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V cosmological models in $f(R, T)$ modified gravity with $\Lambda(T)$ by using generation technique



Nasr Ahmed^{a,b,*}, Anirudh Pradhan^c, M. Fekry^d, Sultan Z. Alamri^a

^a Mathematics Department, Faculty of Science, Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia

^b Astronomy Department, National Research Institute of Astronomy and Geophysics, Helwan, Cairo, Egypt

^c Department of Mathematics, Institute of Applied Sciences and Humanities, GLA University, Mathura 281 406, Uttar Pradesh, India

^d Department of Basic Science, Preparatory Year, King Saud University, Saudi Arabia

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Abstract A new class of cosmological models in $f(R, T)$ modified theories of gravity proposed by Harko et al. (2011), where the gravitational Lagrangian is given by an arbitrary function of Ricci scalar R and the trace of the stress-energy tensor T , has been investigated for a specific choice of $f(R, T) = f_1(R) + f_2(T)$ by generation of new solutions. Motivated by recent work of Pradhan et al. (2015) we have revisited the recent work of Ahmed and Pradhan (2014) by using a generation technique, it is shown that $f(R, T)$ modified field equations are solvable for any arbitrary cosmic scale function. A class of new solutions for particular forms of cosmic scale functions have been investigated. In the present study we consider the cosmological constant Λ as a function of the trace of the stress energy-momentum-tensor, and dub such a model “ $\Lambda(T)$ gravity” where we specified a certain form of $\Lambda(T)$. Such models may exhibit better equability with the cosmological observations. The cosmological constant Λ is found to be a positive decreasing function of time which is supported by results from recent supernovae Ia observations. Expressions for Hubble’s parameter in terms of redshift, luminosity distance redshift, distance modulus redshift and jerk parameter are derived and their significances are described in detail. The physical and geometric properties of the cosmological models are also discussed.

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* Corresponding author at: Mathematics Department, Faculty of Science, Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia.
 E-mail address: nasr.ahmed@nriag.sci.eg (N. Ahmed).

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1. Introduction

Recent observational prediction (Perlmutter et al., 1998, 1999, 2003; Riess et al., 1998, 2004; Clocchiatti et al., 2006) that our universe is going through a phase of accelerated expansion redact new pathway in modern cosmology. It is generally assumed that this cosmic acceleration is due to some kind of exotic matter with negative pressure known as dark energy (DE). The nature of DE and its cosmological origin remains problematic so far. To understand the origin of dark energy and its nature is one of the greatest problems of the 21st century. In order to explain the nature of the DE and the accelerated expansion, a diversity of theoretical models have been proposed in the literature, such as cosmological constant (Padmanabhan, 2003), quintessence (Farooq et al., 2011; Martin, 2008), phantom energy (Nojiri et al., 2003; Alam et al., 2004; Jamil and Hussain, 2011), k-essence (Chiba et al., 2000; Pasqua et al., 2012), tachyon (Padmanabhan and Chaudhury, 2002; Farooq et al., 2010), f-essence (Jamil et al., 2011), Chaplyng gas (Bento et al., 2002; Jamil, 2010), and cosmological nuclear energy (Gupta and Pradhan, 2010).

The Einstein general relativity theory of gravity is well tested and passes all observational local test up to the solar system scale. At large scales the Einstein gravity model of general relativity becomes failure, and a more general action needs to describe the gravitational field. The modification in Einstein–Hilbert action on larger cosmological scales may be a correct explanation of a late time cosmic acceleration of the expanding universe. In this respect, $f(R)$ modified theories of gravity provide a natural unification of the early-time inflation and late-time acceleration (Capozziello and Francaviglia, 2008; Nojiri and Odintsov, 2011). Among the other modified theories, a theory of scalar-Gauss-Bonnet gravity, so called $f(G)$ (Nojiri et al., 2006) and a theory of $f(T)$ gravity (Linder, 2010), where T is the torsion have been proposed to explain the accelerated expansion of the universe.

Recently, Harko et al. (2011) purported a new $f(R, T)$ modified theories of gravity, wherein the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and the trace of the stress energy tensor T . They presented the field equations of several particular models, corresponding to some explicit forms of the function $f(R, T)$.

$$f(R, T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases}$$

The cosmological consequences for the class $f(R, T) = R + 2f(T)$ have been recently discussed in detail by many authors (Houndjo et al., 2013; Pasqua et al., 2013; Adhav, 2012; Chaubey and Shukla, 2013; Sahoo and Mishra, 2014; Sahoo and Mishra, 2014; Reddy et al., 2013; Singh and Singh, 2014; Chakraborty, 2013; Houndjo, 2012; Shabani and Farhoudi, 2013). Recently, Chakraborty (2013) has discussed $f(R, T)$ gravity by considering three cases (a) $f(R, T) = R + h(T)$, (b) $f(R, T) = Rh(T)$ and (c) $f(R, T)$ is arbitrary. Houndjo (2012) has developed the cosmological reconstruction of $f(R, T)$ gravity as $f(R, T) = f_1(R) + f_2(T)$ and discussed the transition of matter dominated phase to an accelerated phase. Shabani and Farhoudi (2013) have studied $f(R, T)$ cosmological models in phase space by choosing $f(R, T) = g(R) + h(T)$. Recently, Ahmed and Pradhan (2014)

have reconstructed the modified $f(R, T)$ gravity by specific choice of $f(R, T) = f_1(R) + f_2(R)$ with “ $A(T)$ gravity” and obtained new accelerating cosmological models in Bianchi type- V space-time. Following this new conception given in Ahmed and Pradhan (2014), Yadav (2013) has obtained Bianchi type- V string cosmological model with power-law expansion in $f(R, T)$ gravity. Recently, Pradhan et al. (2015) studied the reconstruction of modified $f(R, T)$ with $A(T)$ gravity in general class of Bianchi cosmological models following reference Ahmed and Pradhan (2014).

In recent years, several authors (Pradhan and Kumar, 2001; Ellis and MacCallum, 1969; Ryan and Shepley, 1975; Hinshaw et al., 2003) have investigated the solutions of Einstein Field Equations (EFEs) for homogeneous but anisotropic models by using some different generation techniques. Bianchi spaces I – IX are useful tools in constructing models of spatially homogeneous cosmologies (Ellis and MacCallum, 1969; Ryan and Shepley, 1975). From these models, homogeneous Bianchi type V universes are the natural generalization of the open Friedman Robertson Walker (FRW) model which eventually isotropize. Modern observations (Wilkinson Microwave Anisotropy Probe (WMAP) data for example) indicate that the universe is not completely symmetric (Camci et al., 2001; Pradhan et al., 2005; Pradhan et al., 2006). From that point of view Bianchi models (which represents spatially homogeneous and anisotropic spaces) are more appropriate in describing the universe as it has less symmetry than the standard FRW models. Recently, Camci et al. (2001) and Pradhan et al. (2005, 2006) have derived a new technique for generating exact solutions of EFEs with perfect fluid for Bianchi type V space-time.

Motivated by the above discussions, in this paper, we purpose to study the cosmology of the so-called $f(R, T)$ gravity, first introduced in reference Harko et al. (2011) and then studied in references Ahmed and Pradhan (2014) and Pradhan et al. (2015) by using new generating technique (Poplawski, 2006a,b; Magnano, 1995).

2. The basic equations and generation technique

The theory suggests a modified gravity action given by

$$S = \frac{1}{16\pi} \int f(R, T) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x, \quad (1)$$

where $f(R, T)$ is an arbitrary function of the Ricci scalar, R , and the trace T of the stress-energy tensor of the matter, $T_{\mu\nu}$. L_m is the matter Lagrangian density. $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g^{\mu\nu}}, \quad (2)$$

and its trace by $T = g^{\mu\nu} T_{\mu\nu}$. The field equations are obtained (Harko et al., 2011) as

$$f_R(R, T) R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \nabla^i \nabla_i - \nabla_\mu \nabla_\nu) f_R(R, T) = 8\pi T_{\mu\nu} - f_T(R, T) T_{\mu\nu} - f_T(R, T) \Theta_{\mu\nu}. \quad (3)$$

where $f_R(R, T) = \frac{\partial f(R, T)}{\partial R}$, $f_T(R, T) = \frac{\partial f(R, T)}{\partial T}$, $\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}$ and ∇_i denotes the covariant derivative.

The stress-energy tensor of the matter Lagrangian is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (4)$$

where $u^\mu = (0, 0, 0, 1)$ is the four velocity vector satisfying $u^\mu u_\mu = 1$ and $u^\mu \nabla_\nu u_\mu = 0$. ρ and p are the energy density and pressure of the fluid respectively.

