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Modeling sun's radiation effect on restricted four bodies

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A B S T R A C T

In this work, a dynamical system of four bodies is constructed. The forces which govern the motion are mutual gravitational attractions of the primaries, and radiation pressure force emitted from the more massive body. The equations of motion for the four bodies have taken into account the radiation pressure. We have deduced that these equations can be solved by Laplace transformations; the eigenvalues are obtained to study the motion about the libration points which are taken from the classical method, then the stability around the libration points is studied. The results obtained are presented. We remark that this model has special importance in astrodynamics to send spacecraft to stable regions to move in gravitational fields for some planetary system.

1. Introduction

In space dynamics, there are several systems like two-body, three-body, four- body, and N-body problems under considerations. It is very important for the dynamics of binary and multiple stars as well as the planetary systems. The notation of the three-body problem in different forms like restricted three-body problem (RTBP), restricted four-body problem (RFBP) may be considered as an extension either two or of three-body problem. In this research, the motion of a spacecraft or satellite in the Earth-Moon system is studied as a simple example of RTBP in space. The restricted four-body has many possible uses in the dynamical system, for example, the fourth body is very useful for saving fuel time in the trajectory transfers in the restricted four-body problem (Machuy et al., 2007). The masses of these bodies compared to each other are considered, and its motions are considered under their mutual attraction. A few years ago, the planar circular restricted three-body problem was described by Murray (Murray, 1994). Kalvouridis (Kalvouridis et al., 2006) discussed the effect of radiation force due to primaries in the restricted four-body problem using Radzievskill's model and noticed that there are some variations in the result which are unstable for all values of the parameters assumed by him. Also, Reena Kumari (Kumari and Kushvah, 2013) studied the equilibrium points in the restricted four-body problem with solar wind Drag. Jules Verne (Verne, 2008) mentioned that the solar radiation pressure is used as a motive force for solar sailing. The Laplace Transform method is considered as one of the important methods to treat this problem.

It is well known that the Laplace Transform and its inverse help to obtain a direct solution for the ordinary differential equations depends on the algebraic operations (Dawkins, 2007; Elsolts, 1970).

2. Equations of motion

The motion of the fourth body is studied, where its mass m is very insignificant compared with the masses of the other three-bodies and $m_3 > m_1 > m_2 \gg m$. Fig. 1 illustrates the geometric of the problem, r_1, r_2 and r_3 are the position vectors from m_1, m_2 and m_3 to m respectively. The origin is considered at the center of mass of m_1 and m_2 which are called the primaries. The masses of the Earth, the Moon and the Sun in the canonical system are given as $\mu_1 = \frac{M_1}{M_2+M_1} = 0.987871$, $\mu_2 = \frac{M_2}{M_2+M_1} = 0.012151$ and $\mu_3 = \frac{M_3}{M_2+M_1} = 328900.48$ respectively, where the mass of the Earth $(M_1) = 5.98 \times 10^{24} kg$; mass of the Sun$(M_3) = 1.99 \times 10^{30} kg$.

The coordinates of m_1, m_2 and m_3 w.r.t CM as an origin are given by

$x_{1=}-\mu_2 Cost \quad y_{1=}-\mu_2 Sint$

$x_{2=}-\mu_1 Cost \quad y_{2=}-\mu_1 Sint$

$\psi = \psi_0 + \omega_s t$

where ω_s is the angular velocity of the sun which makes an angle ψ with x-axis, ψ_0 is the initial value of ψ and t the instantaneous time. Since the force is defined as F = m $\underline{a}$, where $\underline{a}$ is the acceleration, and F = $-\Delta$ u, u is the potential.

In this system the masses of two primaries are $m_1 = 1-\mu$, $m_2 = \mu$.

