left( \mathbf{u}, \dot{\mathbf{u}}, \frac{\partial V}{\partial \mathbf{u}} \right) \quad (7)$$

The explicit expression of the right hand side of Equation (7) will be given later for Bispherical coordinates.

## 4. Initial value procedures

In what follows, a general procedure will be developed for the final state predictions for J2 gravity perturbed motion of the Earth's artificial satellites using orthogonal curvilinear coordinates. The procedure is described through its basic points: input, output and computational steps

**Input: (1)**  $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$  at  $\mathbf{t} = \mathbf{t}_0$ ,

(2) the final time  $t = t_f$ ;

(3)  $\frac{\partial V}{\partial x} = F_1(x, y, z); \frac{\partial V}{\partial y} = F_2(x, y, z); \frac{\partial V}{\partial z} = F_3(x, y, z)$ .

**Output: (1)**  $u_j; \dot{u}_j; j = 1, 2, 3$  at  $t = t_f$

(2)  $x, y, z; \dot{x}, \dot{y}, \dot{z}$  at  $t = t_f$

Computational steps:

1. Using Eqs. (1) and (6) to find the analytical expressions of the partial derivatives  $\frac{\partial V}{\partial u_j}$ , where  $j = 1, 2, 3$  as:

$$\frac{\partial V}{\partial u_1} = \frac{1}{4(\cos u_1 + \cos h u_2)^4} \left( 1 + \frac{2 \cos u_1}{-\cos u_1 + \cos h u_2} \right)^{1/2} \times$$

$$\begin{aligned} & ((-12c + 5\mu + 2(3c + \mu) \cos 2u_1) \cos h u_2 \\ & + 4 \cos u_1 (3c + \mu + (-3c + \mu) \cos h 2u_2) \\ & + (6c + \mu) \cos h 3u_2) \sin u_1 \end{aligned}$$

$$\frac{\partial V}{\partial u_2} = \frac{1}{4(\cos u_1 + \cos h u_2)^4} \left( 1 + \frac{2 \cos u_1}{-\cos u_1 + \cos h u_2} \right)^{1/2} \times$$

$$\begin{aligned} & ((3c + \mu) \cos 3u_1 + 4(3c + \mu + (-3c + \mu) \cos 2u_1) \cos h u_2 \\ & + \cos u_1 (5(-3c + \mu) + 2(6c + \mu) \cos h 2u_2)) \sin h u_2 \end{aligned}$$

$$\frac{\partial V}{\partial u_3} = 0$$

2. Compute numerically the initial conditions,  $u_{0j}; j = 1, 2, \dots, 6$  for the above system from the result of step 7 by applying the transformations:

$$(x, y, z, u_1, u_2, u_3) \rightarrow (x_0, y_0, z_0, u_{01}, u_{02}, u_{03})$$

$$(\dot{x}, \dot{y}, \dot{z}, \dot{u}_1, \dot{u}_2, \dot{u}_3) \rightarrow (\dot{x}_0, \dot{y}_0, \dot{z}_0, u_{04}, u_{05}, u_{06})$$

3. Using these initial conditions to solve numerically the above differential system for  $u_j; j = 1, 2, \dots, 6$  at  $t = t_f$ , where  $u_4 \equiv \dot{u}_1, u_5 \equiv \dot{u}_2, u_6 \equiv \dot{u}_3$  at  $t = t_f$
4. Using  $u_j; \dot{u}_j; j = 1, 2, 3$  to compute numerically  $x, y, z$  and  $\dot{x}, \dot{y}, \dot{z}$  at  $t = t_f$  from the direct transformation Eq. (1)
5. End.

## 5. Numerical applications

In this section, the numerical applications of the computational developments of the above section will be considered.

### 5.1. Test orbits

For the applications of the above formulations, we consider four test orbits given in the Appendix C of Vinti's book, 1998. All these orbits have the initial time  $t_0 = 0$  and each of different flight time  $t_f$ , they cover the three basic types of conic motion-elliptic, parabolic and hyperbolic orbits.