Assuming $f(R, T) = f_1(R) + f_2(T)$, [Ahmed and Pradhan \(2014\)](#) have recently reconstructed the gravitational field equation of $f(R, T)$ gravity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} - \left(p + \frac{1}{2}T\right)g_{\mu\nu} = \frac{8\pi + \lambda}{\lambda}T_{\mu\nu}. \quad (5)$$

Comparing with Einstein equations

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}. \quad (6)$$

The arbitrary λ is given a negative small value to ensure having the same sign of the RHS of (6), this choice of λ will be kept throughout the article. The term $(p + \frac{1}{2}T)$ can now be considered as a cosmological constant.

$$\Lambda \equiv \Lambda(T) = p + \frac{1}{2}T. \quad (7)$$

The dependence of the cosmological constant Λ on the trace of the energy momentum tensor T has been proposed before by [Poplawski \(2006a\)](#) where the model was denoted “ $\Lambda(T)$ gravity”. $\Lambda(T)$ gravity is more general than the Palatini $f(R)$ and could be reduced to it if the pressure of matter is neglected ([Sahni, 2002; Visser, 2005; Astier, 2006](#)). Considering the perfect fluid case $T = -3p + \rho$, Eq. (7) reduces to

$$\Lambda = \frac{1}{2}(\rho - p). \quad (8)$$

We use the following metric of general class of Bianchi type- V cosmological model:

$$ds^2 = dt^2 - A^2 dx^2 - e^{-2\alpha x} [B^2 dy^2 - C^2 dz^2], \quad (9)$$

where α is a constant and the functions $A(t)$, $B(t)$ and $C(t)$ are the three anisotropic directions of expansion in normal three dimensional space. The average scale factor a , the spatial volume V and the average Hubble's parameter H are defined as

$$a = (ABC)^{\frac{1}{3}}, \quad (10)$$

$$V = a^3 = ABC, \quad (11)$$

and

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \quad (12)$$

respectively with $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$. Here and elsewhere the dot denotes differentiation with respect to cosmic time t . From Eqs. (10)–(12) we get

$$H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \quad (13)$$

Now the cosmological Eq. (5) for the energy momentum tensor (4) and the metric (9) are

$$\frac{\dot{B}\dot{C}}{AC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} = \left(\frac{8\pi + \lambda}{\lambda} \right) p - \Lambda, \quad (14)$$

$$\frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} = \left(\frac{8\pi + \lambda}{\lambda} \right) p - \Lambda, \quad (15)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\alpha^2}{A^2} = \left(\frac{8\pi + \lambda}{\lambda} \right) p - \Lambda. \quad (16)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3\alpha^2}{A^2} = - \left(\frac{8\pi + \lambda}{\lambda} \right) \rho - \Lambda, \quad (17)$$

$$2 \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0. \quad (18)$$

Integrating Eq. (18) and absorbing the integration constant into B or C , we obtain

$$A^2 = BC, \quad (19)$$

without any loss of generality. From Eqs. (14)–(18), we obtain

$$2 \frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B} \right)^2 = 2 \frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C} \right)^2, \quad (20)$$

which on integration yields

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = \frac{k}{(BC)^{\frac{3}{2}}}, \quad (21)$$

where k is a constant of integration. Hence, for the metric function B or C in (21), some scale transformations permit us to get new metric function B or C .

Under the scale transformation $dt = B^{\frac{1}{2}} d\tau$, Eq. (21) becomes

$$CB_\tau - BC_\tau = kC^{-1/2}, \quad (22)$$

where the subscript denotes derivative with respect to τ . Considering Eq. (22) as a linear differential equation for B , where C is an arbitrary function, we get

$$(i) \quad B = k_1 C + kC \int \frac{d\tau}{C^{5/2}}, \quad (23)$$

where k_1 is an integrating constant. Similarly, using the transformations $dt = B^{3/2} d\tilde{\tau}$, $dt = C^{1/2} dT$, and $dt = C^{3/2} d\tilde{T}$ in Eq. (21), we get respectively

$$(ii) \quad B(\tilde{\tau}; k_2, k) = k_2 C \exp \left(k \int \frac{d\tilde{\tau}}{C^{3/2}} \right), \quad (24)$$

$$(iii) \quad C(T; k_3, k) = k_3 B - kB \int \frac{dT}{B^{5/2}}, \quad (25)$$

and

$$(iv) \quad C(\tilde{T}; k_4, k) = k_4 B \exp \left(k \int \frac{d\tilde{T}}{B^{3/2}} \right), \quad (26)$$

where k_2, k_3 and k_4 are constants of integration. Thus choosing any given function B or C in Cases (i), (ii), (iii) and (iv), one can get B or C and hence A from (19).

3. Generation of new solutions

We consider the following four cases:

3.1. Case (i): Let $C = \tau^n$ (n is a real number satisfying $n \neq \frac{2}{3}$)

In this case, Eq. (23) gives

$$B = k_1 \tau^n + \frac{2k}{2-5n} \tau^{1-3n/2} \quad (27)$$

and then from (19), we obtain

$$A^2 = k_1 \tau^{2n} + \frac{2k}{2-5n} \tau^{1-n/2}. \quad (28)$$

Hence the metric (9) reduces to the new form

$$ds^2 = (k_1 \tau^n + 2\ell \tau^{\ell_1}) [d\tau^2 - \tau^n dx^2] - e^{2\alpha x} [(k_1 \tau^n + 2\ell \tau^{\ell_1})^2 dy^2 + \tau^{2n} dz^2], \quad (29)$$

where

$$\ell = \frac{k}{2-5n} \quad \text{and} \quad \ell_1 = 1 - \frac{3n}{2}. \quad (30)$$

The metric (29) is a four-parameter family of solutions to EFEs with a perfect fluid. For this derived model (29), the physical parameters, i.e. the pressure (p), the energy density (ρ) and the cosmological constant (Λ) and the kinematic parameters, i.e. the scalar of expansion (θ), the shear scalar (σ), the proper volume (V^3) and the deceleration parameter (q) are given by

$$p(\tau) = \lambda \frac{F_1(\tau)}{g_1(\tau)} \quad (31)$$

where

$$g_1(\tau) = 16(\lambda^2 + 32\pi^2 + 12\pi\lambda)(-2k_1 \tau^{2n+2} + 5k_1 n \tau^{2n+2} - 2k \tau^{3-\frac{n}{2}}) \left[k_1^2 \tau^{5n} (5n-2)^2 + 4kk_1 \tau^{\frac{5n}{2}+1} (2-5n) + 4k^2 \tau^2 \right]. \quad (32)$$

$$F_1(\tau) = \tau^4 \alpha^2 k^2 \pi (512 - 1280n) + k_1^3 \tau^{7n} (5040\lambda n^3 + 3000\lambda n^5 - 44,800\pi n^4 - 6600\lambda n^4 + 30,720\pi n^3 - 9216\pi n^2 - 1632\lambda n^2 + 1024\pi n + 192\lambda n + 24,000\pi n^5) + k_1^2 \pi \alpha^2 \tau^{5n+2} (512 - 8000n^3 + 9600n^2 - 3840n) + k_1 k^2 \tau^{2n+2} (-128\pi - 8\lambda - 2406\lambda n^2 + 8720\pi n^3 + 3264\pi n + 684\lambda n + 1865\lambda n^3 - 10,848\pi n^2) + k_1^3 \tau^{3-\frac{n}{2}} (-302\lambda n^2 + 1280\pi n + 256\lambda n - 1376\pi n^2 - 8\lambda - 128\pi) + kk_1^2 \tau^{1+\frac{9n}{2}} \times (-3975\lambda n^4 - 22,800\pi n^4 + 600\lambda n - 17,088\pi k n^2 + 2688\pi n + 6930\lambda n^3 + 35,040\pi n^3 - 3636\lambda n^2) + kk_1 \alpha^2 \tau^{\frac{5n}{2}+3} (1024\pi + 6400\pi n^2 - 5120\pi n). \quad (33)$$

$$\rho(\tau) = -\lambda \frac{F_2(\tau)}{g_2(\tau)} \quad (34)$$

where

$$g_2(\tau) = 16(-2k_1 \tau^{2n+2} + 5nk_1 \tau^{2n+2} - 2k \tau^{3-\frac{n}{2}}) \times \left[kk_1 \tau^{\frac{5n}{2}+1} (-240n\pi\lambda + 8\lambda^2 + 256\pi^2 - 20n\lambda^2 + 96\pi\lambda - 640n\pi^2) + k_1^2 \tau^{5n} (48\lambda\pi - 640n\pi^2 - 20n\lambda^2 + 25\lambda^2 n^2 + 800\pi^2 n^2 + 4\lambda^2 + 128\pi^2 + 300\pi\lambda n^2 - 240\lambda n\pi) + k^2 \tau^2 (48\lambda\pi + 4\lambda + 128\pi^2) \right]. \quad (35)$$

$$F_2(\tau) = \alpha^2 k^2 \tau^4 (1536\pi - 3840n\pi + 256\lambda - 640n\lambda) + k_1^3 \tau^{7n} (240\lambda n^3 + 3000\lambda n^5 - 28,800\pi n^4 - 2600\lambda n^4 + 11,520\pi n^3 - 1536\pi n^2 + 288\lambda n^2 - 64\lambda n + 24,000\pi n^5) + \alpha^2 k_1^2 \tau^{5n+2} (1536\pi - 24,000\pi n^3 - 11,520\pi n + 28,800\pi n^2 + 256\lambda - 4000\lambda n^3 + 4800\lambda n^2 - 1920\lambda n) + k_1 k^2 \tau^{2n+2} (-256\pi - 40\lambda + 1794\lambda n^2 - 3680\pi n^3 - 1152n\pi - 420n\lambda - 1235\lambda n^3 + 5952n^2) + k_1^3 \tau^{3-\frac{n}{2}} (218\lambda n^3 - 256n\pi - 128n\lambda + 704\pi n^2 - 40\lambda - 256\pi) + kk_1^2 \tau^{1+\frac{9n}{2}} (525\lambda n^4 - 4800\pi n^4 - 456n\lambda + 6912\pi n^2 - 1536n\pi - 3270\lambda n^3 - 5760\pi n^3 + 2364\lambda n^2) + kk_1 \alpha^2 \tau^{\frac{5n}{2}+3} (3072 + 19,200\pi n^2 - 15,360n\pi + 512\lambda + 3200\lambda n^2 - 2560n\lambda) \quad (36)$$

$$\Lambda = -\lambda \frac{F_3(\tau)}{g_3(\tau)} \quad (37)$$

where

$$g_3(\tau) = 16(4\pi + \lambda)(-2k_1 \tau^{2n+2} + 5nk_1 \tau^{2n+2} - 2k \tau^{3-\frac{n}{2}}) \times (4k_1^2 \tau^{5n} - 20nk_1^2 \tau^{5n} + 8kk_1 \tau^{\frac{5n}{2}+1} + 25n^2 k_1^2 \tau^{5n} - 20k_1 \tau^{\frac{5n}{2}+1} nk + 4k^2 \tau^2) \quad (38)$$

and

$$F_3(\tau) = k_1 k^2 \tau^{2n+2} (315n^3 - 306n^2 + 132n - 24) + k^3 \tau^{3-\frac{n}{2}} (-42n^2 + 64n - 24) + \alpha^2 k_1^2 \tau^{5n+2} (-2000n^3 + 2400n^2 - 960n + 128) + \alpha^2 k^2 \tau^4 (128 - 320n) + kk_1 \alpha^2 \tau^{\frac{5n}{2}+3} (1600n^2 - 1280n + 256) + kk_1^2 \tau^{\frac{9n}{2}+1} (-1725n^4 - 636n^2 + 72n + 1830n^3) + k_1^3 \tau^{7n} (3000n^5 - 4600n^4 + 2640n^3 - 672n^2 + 64n). \quad (39)$$

The variation of pressure versus time is plotted in Fig. 1(a) for $\lambda = -0.1$, $k_1 = 1$, $k_2 = -1$, $\alpha = 0.1$ and $n = 0.25$. We can see that pressure is an increasing function of time where It starts from a large negative value and approaches to zero at the present epoch. It is generally assumed that the discovered accelerated expansion of the universe is due to some kind of energy-matter with negative pressure known as ‘dark energy’. Thus, the nature of pressure in our model is in a good agreement with this assumption.