where μ is the mass ratio of the system $\mu = \frac{m_2}{m_1 + m_2}$. The distance of two primaries is unity.
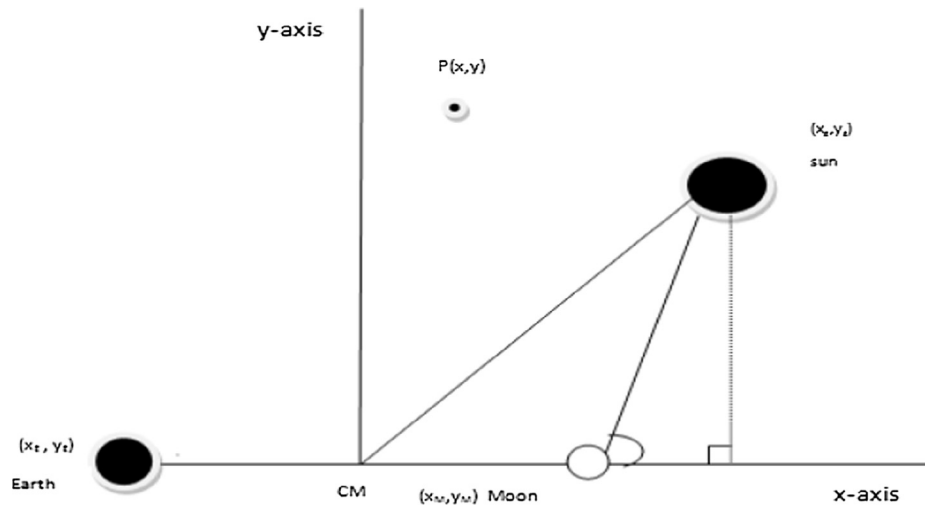
Fig. 1. The geometric of the problem.

$$r_1 = \sqrt{(x + \mu)^2 + (y - y_1)^2 + z^2}$$

$$r_2 = \sqrt{(x + \mu - 1)^2 + (y - y_2)^2 + z^2}$$

$$r_3 = \sqrt{(x - x_3)^2 + (y - y_3)^2 + z^2}$$

then the components of the force are $(\ddot{x}, \ddot{y}, \ddot{z})$ for a body of unit mass which are given by

$$\ddot{x} = \frac{-(1-\mu)(\,x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} - \frac{\mu_s(1-\beta)(x-R_s Cos\psi)}{r_3^3} \quad (1)$$

$$\ddot{y} = \frac{-(1-\mu)(y-y_1)}{r_1^3} - \frac{\mu(y-y_2)}{r_2^3} - \frac{\mu_s(1-\beta)(y-R_s Sin\psi)}{r_3^3} \quad (2)$$

$$\ddot{z} = \left(\frac{-(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{1}{r_3^3} \right) \quad (3)$$

where

$$Fg - Fr = Fg(1-\beta) \quad \beta = \frac{radiation\ pressure\ force}{gravitional\ force} = \frac{Fr}{Fg}$$

Using the coordinate transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} Cost & -Sint & 0 \\ Sint & Cost & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad (4)$$

$x = \xi Cos\, t - \eta Sin\, t, \quad y = \xi Sint + \eta Cost$ and $z = \zeta$
Using Eq. (4) into (1), (2) and (3), this yield

$$\ddot{\xi} - 2\dot{\eta} = \xi - \frac{(1-\mu)(\xi+\mu)}{r_1^3} - \frac{\mu(\xi-1+\mu)}{r_2^3} - \frac{\mu_s(1-\beta)(\xi-R_s Cos\psi)}{r_3^3} \quad (5)$$

$$\ddot{\eta} + 2\dot{\xi} = \eta - \frac{(1-\mu)(\eta-y_1)}{r_1^3} - \frac{\mu(\eta-y_2)}{r_2^3} - \frac{\mu_s(1-\beta)(\eta-R_s Sin\psi)}{r_3^3} \quad (6)$$

$$\ddot{\zeta} = \frac{-(1-\mu)\zeta}{r_1^3} - \frac{\mu\zeta}{r_2^3} - \frac{\zeta}{r_3^3} \quad (7)$$

With

$$u = \frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} - \frac{\mu_s(1-\beta)}{r_3^3} + \frac{\xi^2 + \eta^2}{2} \quad (8)$$

the potential in the rotating coordinate system.
It is clear, that the right-hand sides in these equations represent $\frac{\partial u}{\partial \xi} = u_\xi$, $\frac{\partial u}{\partial \eta} = u_\eta$ and $\frac{\partial u}{\partial \zeta} = u_\zeta$ respectively, then