### 5.2. Reference orbits

For each orbit, the J2 gravity perturbed equations of motion in Cartesian coordinate are solved by any differential integrator.

**Table 1** Initial and Reference Final State Solutions of the Test Orbits.

Initial condition	Reference solution
<i>Low-earth orbit</i>	
$x_o = 2328.9694$	$x_R = -516.450939$
$y_o = -5995.21600$	$y_R = -3026.5115474$
$z_o = 1719.97894$	$z_R = 5848.117544$
$\dot{x}_o = 2.911101130$	$\dot{x}_R = 3.96699$
$\dot{y}_o = -0.98164053$	$\dot{y}_R = -6.121618$
$\dot{z}_o = -7.090499220$	$\dot{z}_R = -2.754866$
$t_f = 10\ 000\ s$	
$e = 0.00949621$	
<i>Geosynchronous orbit</i>	
$x_o = -14420.99601$	$x_R = -13755.32790$
$y_o = -39621.36091$	$y_R = -39857.2791670$
$z_o = 0$	$z_R = 0$
$\dot{x}_o = 2.88923555010$	$\dot{x}_R = 2.906438$
$\dot{y}_o = -1.0515957400$	$\dot{y}_R = -1.003071$
$\dot{z}_o = 0$	$\dot{z}_R = 0$
$t_f = 86400\ s$	
$e = 0$	
<i>Parabolic orbit of zero inclination</i>	
$x_o = 10\ 000.00$	$x_R = -65357.0633677$
$y_o = 0$	$y_R = 54991.369699$
$z_o = 0$	$z_R = 0$
$\dot{x}_o = 0$	$\dot{x}_R = -2.871888$
$\dot{y}_o = 8.9286113142$	$\dot{y}_R = 1.050276$
$\dot{z}_o = 0$	$\dot{z}_R = 0$
$t_f = 21\ 600\ s$	
$e = 1$	
<i>Hyperbolic orbit of zero inclination</i>	
$x_o = 10\ 000.00000$	$x_R = -1.898682002201 \times 10^6$
$y_o = 0$	$y_R = 1.020654164530 \times 10^6$
$z_o = 0$	$z_R = 0$
$\dot{x}_o = 0$	$\dot{x}_R = -2.049040$
$\dot{y}_o = 9.20000000$	$\dot{y}_R = 1.052929$
$\dot{z}_o = 0$	$\dot{z}_R = 0.0 \times 10^{-9}$
$t_f = 864\ 000\ s$	
$e = 1.12343$	

A final state prediction was determined by reducing the step size until at least five decimal places ( $< 10^{-2}$  meter (m)) stabilized in  $x(t_f)$ ,  $y(t_f)$  and  $z(t_f)$ . These values are considered as reference final state solutions to the orbit they refer and are denoted by:

$$\begin{aligned} \mathbf{r}_R &\equiv (x_R(t_f), y_R(t_f), z_R(t_f)) \text{ and } \dot{\mathbf{r}}_R \\ &\equiv (\dot{x}_R(t_f), \dot{y}_R(t_f), \dot{z}_R(t_f)) \end{aligned} \quad (8)$$

for the reference position and velocity vectors respectively.

Table 1 gives the initial and the reference final state solutions of the test orbits. The length is in km, while the velocities are in km/sec

### 5.3. Efficiency of Bispherical coordinates

Upon the above reference solutions the efficiency of the initial value problem for J2 gravity perturbed trajectories using curvilinear coordinates (CL- solution) may be checked by testing its ability in predicting final states within certain tolerances as follows:

**Table 2** Efficiency of the initial value problem for J2 gravity perturbed motion using Bispherical coordinates.

Orbit	$\Delta R$ (m)	$\Delta v$ (m/sec)
Low-earth orbit (elliptic)	0.02251	0.00003
Geosynchronous orbit(circular)	0.00103388	0.0
Parabolic orbit of $0^\circ$	0.013024	0.0
Inclination(parabolic)		
Hyperbolic orbit of $0^\circ$	0.03469	0.0
inclination(hyperbolic)		