Fig. 1(b) indicates the behavior of the energy density versus time. The energy density remains always positive and decreasing function of time. It converges to zero as $t \rightarrow \infty$ as expected.

The cosmological term Λ versus time is plotted in Fig. 1(c). We see that Λ is a decreasing function of time t and it approaches a small positive value at the present epoch. Recent cosmological observations (Perlmutter et al., 1998, 1999, 2003; Riess et al., 1998, 2004; Clocchiatti et al., 2006) suggest a very tiny positive cosmological constant Λ with a magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ -term. Thus, the nature of Λ in our derived models is supported by observations.

The physical parameters such as Hubble’s parameters (H), expansion scalar (θ), shear scalar (σ), spatial volume (V),

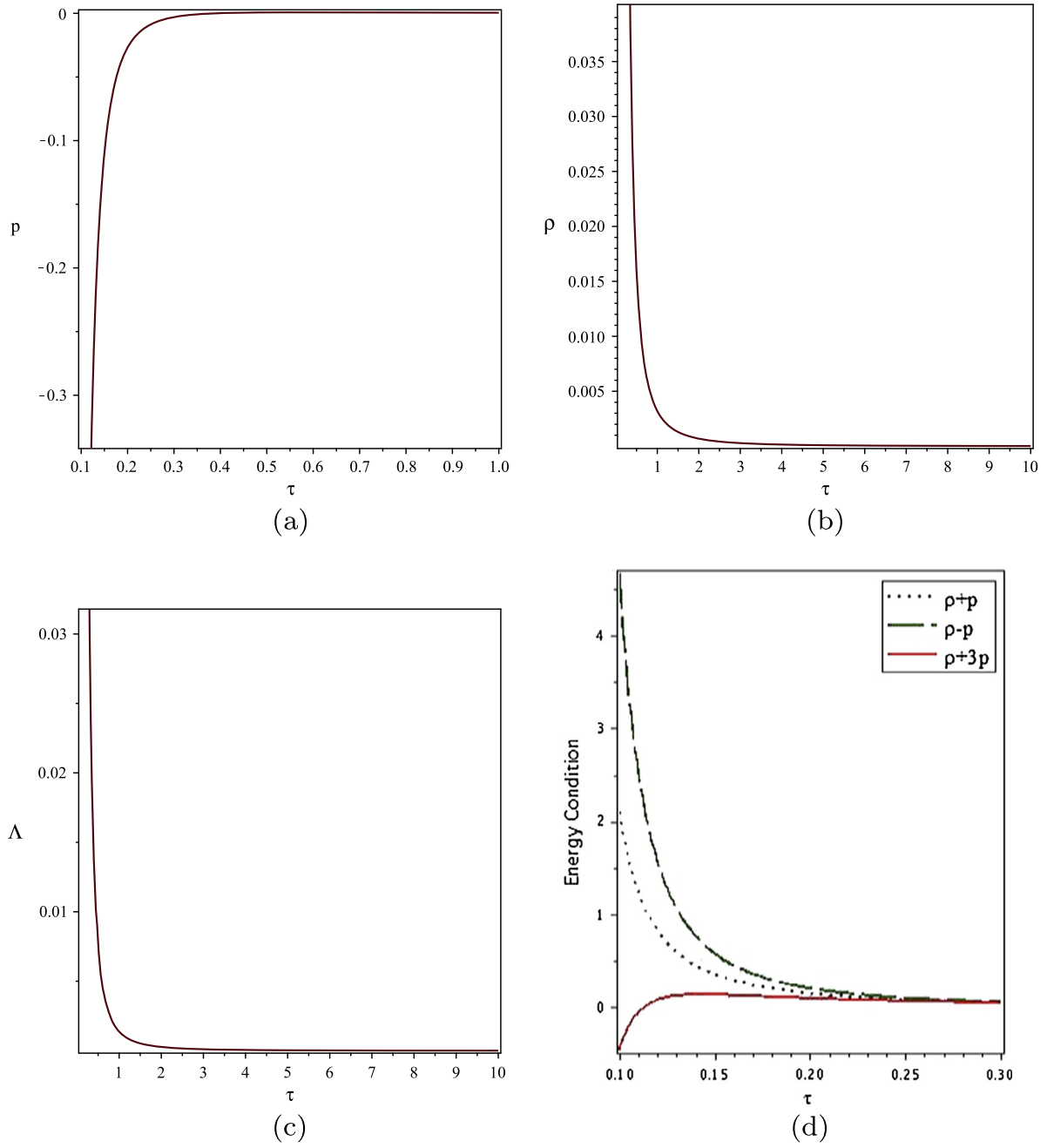


Figure 1 Case 1: Plots of p, ρ, Λ and energy conditions. The energy density and cosmological constant $\Lambda(t)$ are positive decreasing functions while the pressure is negative. Here $\lambda = -0.1$, $k = 1$, $k_1 = -1$, $\alpha = 0.1$ and $n = 0.25$.

deceleration parameter (q) and scale factor (a) are, respectively, given by

$$\theta = 3 \left[k_1 n \tau^{n-1} + \frac{\ell(2-n)}{2} \tau^{-3n/2} \right] (k_1 \tau^n + 2\ell \tau^{\ell_1})^{-3/2} \quad (40)$$

$$\sigma = \frac{1}{2} k \tau^{-3n/2} (k_1 \tau^n + 2\ell \tau^{\ell_1})^{-3/2} \quad (41)$$

$$V^3 = (k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1})^{\frac{3}{2}} \quad (42)$$

$$q = -\frac{F_4(\tau)}{g_4(\tau)} \quad (43)$$

$$a(\tau) = (k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1})^{\frac{1}{6}} \quad (44)$$

where

$$g_4(\tau) = \left(50n^3 k_1^2 \tau^{5n} - 15n^2 k k_1 \tau^{\frac{5n}{2}+1} - 40n^2 k_1^2 \tau^{5n} + 8n k_1^2 \tau^{5n} - 2n k^2 \tau^2 - 4n k k_1 \tau^{\frac{5n}{2}+1} + 4k^2 \tau^2 + 4k k_1 \tau^{\frac{5n}{2}+1} \right)^2. \quad (45)$$

and

$$\begin{aligned}
 F_4(\tau) = & k_1 k^3 \tau^{\frac{5n}{2}+3} (-440n^3 + 536n^2 - 64n - 32) \\
 & + k_1^3 k \tau^{\frac{15n}{2}+1} (-4625n^5 + 9800n^4 - 7320n^3 + 2336n^2 \\
 & - 272n) + k_1^2 k^2 \tau^{5n+2} (2525n^4 - 4220n^3 + 2064n^2 - 272n \\
 & - 16) + k^4 \tau^4 (4n^2 - 16) + k^4 \tau^{10n} (2500n^6 - 6500n^5 \\
 & + 6400n^4 - 3040n^3 + 704n^2 - 64n). \quad (46)
 \end{aligned}$$

Eqs. (40) and (41) lead to

$$\frac{\sigma}{\theta} = \frac{k}{6} \left[k_1 n \tau^{n-\ell_1} + \frac{\ell(2-n)}{2} \right]^{-1} \quad (47)$$

Eqs. (42) and (40) indicate that the spatial volume is zero at $\tau = 0$ and the expansion scalar is infinite. This shows that the evolution of the universe starts with zero volume at $\tau = 0$ (big bang scenario). We can also see that the spatial scale factors are zero at the initial epoch $\tau = 0$ which is a point type singularity (MacCallum, 1971). The proper volume increases with time. The physical quantities isotropic pressure (p), proper energy density (ρ), Hubble factor (H) and shear scalar (σ) diverge at $\tau = 0$. As $\tau \rightarrow \infty$, volume becomes infinite where as p, ρ, H, θ approach to zero. It is interesting to see that $\lim_{\tau \rightarrow \infty} \left(\frac{\rho}{\theta^2} \right)$ becomes a constant. Therefore, the model of the universe goes up homogeneity and matter is dynamically negligible near the origin. This agrees with the result obtained by Collins (1977). The variation of deceleration parameter q versus τ is plotted in Fig. 2. It shows that q is a decreasing function of time and approaches a small positive value at late time. We find that $\lim_{\tau \rightarrow \infty} \frac{\sigma^2}{\theta^2} = 0$, which indicates that the model eventually approaches isotropy for large values of τ . Our model represents a shearing, non-rotating, expanding and decelerating universe that starts with a big bang singularity and approaches to isotropy at the present epoch.