$$\ddot{\xi} - 2\dot{\eta} = u_\xi \quad (9)$$

$$\ddot{\eta} + 2\dot{\xi} = u_\eta \quad (10)$$

$$\ddot{\zeta} = u_\zeta \quad (11)$$

3. Laplace transformation

Eqs. (9), (10) and (11) can be solved by Laplace transforms, by putting

$$L[\ddot{\xi}] - L[2\dot{\eta}] = L[u_\xi] \quad (12)$$

$$L[\ddot{\eta}] + L[2\dot{\xi}] = L[u_\eta] \quad (13)$$

$$L[\ddot{\zeta}] = L[u_\zeta] \quad (14)$$

These equations yield to

$$s^2\xi(s) - s\xi(0) - \dot{\xi}(0) - 2s\eta(s) + 2\eta(0) = \frac{u_\xi}{s} \quad (15)$$

$$s^2\eta(s) - s\eta(0) - \dot{\eta}(0) + 2s\xi(s) - 2\xi(0) = \frac{u_\eta}{s} \quad (16)$$

$$s^2\zeta(s) - s\zeta(0) - \dot{\zeta}(0) = \frac{u_\zeta}{s} \quad (17)$$

To solve this system of Eqs. (15), (16) and (17), it is more convenient to consider $\dot{\xi}(0) = \dot{\eta}(0) = \dot{\zeta}(0) = 0$, then the solution will be

$$\eta(s) = \frac{-2u_\xi + su_\eta + (s^3 + 4s)\eta(0)}{s^4 + 4s^2} \quad (18)$$

$$\xi(s) = \frac{su_\xi + 2u_\eta + (s^3 + 4s)\xi(0)}{s^4 + 4s^2} \quad (19)$$

$$\zeta(s) = \frac{u_\zeta + s^2\zeta(0)}{s^3} \quad (20)$$

which is the solution of the system of Eqs. (18), (19) and (20) by Laplace transformation.

4. Stability points

4.1. Collinear point

To apply Eqs. (18), (19) and (20) on the Collinear point, the positions of these points are taken from (Ibrahim, 2017) so that
$L_1(0.9, 0)$, $L_2(1.2, 0)$ and $L_3(-L_3(-0.9, 0)$
Stability at L_2 at $\xi = 1.2$, $\eta = 0$

$$\eta(s) = -\frac{1.221}{s^2} + \frac{1.221}{4. + 1. \, s^2}$$

$$\xi(s) = -\frac{1.811}{s} - \frac{0.611s}{4. + 1. \, s^2}$$

We take the inverse Laplace transforms

$$\eta(t) = L^{-1}\left[-\frac{1.221}{s^2}\right] + L^{-1}\left[\frac{1.221}{4. + 1. \, s^2}\right]$$

$$\xi(t) = L^{-1}\left[-\frac{1.811}{s}\right] + L^{-1}\left[-\frac{0.611s}{4. + 1. \, s^2}\right]$$

then

$\eta(t) = (-0.305 \, i) \, e^{(-2i)t} \, ((-1 + i) + (1 + i)e^{(4 \, i) \, t}) - 1.221t$
$\xi(t) = 1.811 - 0.305 \, e^{(-2. \, i) \, t} (1 + e^{(4. \, i) \, t})$

which represent the coordinates about L_2 as a function of time and will give a periodicity about L_2 referred to the existence of trigonometric functions, a code of Mathematica was constructed to solve the system of Eqs. (18), (19) and (20) taken into account $\beta = 0.0001$. This code is applied on the five libration points.

5. Results and discussion

From the results at $L_2(1.2, 0)$. Fig. 2 shows the behavior of ξ versus with time ($0 < t < 100$), and it illustrates that there is periodicity around L_2. Fig. 3 shows that the phase space, for the motion of the fourth body about L_2 and it is found that it takes an ellipse.

Stability about L_1, L_3 at (± 0.9, 0)
The solution of the system at $\xi = 0.9$, $\eta = 0$

$$\eta(s) = -\frac{1.734}{s^2} + \frac{1.734}{4. + 1. \, s^2}$$

$$\xi(s) = \frac{1.767}{s} - \frac{0.866s}{4. + 1. \, s^2}$$

We take the inverse Laplace transforms

$$\eta(t) = L^{-1}\left[-\frac{1.734}{s^2}\right] + L^{-1}\left[\frac{1.734}{4. + 1. \, s^2}\right]$$

$$\xi(t) = L^{-1}\left[\frac{1.767}{s}\right] + L^{-1}\left[-\frac{0.866s}{4. + 1. \, s^2}\right]$$