Let  $\mathbf{r} \equiv (x(t_f), y(t_f), z(t_f))$  and  $\dot{\mathbf{r}} \equiv (\dot{x}(t_f), \dot{y}(t_f), \dot{z}(t_f))$  are the final state of the CL- solution of a given orbit .The efficiency of the CL- solution is then checked by the magnitude of the error criteria  $\Delta R$  and  $\Delta v$  as:

$$\Delta R = \{(x - x_R)^2 + (y - y_R)^2 + (z - z_R)^2\}^{1/2} \times 1000(\text{in m}), \quad (9.1)$$

$$\Delta v = \{(\dot{x} - \dot{x}_R)^2 + (\dot{y} - \dot{y}_R)^2 + (\dot{z} - \dot{z}_R)^2\}^{1/2} \times 1000(\text{in m/sec}), \quad (9.2)$$

such that, the small the values of  $\Delta R$  and  $\Delta v$ , the higher the efficiency will be, in this respect, we may define an acceptable solution set (**S.S**) to the problem at hand as:

$$\mathbf{S.S} = ((\mathbf{r}, \dot{\mathbf{r}}) : \Delta R \leq \varepsilon_1, \Delta v \leq \varepsilon_2) \quad (10)$$

where  $\varepsilon_{1,2}$  are given tolerances. For the very accurate predictions required nowadays we may consider the tolerances  $\varepsilon_{1,2}$  as:

$$\varepsilon_1 = 2 \text{ meter} \pm 10 \text{ centimeter}, \quad (11.1)$$

$$\varepsilon_2 = .259 \text{ m/sec}. \quad (11.2)$$

### 5.4. Numerical results

(see Table 2).

## 6. Conclusion

In concluded the present paper we stress that initial value problem for dynamical astronomy was established using Bispherical coordinates A computational algorithm is developed for the final state predictions for J2 gravity perturbed motion of the Earth's artificial satellites using Bispherical orthogonal curvilinear coordinates. This algorithm is important in targeting, rendezvous maneuvers as well for scientific researches.

The applications of the algorithm were illustrated by numerical examples of some test orbits of different eccentricities. The numerical results are extremely accurate and efficient.

## References

- Sharaf, M.A., Arafah, M.R., Awad, M.E., 1987. Prediction of satellites in earth's gravitational field with axial symmetry using burdet's regularized theory earth. Moon and Planets 38, 21–36.
- Sharaf, M.A., Awad, M.E., Banajh, M.A., 1989. Efficient algorithms of the earth's gravitational potential and its partial derivatives for satellite prediction earth. Moon and Planets 47, 171–192.

- Sharaf, M.A., Awad, M.E., Najmuldeen, S.A., 1991a. Motion of artificial satellites in the set of eulerian redundant parameters earth. *Moon and Planets* 55, 21–44.
- Sharaf, M.A., Awad, M.E., Najmuldeen, S.A., 1991b. Motion of artificial satellites in the set of eulerian redundant parameters II earth. *Moon and Planets* 55, 223–231.
- Sharaf, M.A., Awad, M.E., Najmuldeen, S.A., 1992. Motion of artificial satellites in the set of eulerian redundant parameters III earth. *Moon and Planets* 56, 141–164.
- Sharaf, M.A., Sharaf, A.A., 1995. Motion of artificial earth satellites in the general gravity field using KS regularized theory. *Bull. Fac. Sci. Cairo Univ. Egypt* 63, 157–181.
- Sharaf, M.A., Selim, H.H., 2006. Motion in parabolic cylindricl coordinates: application to J2 gravity pertubed trajectories. *J. KAS*, 39.
- Sharaf, M.A., Selim, H.H., 2011a. Motion in cylindrical coordinates: applications to J2 gravity perturbed trajectories of space dynamics. *J. CAOSP*, 41.
- Sharaf, M.A., Selim, H.H. 2011b. Motion in Paraboloidal Coordinates: Applications to J2 Gravity Perturbed Trajectories of Space Dynamics. *IJOPMA*, 1.
- Stiefel, E.L., Scheifele, G., 1971. *Linear and Regular Celestial Mechanics*. Springer-Verlag, Berlin.