Energy conditions:

The weak energy condition (WEC) and dominant energy condition (DEC) are written as

(i) $\rho \geq 0$, (ii) $\rho - p \geq 0$ and (iii) $\rho + p \geq 0$.

The strong energy condition (SEC) is written as $\rho + 3p \geq 0$.

The left hand side of energy conditions has been graphed in Fig. 1(d) in Case (i). From this figure, we observe that

- The WEC and DEC are valid for our model.
- The SEC is not valid.

3.1.1. Expressions for some observable parameters

(a) $H(z)$ and $\mu(z)$ parameters

The Hubble parameter H is used to estimate the size and age of the Universe. It also indicates the expanding rate of the universe. From Eq. (44), the Hubble's parameter is computed as

$$H = \frac{nk_1 \tau^{2n-1} + \ell(n + \ell_1) \tau^{n+\ell_1-1}}{3(k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1})}. \quad (48)$$

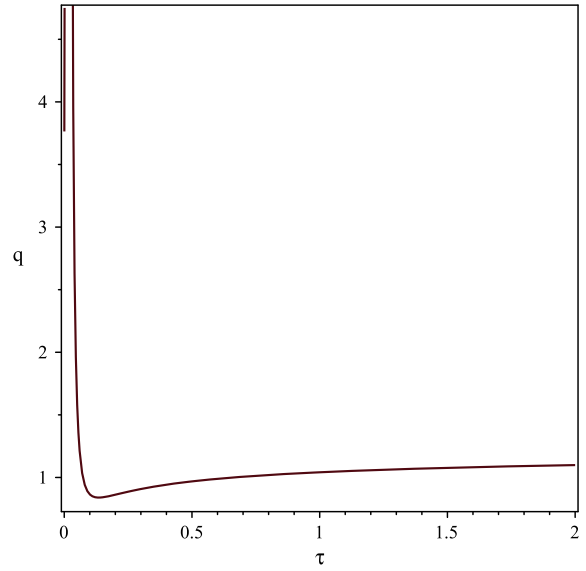


Figure 2 Deceleration parameter for Case 1. Here $\lambda = -0.1$, $k = 1$, $k_1 = -1$, $\alpha = 0.1$ and $n = 0.25$.

Hence

$$\frac{H}{H_0} = \frac{k_1 \tau_0^{2n} + 2\ell \tau_0^{n+\ell_1}}{k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1}} \times \frac{nk_1 \tau^{2n-1} + \ell(n + \ell_1) \tau^{n+\ell_1-1}}{nk_1 \tau_0^{2n-1} + \ell(n + \ell_1) \tau_0^{n+\ell_1-1}}, \quad (49)$$

where H_0 is the present value of Hubble's parameter.

The redshift we measure for a distant source is directly related to the scale factor of the universe at the time of the photons were emitted from the source. The scale factor a and redshift z are related through the equation

$$a = \frac{a_0}{1+z}, \quad (50)$$

where a_0 is the present value of scale factor. The above Eq. (50) can be rewritten as

$$\frac{a_0}{a} = 1+z = \left(\frac{k_1 \tau_0^{2n} + 2\ell \tau_0^{n+\ell_1}}{k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1}} \right)^{\frac{1}{6}}, \quad (51)$$

which leads to

$$H = H_0 (1+z)^6 \left(\frac{\tau_0}{\tau} \right)^6 \left(\frac{nk_1 \tau^{2n} + \ell(n + \ell_1) \tau^{n+\ell_1}}{nk_1 \tau_0^{2n} + \ell(n + \ell_1) \tau_0^{n+\ell_1}} \right). \quad (52)$$

This is the value of Hubble's parameter in terms of redshift parameter.

The distance modulus (μ) is given by

$$\mu(z) = 5 \log d_L + 25, \quad (53)$$

where d_L stands for the luminosity distance defined by

$$d_L = r_1 (1+z) a_0. \quad (54)$$

A photon emitted by a source with coordinate $r = r_1$ and $t = \tau_0$ and received at a time τ by an observer located at $r = 0$, then we determine r_1 from the following relation:

$$r_1 = \int_{\tau}^{\tau_0} \frac{d\tau}{a} = \int_{\tau}^{\tau_0} \frac{dt}{(k_1 \tau^{2n} + 2\ell \tau^{n+\ell_1})^{\frac{1}{6}}}. \quad (55)$$

To solve this integral, we take $k_1 = 1$ without any loss of generality. Using the values of ℓ and ℓ_1 given in Eq. (30), we obtain the value of r_1 in terms of hyper-geometric functions as

$$r_1 = \frac{3}{n-3} \left[{}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau_0^{1-\frac{5n}{2}}}{5n-2} \right) \times \tau_0^{1-2n} \left(2k \frac{\tau_0^{1-\frac{5n}{2}}}{2-5n} + \tau_0^{2n} \right)^{\frac{5}{6}} - {}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau^{1-\frac{5n}{2}}}{5n-2} \right) \times \tau^{1-2n} \left(2k \frac{\tau^{1-\frac{5n}{2}}}{2-5n} + \tau^{2n} \right)^{\frac{5}{6}} \right] \quad (56)$$

Hence from Eqs. (54) and (56), we obtain the expression for luminosity distance as

$$d_L = \frac{3(1+z)a_0}{n-3} \left[{}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau_0^{1-\frac{5n}{2}}}{5n-2} \right) \times \tau_0^{1-2n} \left(2k \frac{\tau_0^{1-\frac{5n}{2}}}{2-5n} + \tau_0^{2n} \right)^{\frac{5}{6}} - {}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau^{1-\frac{5n}{2}}}{5n-2} \right) \times \tau^{1-2n} \left(2k \frac{\tau^{1-\frac{5n}{2}}}{2-5n} + \tau^{2n} \right)^{\frac{5}{6}} \right] \quad (57)$$

From Eqs. (53) and (57), we can obtain the expression for distance modulus.

(b) Jerk parameter

A convenient method to describe models close to Λ CDM is based on the cosmic jerk parameter j (Sahni, 2002; Visser, 2005). A deceleration-to-acceleration transition occurs for models with a positive value of j_0 and negative q_0 . Flat Λ CDM models have a constant jerk $j = 1$. The jerk parameter in cosmology is defined as the dimensionless third derivative of the scale factor with respect to cosmic time

$$j(t) = \frac{1}{H^3} \frac{\ddot{a}}{a}. \quad (58)$$

and in terms of the scale factor to cosmic time

$$j(t) = \frac{(a^2 H^2)''}{2H^2}. \quad (59)$$

where the 'dots' and 'primes' denote derivatives with respect to cosmic time and scale factor, respectively. One can rewrite Eq. (58) as

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H}. \quad (60)$$

Therefore, the expression for Jerk parameter is computed and is given by

$$j(\tau) = 36(k_1\tau^{2n} + 2\ell\tau^{n+\ell_1})^2 \times \frac{4k_1\tau^{2n}(2n^3 - 3n^2 + n) + 2\ell\tau^{(n+\ell_1)}((n+\ell_1)^3 - 3(n+\ell_1)^2 + 2(n+\ell_1))}{(2nk_1\tau^{2n} + 2\ell(n+\ell_1)\tau^{n+\ell_1})^3} - 90(k_1\tau^{2n} + 2\ell\tau^{n+\ell_1})(2nk_1\tau^{2n} + 2\ell(n+\ell_1)\tau^{n+\ell_1}) \times (2nk_1\tau^{2n}(2n-1) + 2\ell\tau^{n+\ell_1}((n+\ell_1)^2 - n - \ell_1)) + 255 \quad (61)$$

This value overlaps with the value $j \simeq 2.16$ obtained from the combination of three kinematic data sets: the gold sample of type Ia supernovae (Riess et al., 2004), the SNIa data from the SNLS project (Astier, 2006), and the X-ray galaxy cluster distance measurements (Rapetti et al., 2007) for $\tau = 1.073555545$, $n = 4$, $k = 1$, $k_1 = 1$, $\ell = -\frac{1}{18}$, $\ell_1 = -5$. In addition to this choice, one can select other sets of values of different quantities to obtain the observed value of j .

3.2. Case (ii): Let $C = \tilde{\tau}^n$ (n is a real number satisfying $n \neq 2/3$)

In this case Eq. (24) gives

$$B = k_2 \tilde{\tau}^n \exp(M \tilde{\tau}^{\ell_1}) \quad (62)$$

and from (19), we obtain

$$A^2 = k_2 \tilde{\tau}^{2n} \exp(M \tilde{\tau}^{\ell_1}) \quad (63)$$

where $M = \frac{k}{\ell_1}$. Hence the metric (9) reduces to the form

$$ds^2 = \tilde{\tau}^{4(1-\ell_1)/3} \left[\tilde{\tau}^{2(1-\ell_1)/3} e^{3M\ell_1} d\tilde{\tau}^2 - e^{M\ell_1} dx^2 - e^{2\alpha x} (e^{2M\ell_1} dy^2 + dz^2) \right], \quad (64)$$

The constant k_2 can be chosen equal to 1 without loss of generality.