Then

$\eta(t) = -(-0.434 \, i) \, e^{(-2i) \, t} \, ((-1 + i) + (1 + i) \, e^{(4i) \, t}) - 1.733t$
$\xi(t) = 1.767 - 0.434 \, e^{(-2i) \, t} (1 + e^{(4i) \, t})$
The solution of the system at $\xi = -0.9$, $\eta = 0$

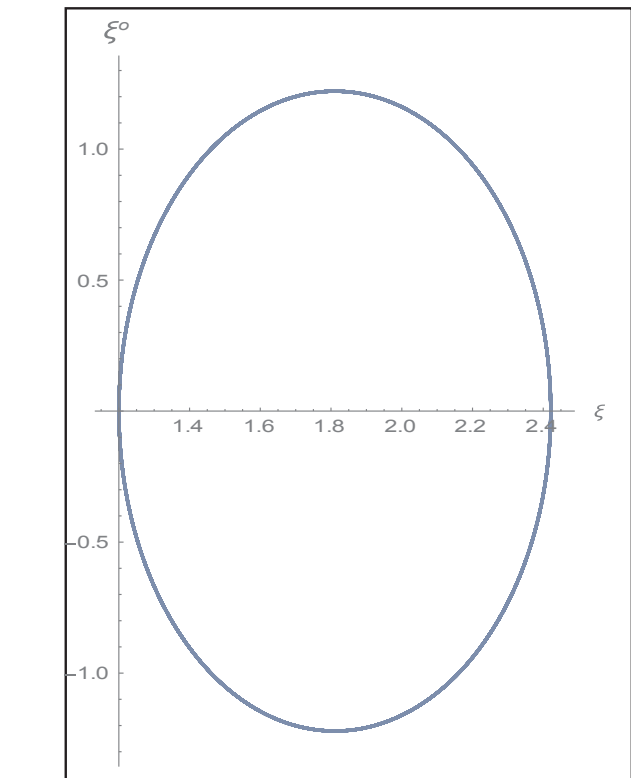$$\eta(s) = -\frac{1.259}{s^2} + \frac{1.259}{4. + 1. \, s^2}$$

Fig. 3. The phase space of the body about.

$$\xi(s) = -\frac{0.271}{s} - \frac{0.629s}{4. + 1. \, s^2}$$

We take the inverse Laplace transforms

$$\eta(t) = L^{-1}\left[-\frac{1.259}{s^2}\right] + L^{-1}\left[\frac{1.259}{4. + 1. \, s^2}\right]$$

$$\xi(t) = L^{-1}\left[\frac{-0.271}{s}\right] + L^{-1}\left[-\frac{0.629s}{4. + 1. \, s^2}\right]$$

Then

$\eta(t) = (-0.315 \, i) \, e^{(-2i) \, t} \, ((-1 + i) + (1 + i) \, e^{(4.i) \, t}) - 1.259t$
$\xi(t) = -0.271 - 0.315 \, e^{(-2 \, i) \, t} (1 + e^{(4 \, i) \, t})$
From the results obtained it is found that L_1, L_3 at (± 0.9, 0).

Figs. 4 and 6 shows the variation for the general solution with time, a periodicity behavior is found. Figs. 5 and 7 the body moves in an elliptical orbit about L_1, L_3.
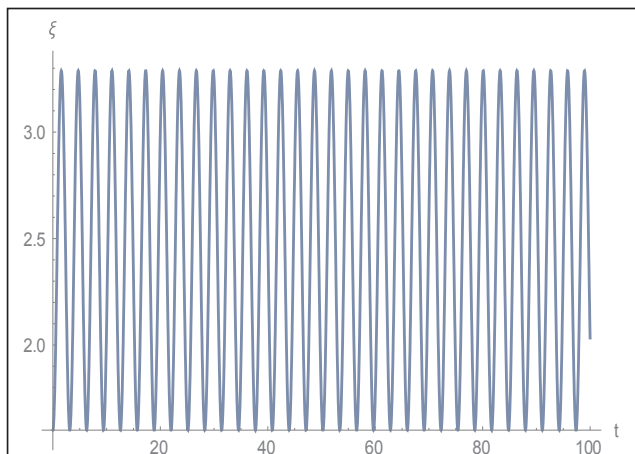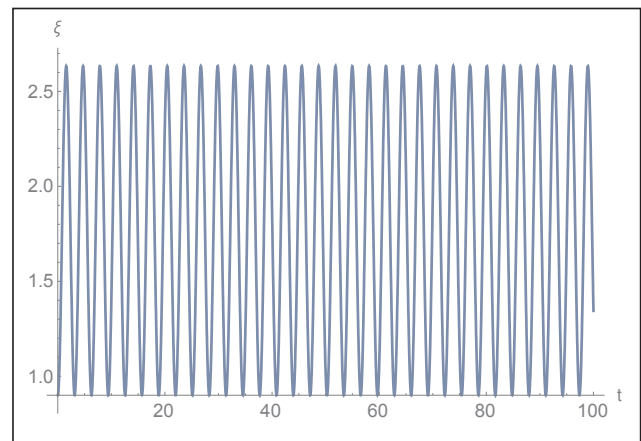