For this derived model (64), the physical parameters, i.e. the pressure (p), the energy density (ρ) and the cosmological constant (Λ) and the kinematic parameters, i.e. the scalar of expansion (θ), the shear scalar (σ), the proper volume (V^3) and the deceleration parameter (q) can be written as

$$p(\tilde{\tau}) = \frac{\lambda}{16k_2\tilde{\tau}^2(32\pi^2 + 12\pi\lambda + \lambda^2)} \times \left[-64\pi\alpha^2\tilde{\tau}^{-2n+2}e^{\frac{2k\ell_1}{-2+3n}} + 112\pi k_2 k^2 \tilde{\tau}^{2-3n} + 15nk_2 k \lambda \tilde{\tau}^{1-\frac{3n}{2}} + 24\lambda k_2(n^2 - n) + k_2\pi(192n^2 - 128n) + 19\lambda k_2 k^2 \tilde{\tau}^{2-3n} + 144n\pi k_2 k \tilde{\tau}^{1-\frac{3n}{2}} \right]. \quad (65)$$

$$\rho(\tilde{\tau}) = \frac{-\lambda}{16k_2\tilde{\tau}^2(32\pi^2 + 12\pi\lambda + \lambda^2)} \left[\alpha^2\tilde{\tau}^{-2n+2}e^{\frac{2k\ell_1}{-2+3n}}(-192\pi - 32\lambda) + nk_2\tilde{\tau}^{1-\frac{3n}{2}}(192\pi + 27\lambda) + k_2k^2\tilde{\tau}^{2-3n}(32\pi - \lambda) + 192\pi k_2 n^2 + 24\lambda k_2 n^2 + 8n\lambda k_2 \right]. \quad (66)$$

$$\Lambda = -\frac{\lambda}{16k_2\tilde{\tau}^2(4\pi + \lambda)} \left[-16\alpha^2\tilde{\tau}^{-2n+2}e^{\frac{2k\ell_1}{-2+3n}} + 24n^2k_2 + 21nk_2\tilde{\tau}^{1-\frac{3n}{2}} + 9k_2k^2\tilde{\tau}^{2-3n} - 8nk_2 \right]. \quad (67)$$

$$\theta = 3 \left[n\tilde{\tau}^{\ell_1-2} + \frac{k}{2}\tilde{\tau}^{2(\ell_1-1)} \right], \quad (68)$$

$$\sigma = \frac{k}{2}\tilde{\tau}^{2(\ell_1-1)}e^{-3M\ell_1}, \quad (69)$$

$$V^3 = [k_2\tilde{\tau}^{2n}e^{M\ell_1}]^{\frac{3}{2}}e^{2\alpha x} \quad (70)$$

$$q = -\frac{1}{(2n + k\tilde{\tau}^{1-\frac{3n}{2}})^2} (4n^2 - 4n + nk\tilde{\tau}^{1-\frac{3n}{2}} + k^2\tilde{\tau}^{2-3n}) \quad (71)$$

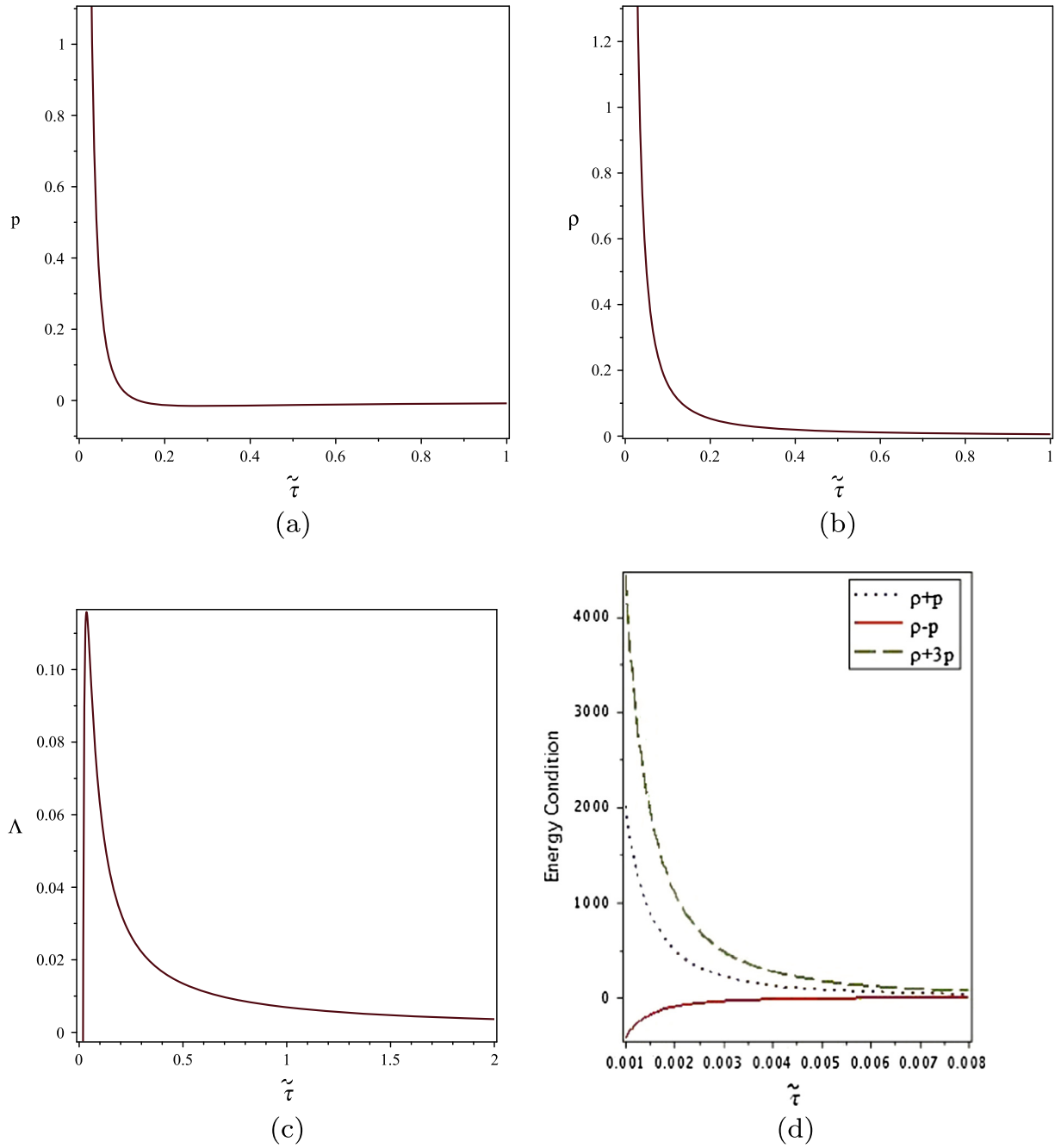


Figure 3 Case 2: Plots of p, ρ, Λ and energy conditions. Here $\lambda = -0.1$, $k = 1$, $k_2 = -1$, $\alpha = 0.1$ and $n = 0.25$.

$$a(\tilde{\tau}) = [k_2 \tilde{\tau}^{2n} e^{M\tilde{\tau}^{\ell_1}}]^{\frac{1}{6}} e^{\frac{2\alpha x}{3}}. \quad (72)$$

From Eqs. (68) and (69), we have

$$\frac{\sigma}{\theta} = \frac{k}{6(n\tilde{\tau}^{-\ell_1} + \frac{k}{2})}. \quad (73)$$

Fig. 3(a) illustrates the variation of pressure versus time for $\lambda = -0.1$, $k_1 = 1$, $k_2 = -1$, $\alpha = 0.1$ and $n = 0.25$. From the figure we observe that pressure is decreasing function of time and it tends to zero at the present epoch. Thus, we see that at early time (i.e. in early universe) p was large but it decreases as time increases.

Fig. 3(b) shows that the energy density remains always positive and decreasing function of time and it tends to zero as $t \rightarrow \infty$.

Fig. 3(c) shows that Λ takes a very large value in the early universe then starts decreasing as time increases. It approaches a small positive value at the present epoch. So the nature of Λ in our models agrees with the observations (Perlmutter et al., 1998, 1999, 2003; Riess et al., 1998, 2004; Clocchiatti et al., 2006).

Fig. 3(d) Case (ii) shows that SEC is satisfied whereas DEC violates. Fig. 4 indicates that q is a decreasing function of time and approaches to a small positive value at late time. Hence the model is decelerating.

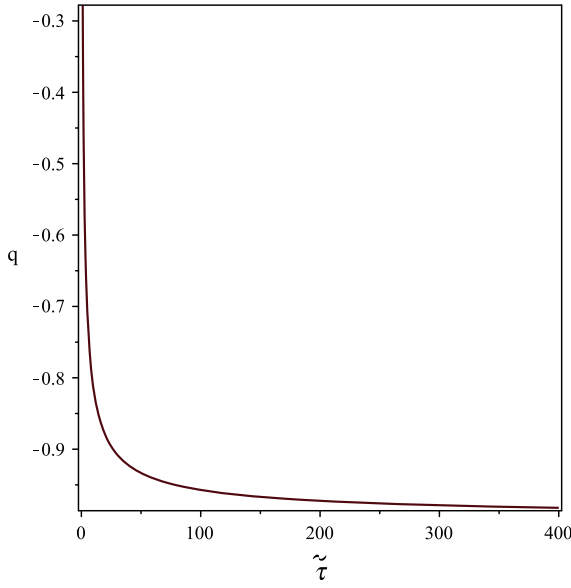


Figure 4 Deceleration parameter for Case 2. Here $\lambda = -0.1$, $k = 1$, $k_1 = -1$, $\alpha = 0.1$ and $n = 0.25$.

The physical and kinematic quantities in Case (ii) have the similar properties as the model discussed in Case (i).

3.2.1. Expressions for some observable parameters

(a) $H(z)$ and $\mu(z)$ parameters

From Eq. (72), the Hubble's parameter is obtained as

$$H = \frac{2n + M\ell_1 \tilde{\tau}^{\ell_1}}{6\tilde{\tau}} \quad (74)$$

Hence

$$\frac{H}{H_0} = \left(\frac{\tilde{\tau}_0}{\tilde{\tau}}\right) \left(\frac{2n + M\ell_1 \tilde{\tau}^{\ell_1}}{2n + M\ell_1 \tilde{\tau}_0^{\ell_1}}\right) \quad (75)$$

where H_0 is the present value of Hubble's parameter.

Since

$$\frac{a_0}{a} = (1+z) = \left(\frac{\tilde{\tau}_0}{\tilde{\tau}}\right)^{\frac{n}{6}} e^{\frac{M}{6}(\frac{\tilde{\tau}_0^{\ell_1}}{\tilde{\tau}^{\ell_1}} - \tilde{\tau}^{\ell_1})}, \quad (76)$$

which leads to

$$H = H_0(1+z)^3 e^{\frac{M}{6}(\frac{\tilde{\tau}_0^{\ell_1}}{\tilde{\tau}^{\ell_1}} - \tilde{\tau}^{\ell_1})} \left(\frac{2n + M\ell_1 \tilde{\tau}^{\ell_1}}{2n + M\ell_1 \tilde{\tau}_0^{\ell_1}}\right). \quad (77)$$

This is the value of Hubble's parameter in terms of redshift parameter.

To get the distance modulus μ , we first calculate r_1 which is given for this case by

$$r_1 = \int_{\tilde{\tau}}^{\tilde{\tau}_0} \frac{d\tilde{\tau}}{a} = \int_{\tilde{\tau}}^{\tilde{\tau}_0} \frac{d\tilde{\tau}}{(k_2 \tilde{\tau}^{2n} e^{M\tilde{\tau}^{\ell_1}})^{\frac{1}{6}} e^{\frac{\alpha}{6} \tilde{\tau}^{\ell_1}}}. \quad (78)$$

We take $k_1 = 1$ without any loss of generality. Using the values of ℓ and ℓ_1 given in Eq. (30), we obtain the value of r_1 in terms of Gamma functions as

$$\begin{aligned} r_1 = & \frac{\tilde{\tau}_0^{2-n}}{k(3n-2)e^{\frac{\alpha}{6} \tilde{\tau}_0^{\ell_1}}} \left((3n-2)e^{\frac{2k\tilde{\tau}_0^{1-\frac{3n}{2}}}{3n-2}} + n2^{\frac{n}{2-3n}} \Gamma\left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_0^{1-\frac{3n}{2}}}{2-3n}\right) \right) \\ & \times \left(\frac{k\tilde{\tau}_0^{1-\frac{3n}{2}}}{2-3n} \right)^{\frac{n}{2-3n}} + \frac{\tilde{\tau}_1^{2-n}}{k(3n-2)e^{\frac{\alpha}{6} \tilde{\tau}_1^{\ell_1}}} \left((2-3n)e^{\frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{3n-2}} - n2^{\frac{n}{2-3n}} \Gamma\left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n}\right) \right) \\ & \times \left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n} \right) \left(\frac{k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n} \right)^{\frac{n}{2-3n}}. \end{aligned} \quad (79)$$

Therefore, the expression for luminosity distance is obtained as

$$\begin{aligned} d_L = & \frac{\tilde{\tau}_0^{2-n}(1+z)a_0}{k(3n-2)e^{\frac{\alpha}{6} \tilde{\tau}_0^{\ell_1}}} \left((3n-2)e^{\frac{2k\tilde{\tau}_0^{1-\frac{3n}{2}}}{3n-2}} + n2^{\frac{n}{2-3n}} \Gamma\left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_0^{1-\frac{3n}{2}}}{2-3n}\right) \right) \\ & \times \left(\frac{k\tilde{\tau}_0^{1-\frac{3n}{2}}}{2-3n} \right)^{\frac{n}{2-3n}} + \frac{\tilde{\tau}_1^{2-n}(1+z)a_0}{k(3n-2)e^{\frac{\alpha}{6} \tilde{\tau}_1^{\ell_1}}} \left((2-3n)e^{\frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{3n-2}} - n2^{\frac{n}{2-3n}} \Gamma\left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n}\right) \right) \\ & \times \left(\frac{n}{3n-2}, \frac{2k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n} \right) \left(\frac{k\tilde{\tau}_1^{1-\frac{3n}{2}}}{2-3n} \right)^{\frac{n}{2-3n}}. \end{aligned} \quad (80)$$

From Eqs. (53) and (80), we can obtain the expression for distance modulus.

(b) Jerk parameter

In this case, the jerk parameter $j = \frac{1}{H^3} \frac{d^3 H}{dt^3}$ is computed as

$$\begin{aligned} j(\tau) = & \frac{1}{(2n + M\ell_1 \tilde{\tau}^{\ell_1})^3} (12M\ell_1 \tilde{\tau}^{\ell_1} (n^2 - 6n - 1) \\ & + 6M\ell_1^2 \tilde{\tau}^{\ell_1} (nM + 6n + 6\ell_1 - 18) + 18M^2 \ell_1^2 \tilde{\tau}^{2\ell_1} (\ell_1 - 1) \\ & + M^3 \ell_1^3 \tilde{\tau}^{3\ell_1} + 8(n^3 - 6n^2 + 12n)) \end{aligned} \quad (81)$$

This value overlaps with the value $j \simeq 2.16$ obtained from the combination of three kinematic data sets: the gold sample of type Ia supernovae (Riess et al., 2004), the SNIa data from the SNLS project (Astier, 2006), and the X-ray galaxy cluster distance measurements (Rapetti et al., 2007) for $\tilde{\tau} = 2.628716481$, $n = 0.25$, $k = k_2 = 1$, $\ell_1 = 0.625$, $M = 1.6$.

3.3. Case (iii): Let $B = T^n$ (n is a real number)

In this case Eq. (25) gives

$$C = k_3 T^n - 2\ell T^{\ell_1} \quad (82)$$

and then from (19), we obtain

$$A^2 = k_3 T^{2n} - 2\ell T^{\ell_1 + n} \quad (83)$$

Hence the metric (9) takes the new form

$$\begin{aligned} ds^2 = & (k_3 T^n - 2\ell T^{\ell_1}) [dt^2 - T^n dx^2] \\ & - e^{2\alpha x} [T^{2n} dy^2 + (k_3 T^n - 2\ell T^{\ell_1})^2 dz^2] \end{aligned} \quad (84)$$

For this derived model (84), the physical parameters, i.e. the pressure (p), the energy density (ρ) and the cosmological constant (Λ) and the kinematic parameters, i.e. the scalar of expansion (θ), the shear scalar (σ), the proper volume (V^3) and the deceleration parameter (q) are given by

$$p(T) = \lambda \frac{F_5(T)}{g_5(T)} \quad (85)$$

where

$$F_5(T) = kk_3^2 T^{\frac{9n}{2}+1} (4725\lambda n^4 - 39,840\pi n^3 - 7830\lambda n^3 + 19,008\pi n^2 + 3996\lambda n^2 - 2944\pi n - 648\lambda n + 26,800\pi n^4) + k_3 k^2 T^{2n+2} (7920\pi n^3 + 1715\lambda n^3 + 1856\pi n + 420n\lambda - 8608\pi n^2 - 1986\lambda n^2 + 128\pi + 40\lambda) + kk_3 \alpha^2 T^{\frac{5n}{2}+3} (6400n^2\pi + 5120\pi n - 1024\pi) + k_3^3 T^{7n} (3072\pi n^3 5040\lambda n^3 - 9216\pi n^2 + 3000\lambda n^5 - 44,800\pi n^4 - 6600\lambda n^4 - 1632\lambda n^2 + 1024\pi n + 192n\lambda + 24,000\pi n^5) + k^2 \alpha^2 T^4 (512\pi - 1280\pi n) + k^3 T^{-\frac{n}{2}+3} (416\pi n^2 + 122\lambda n^2 - 256\pi n - 64n\lambda - 40\lambda - 128\pi) + k_3^2 \alpha^2 T^{5n+2} (512\pi - 3840\pi n - 9600\pi n^2 - 8000\pi n^3). \quad (86)$$

and

$$g_5(T) = 16(-2k_3 T^{2n+2} + 5nk_3 T^{2n+2} + 2k T^{-\frac{n}{2}+3}) \times [k_3^2 T^{5n} (4\lambda^2 + 128\pi^2 - 20n\lambda^2 + 25n^2\lambda^2 + 800n^2\pi^2 + 48\pi\lambda - 640\pi n^2 + 300n^2\pi\lambda - 240\pi n\lambda) + k^2 T^2 (4\lambda^2 + 128\pi^2 + 48\pi\lambda) + kk_3 T^{\frac{5n}{2}+1} (-8\lambda^2 - 256\pi^2 + 640\pi n^2 + 20n\lambda^2 - 96\pi\lambda + 240\pi n\lambda)]. \quad (87)$$

$$\rho(T) = -\lambda \frac{F_6(T)}{g_6(T)} \quad (88)$$

where

$$g_6(T) = 16(-2k_3 T^{2n+2} + 5nk_3 T^{2n+2} + 2k T^{-\frac{n}{2}+3})^2 \times [k_3 T^n (60n\lambda\pi + 160\pi n^2 + 5n\lambda^2 - 64\pi^2 - 2\lambda^2 - 24\pi\lambda) + k T^{1-\frac{3n}{2}} (2\lambda^2 + 64\pi^2 + 24\pi\lambda)]. \quad (89)$$