Fig. 2. The general solution about L2.

Fig. 4. The general solution about L1.
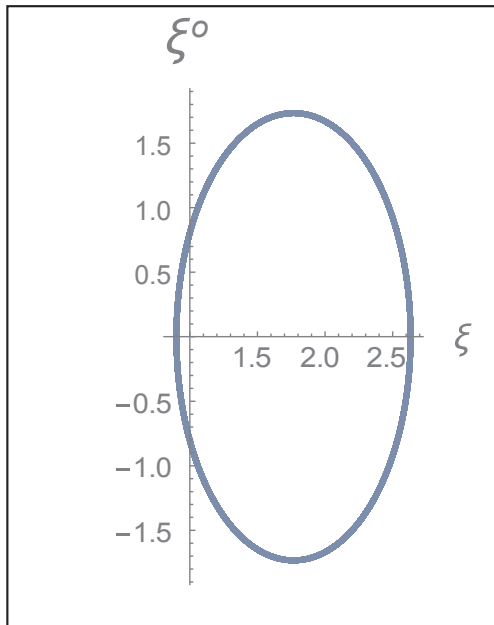
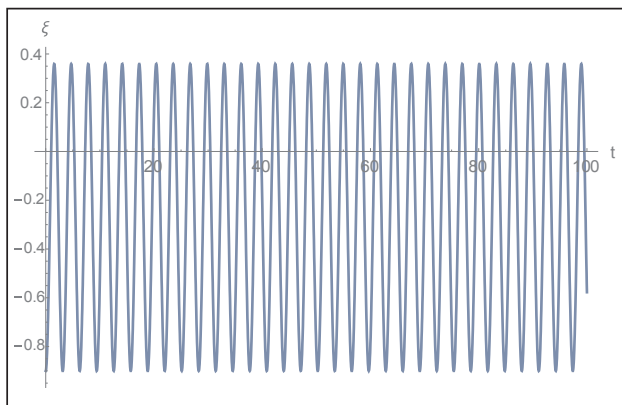**Fig. 5.** The phase space of the body about L1.
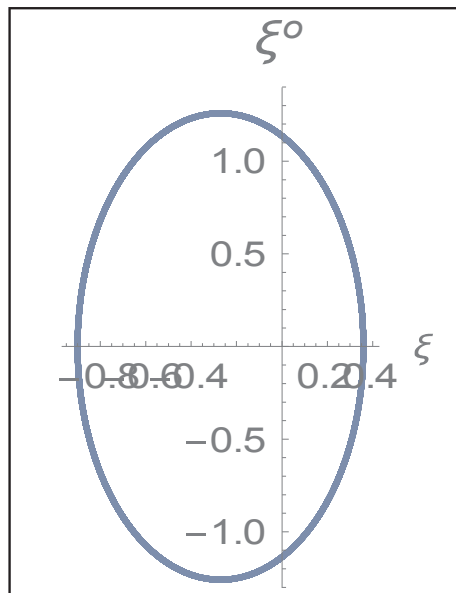

Fig. 6. The general solution about L3.

Fig. 7. The phase space of the body about L3.