and

$$F_6(T) = k^3 T^{3-\frac{5}{2}} (56\lambda + 256\pi + 64n\lambda + 256\pi n - 704\pi n^2 - 158\lambda n^2) + k_3 k^2 T^2 (-256\pi - 56\lambda - 3680\pi n^3 - 1185\lambda n^3 - 1152\pi n - 332n\lambda + 5952\pi n^2 + 1654\lambda n^2) + \alpha^2 k^2 T^{4-2n} (1536\pi - 3840\pi n + 256\lambda - 640n\lambda) + k_3^3 T^{5n} (24,000\pi n^5 + 3000\lambda n^5 - 28,800\pi n^4 - 2600\lambda n^4 + 11,520\pi n^3 + 240\lambda n^3 - 1536\pi n^2 + 288\lambda n^2 - 64n\lambda) + kk_1 \alpha^2 T^{3+\frac{n}{2}} (-3072\pi - 3200\lambda n^2 + 2560n\lambda - 512\lambda - 19,200\pi n^2 + 15,360\pi n) + k_3^2 \alpha^2 T^{3n+2} (1536\pi - 11,520\pi n + 28,800\pi n^2 - 24,000\pi n^3 + 256\lambda - 1920n\lambda - 4000\lambda n^3) + kk_3^2 T^{\frac{5n}{2}+1} (4800\pi n^4 - 775\lambda n^4 + 5760\pi n^3 + 3570\lambda n^3 - 6912\pi n^2 - 2484\lambda n^2 + 1536\pi n + 472n\lambda). \quad (90)$$

$$A = -\lambda \frac{F_7(T)}{g_7(T)} \quad (91)$$

where

$$g_7(T) = 16(-2k_3 T^{2n} + 5nk_3 T^{2n} + 2k T^{1-\frac{n}{2}})^2 (-8kk_3 T^{\frac{5n}{2}+3} + 25n^2 k_3^2 T^{5n+2} + 20nk k_3 T^{\frac{5n}{2}+3} + 2k^2 T^4 + 4k_3^2 T^{5n+2} - 20nk_3^2 T^{5n+2}) (20n\pi k_3 T^n - 8\pi k_3 T^n + 8\pi k T^{1-\frac{3n}{2}} - 2\lambda k_3 T^n + 5n\lambda k_3 T^n + 2\lambda k T^{1-\frac{3n}{2}}). \quad (92)$$

and

$$F_7(T) = \alpha^2 k_3^4 T^{8n+2} (-5000n^5 + 100,000n^4 - 8000n^3 + 32,000n^2 - 6400n + 512) + \alpha^2 k^4 T^{6-2n} (-1280n + 512) + \alpha^2 k^2 k_3^2 T^{3n+4} (3072 - 23,040n - 57,600n^2 - 48,000n^3) + k^2 k_3^2 T^{5n+2} (58,125n^5 - 86,250n^4 + 48,200n^3 - 12,240n^2 + 1296n - 32) + kk_3^4 T^{\frac{15n}{2}+1} (109,375n^6 - 208,750n^5 + 159,000n^4 - 60,400n^3 + 11,440n^2 - 864n) + \alpha^2 k_3 k^3 T^{\frac{5n}{2}+3} (-12,800n^2 + 10,240n - 2048) + k_3^2 k^3 T^{\frac{5n}{2}+3} (12,750n^4 - 13,600n^3 + 5360n^2 - 1024n) + \alpha^2 k k_3^3 T^{\frac{11n}{2}+3} (-8000n^4 + 128,000n^3 - 76,800n^2 + 20,480n - 2048) + k_3^5 T^{10n} (75,000n^7 - 175,000n^6 + 170,000n^5 - 88,000n^4 + 25,600n^3 - 3968n^2 + 256n) + k^5 T^{6-\frac{5n}{2}} (-72n^2 + 32) + \alpha^2 k_3^2 k^2 T^{3n+4} (-48,000n^3 + 57,600n^2 - 23,040n) + k_3 k^4 T^4 (700n^3 - 520n^2 + 336n - 96). \quad (93)$$

$$\theta = 3 \left[\frac{\ell(n-2)}{2} T^{-3n/2} + k_3 n T^{n-1} \right] (k_3 T^n - 2\ell T^{\ell_1})^{-3/2}, \quad (94)$$

$$\sigma = \frac{k T^{-3n/2}}{2} (k_3 T^n - 2\ell T^{\ell_1})^{-3/2}, \quad (95)$$

$$V^3 = [k_3 T^{2n} - 2\ell T^{\ell_1+n}]^{\frac{3}{2}} e^{2xx}, \quad (96)$$

$$a = [k_3 T_0^{2n} 2\ell T^{\ell_1+n}]^{\frac{1}{6}} e^{\frac{2}{3}xx} \quad (97)$$

$$q = \frac{F_8(T)}{g_8(T)} \quad (98)$$

where

$$g_8(T) = (50k_3^2 n^3 T^{5n} + 15n^2 k k_3 T^{\frac{5n}{2}+1} - 40n^2 k_3^2 T^{5n} + 4n k k_3 T^{\frac{5n}{2}+1} - 2n k^2 T^2 + 8n k_3^2 T^{5n} + 4k^2 T^2 - 4k k_3 T^{\frac{5n}{2}+1})^2. \quad (99)$$

and

$$F_8(T) = k k_3^3 T^{\frac{15n}{2}+1} (4625n^5 - 2336n^2 + 7320n^3 - 9800n^4 + 272n) + k^2 k_3^2 T^{5n+2} (2525n^4 - 4220n^3 + 2064n^2 - 272n - 16) + k_3^4 T^{10n} (2500n^6 - 6500n^5 + 6400n^4 - 3040n^3 + 704n^2 - 64n) + k^4 T^4 (4n^2 - 16) + k_3 k^3 T^{\frac{5n}{2}+3} (440n^3 - 536n^2 + 64n + 32). \quad (100)$$

From (94) and (95), we get

$$\frac{\sigma}{\theta} = \frac{k}{6} \left[k_3 n T^{-\ell_1+n} + \frac{\ell(n-2)}{2} \right]^{-1}. \quad (101)$$

Fig. 5(a) shows the variation of pressure versus time for $\lambda = -0.1$, $k_1 = 1$, $k_2 = -1$, $\alpha = 0.1$ and $n = 0.25$ as a representative case. From the figure we see that pressure is positive decreasing function of time and it approaches to a small positive value at the present epoch.

Fig. 5(b) shows the variation of energy density with cosmic time. It is evident that the energy density remains always positive and decreasing function of time and it converges to zero as $t \rightarrow \infty$ as expected.

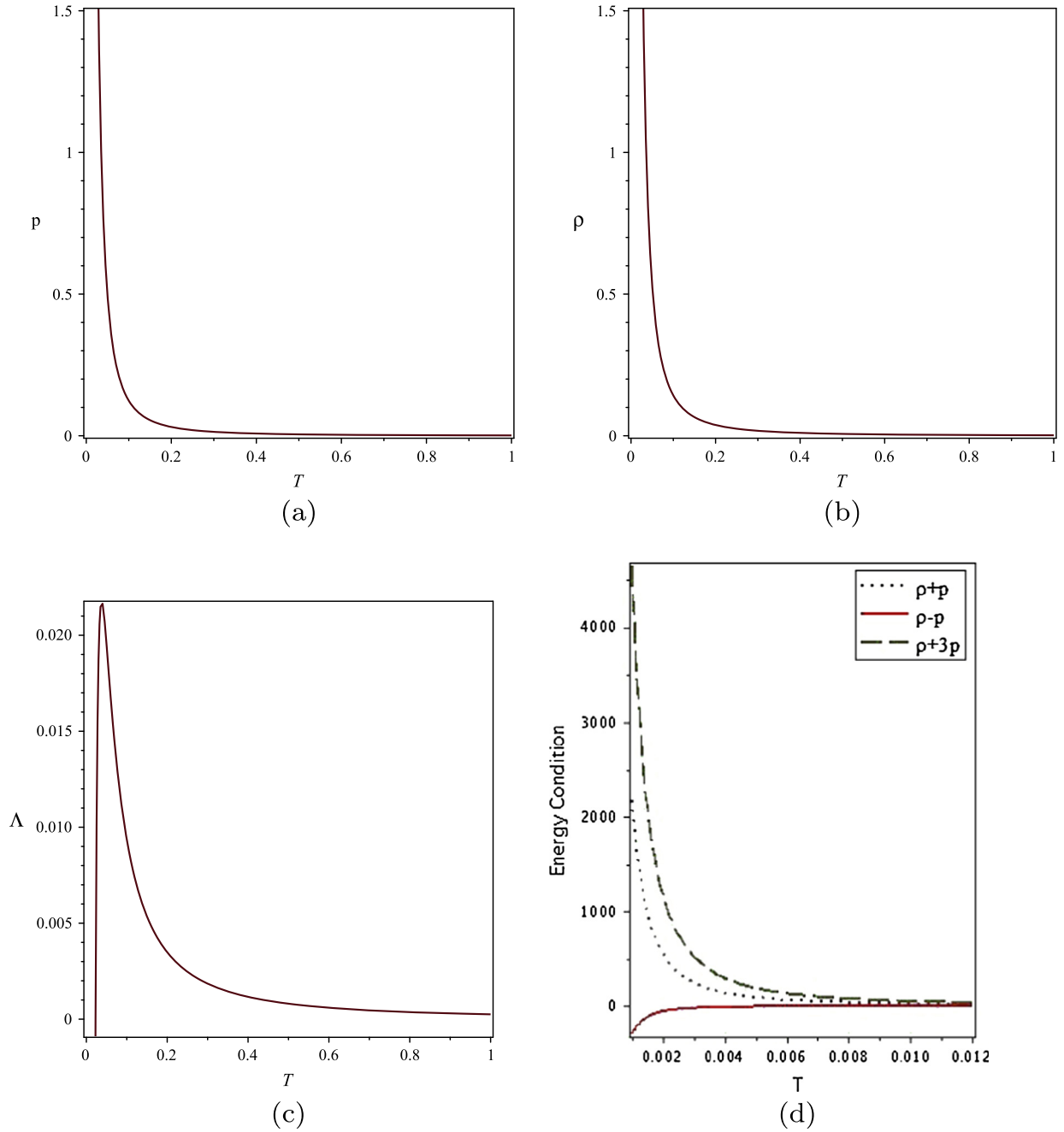


Figure 5 Case 3: Plots of p, ρ, Λ and energy conditions. Here $\lambda = -0.1$, $k = 1$, $k_3 = -1$, $\alpha = 0.1$ and $n = 0.25$.

Fig. 5(c) is the plot of cosmological term Λ versus time. From this figure, we observe that Λ is very large value in the early universe but it starts decreasing as time increases and it approaches a small positive value at the present epoch. Thus, the nature of Λ in our models is supported by observations (Perlmutter et al., 1998, 1999, 2003; Riess et al., 1998, 2004; Clocchiatti et al., 2006).

The left hand side of energy conditions is plotted in Fig. 5 (d) in Case (iii). From this figure, we observe that SEC is satisfied whereas DEC violates in Case (iii).

Fig. 5 plots the variation of decelerating parameter q versus $\bar{\tau}$. We see that q is a decreasing function of time and

approaches to a small positive value at late time. Hence the model is decelerating.

The physical and kinematic quantities in Case (iii) have the similar properties as the model discussed in Case (i) (see Fig. 6).

3.3.1. Expressions for some observable parameters

(a) $H(z)$ and $\mu(z)$ parameters

In this case, from Eq. (94), we obtain the value of the Hubble's parameter as

$$H = \frac{nk_3 T^{2n-1} - 2\ell(n + \ell_1) T^{\ell_1+n-1}}{3(k_3 T^{2n} - 2\ell T^{\ell_1+n})} \quad (102)$$

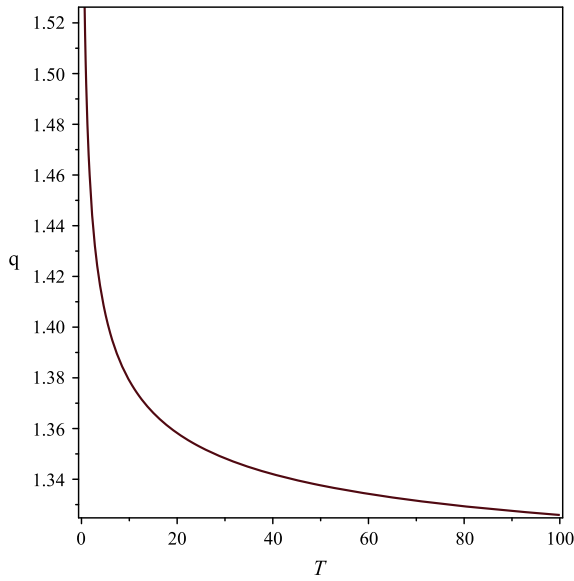


Figure 6 Deceleration parameter for Case 3. Here $\lambda = -0.1$, $k = 1$, $k_1 = -1$, $\alpha = 0.1$ and $n = 0.25$.

Since

$$\frac{a_0}{a} = (1+z) = \left(\frac{k_3 T_0^{2n} - 2\ell T_0^{\ell_1+n}}{k_3 T^{2n} - 2\ell T^{\ell_1+n}} \right)^{\frac{1}{6}} \quad (103)$$

which leads to

$$H = H_0(1+z)^6 \left(\frac{T_0}{T} \right)^{\frac{1}{6}} \left[\frac{nk_1 T^{2n} + \ell(n+\ell_1) T^{n+\ell_1}}{nk_1 T_0^{2n} + \ell(n+\ell_1) T_0^{n+\ell_1}} \right] \quad (104)$$

This is the value of Hubble's parameter in terms of redshift parameter.

To get the distance modulus μ , we first calculate r_1 which is given for this case by

$$r_1 = \int_T^{T_0} \frac{dT}{a} = \int_T^{T_0} \frac{dT}{(k_3 T^{2n} + 2\ell T^{n+\ell_1})^{\frac{1}{6}} e^{\frac{2\alpha}{3} T}} \quad (105)$$

Setting $k_3 = 1$ without any loss of generality and using the values of ℓ and ℓ_1 given in Eq. (30), we obtain the value of r_1 in terms of Hyper-geometric functions as

$$\begin{aligned} r_1 = & \frac{3}{(n-3)e^{\frac{2\alpha}{3} T_0}} \left[{}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau_0^{1-\frac{5n}{2}}}{5n-2} \right) \right. \\ & \times \tau_0^{1-2n} \left(2k \frac{\tau_0^{1-\frac{n}{2}}}{2-5n} + \tau_0^{2n} \right)^{\frac{5}{6}} \\ & - {}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau^{1-\frac{5n}{2}}}{5n-2} \right) \\ & \left. \times \tau^{1-2n} \left(2k \frac{\tau^{1-\frac{n}{2}}}{2-5n} + \tau^{2n} \right)^{\frac{5}{6}} \right]. \end{aligned} \quad (106)$$

Therefore, the expression for luminosity distance is obtained as

$$\begin{aligned} d_L = & \frac{3(1+z)a_0}{(n-3)e^{\frac{2\alpha}{3} T_0}} \left[{}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau_0^{1-\frac{5n}{2}}}{5n-2} \right) \right. \\ & \times \tau_0^{1-2n} \left(2k \frac{\tau_0^{1-\frac{n}{2}}}{2-5n} + \tau_0^{2n} \right)^{\frac{5}{6}} \\ & - {}_2F_1 \left(1, \frac{22-29n}{12-30n}, \frac{12-17n}{6-15n}, \frac{2k\tau^{1-\frac{5n}{2}}}{5n-2} \right) \\ & \left. \times \tau^{1-2n} \left(2k \frac{\tau^{1-\frac{n}{2}}}{2-5n} + \tau^{2n} \right)^{\frac{5}{6}} \right]. \end{aligned} \quad (107)$$

From Eqs. (53) and (107), we can obtain the expression for distance modulus.

(b) Jerk parameter

In this case, the jerk parameter $j = \frac{1}{H^3} \frac{dH}{dt}$ is computed as

$$\begin{aligned} j(T) = & 36(k_3 T^{2n} - 2\ell T^{n+\ell_1})^2 \\ & \times \frac{4k_3 T^{2n}(n-1)(2n^2) - 2\ell T^{n+\ell_1}(n+\ell_1)[(n+\ell_1)(n+\ell_1-3)+2]}{(2nk_3 T^{2n} - 2\ell(n+\ell_1) T^{n+\ell_1})^3} \\ & - 90(k_3 T^{2n} - 2\ell T^{n+\ell_1})(2nk_3 T^{2n} - 2\ell(n+\ell_1) T^{n+\ell_1}) \\ & \times [2nk_3 T^{2n}(2n-1) - 2\ell T^{n+\ell_1}(n+\ell_1)(n+\ell_1-1)] + 255 \end{aligned} \quad (108)$$

This value overlaps with the value $j \simeq 2.16$ obtained from the combination of three kinematic data sets: the gold sample of type Ia supernovae (Riess et al., 2004), the SNIa data from the SNLS project (Astier, 2006), and the X-ray galaxy cluster distance measurements (Rapetti et al., 2007) for $T = 0.3201378421$, $n = 0.25$, $\ell = 1.3$, $\ell_1 = 0.625$, $k_3 = 1$, $M = 1.6$.

3.4. Case (iv): Let $B = \tilde{\tau}^n$, where n is any real number

In this case Eq. (26) gives

$$C = k_4 \tilde{\tau}^n \exp \left(\frac{k}{\ell_1} \tilde{\tau}^{\ell_1} \right) \quad (109)$$

and then from (19), we obtain

$$A^2 = k_4 \tilde{\tau}^{2n} \exp \left(\frac{k}{\ell_1} \tilde{\tau}^{\ell_1} \right) \quad (110)$$

Hence the metric (9) reduces to

$$\begin{aligned} ds^2 = & \tilde{\tau}^{2n} \exp \left(\frac{k}{\ell_1} \tilde{\tau}^{\ell_1} \right) \left[\tilde{\tau}^n \exp \left(\frac{2k}{\ell_1} \tilde{\tau}^{\ell_1} \right) - dx^2 \right] \\ & - e^{2\alpha x} \left[dy^2 + \exp \left(\frac{2k}{\ell_1} \tilde{\tau}^{\ell_1} \right) - dz^2 \right], \end{aligned} \quad (111)$$

where the constant k_4 is equal to 1 without loss of generality.

4. Discussions

In this paper, we have studied the evolution of Bianchi type- V cosmological model in presence of perfect fluid and variable cosmological constant in $f(R, T)$ theory of gravity (Harko et al., 2011). In this paper, the field equations has been constructed by taking the case $f(R, T) = f_1(R) + f_2(T)$ into consideration. We have reexamined the recent work (Ahmed and Pradhan, 2014) by using a generation technique (Poplawski, 2006a,b; Magnano, 1995) and shown that the $f(R, T)$ gravity field equations are solvable for any arbitrary cosmic scale func-

tion. Solutions for four particular forms of cosmic scale functions are obtained in this paper.

We have also established the expressions of observational parameter, namely Hubble's parameter $H(z)$, luminosity distance d_L and distance modulus $\mu(z)$ with redshift and discussed its significances. We have also found out the expressions for Jerk parameter which describes models close to Λ CDM.

- we have proposed a new method to construct four particular models of $f(R, T)$ gravity which naturally unifies two expansion phases of the universe: inflation at early times and cosmic acceleration at current epoch.
- The models are based on exact solutions of the $f(R, T)$ gravity field equations for the anisotropic Bianchi- V space-time filled with perfect fluid with time dependent Λ -term which are perfectly new and physically acceptable.
- The model represents an expanding, shearing, non-rotating and decelerating universe.
- Λ in this model is a decreasing function of time and it tends to a small positive value at late time which agrees with the recent cosmological observations (Perlmutter et al., 1998, 1999, 2003; Riess et al., 1998, 2004; Clocchiatti et al., 2006).
- We would like to note that all results of this paper are new and different from the results of recent paper (Ahmed and Pradhan, 2014) and other papers on the subject.

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