5.1. Triangular points $L_4(0.8, \frac{\sqrt{3}}{2})$ and $L_5\left(0.8, -\frac{\sqrt{3}}{2}\right)$

The solution of the system at $\xi = 0.8$, $\eta = \frac{\sqrt{3}}{2}$

$$\eta(s) = -\frac{1.252}{s^2} + \frac{0.949}{s} - \frac{0.084(-14.957 + 1.s)}{4. + 1. s^2}$$

$$\xi(s) = \frac{1.426}{s} + \frac{0.167}{s^2} - \frac{0.167 - 0.626s}{4. + 1. s^2}$$

We take the inverse Laplace transforms

$$\eta(t) = L^{-1}\left[-\frac{1.252}{s^2}\right] + L^{-1}\left[\frac{0.949}{s}\right] - L^{-1}\left[\frac{0.084(-14.957 + 1.s)}{4. + 1. s^2}\right]$$

$$\xi(t) = L^{-1}\left[\frac{1.426}{s}\right] + L^{-1}\left[\frac{0.167}{s^2}\right] - L^{-1}\left[\frac{0.084(-14.957 + 1.s)}{4. + 1. s^2}\right]$$

Then

$$\eta(t) = 0.949 - (0.042 + 0.313i)e^{(-2.i)t}((-0.965 - 0.263i) + (1 + i)e^{(4.i)t})$$
$$- 1.252t$$

$$\xi(t) = 1.426 - (0.313 - 0.042i)e^{(-2.i)t}((0.965 + 0.263i) + (1 + i)e^{(4.i)t})$$
$$+ 0.1677t$$

The solution of the system at $\xi = 0.8$, $\eta = -\frac{\sqrt{3}}{2}$

$$\eta(s) = -\frac{1.252}{s^2} - \frac{0.949}{s} + \frac{(0.084(14.957 + 1.s))}{4. + 1. s^2}$$

$$\xi(s) = \frac{1.426}{s} - \frac{0.167}{s^2} + \frac{0.167 - 0.626s}{4. + 1. s^2}$$

We take the inverse Laplace transforms

$$\eta(t) = L^{-1}\left[-\frac{1.252}{s^2}\right] - L^{-1}\left[\frac{0.949}{s}\right] + L^{-1}\left[\frac{(0.084(14.957 + 1.s))}{4. + 1. s^2}\right]$$

$$\xi(t) = L^{-1}\left[\frac{1.426}{s}\right] - L^{-1}\left[\frac{0.167}{s^2}\right] + L^{-1}\left[\frac{0.167 - 0.626s}{4. + 1. s^2}\right]$$

Then
$$\eta(t) = -0.949 + (0.313 + 0.042i) \quad e^{(-2 \quad i) \quad t}$$
$$((0.263 + 0.964i) - (0 + i) e^{(4 i) t}) - 1.252t$$
$$\xi(t) = 1.426 - (0.313 + 0.042i) \quad e^{(-2 \quad i) \quad t}$$
$$((0.965 - 0.263i) + (1 + i) e^{(0. + 4. i) t}) - 0.167 t$$

But L_4 at $(0.8, \frac{\sqrt{3}}{2})$ the behavior is periodicity increasing as time increasing as shown in Fig. 8 the body moves in circular trajectory about L_4 as shown in Fig. 9 and its more stable, as shown in Fig. 10 the behavior around L_5 at $(0.8, -\frac{\sqrt{3}}{2})$ is periodicity decreasing at time increase and the body moves in circular orbits as shown in Fig. 11.
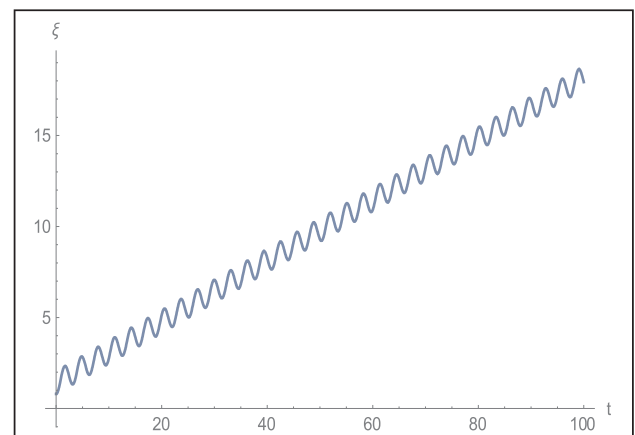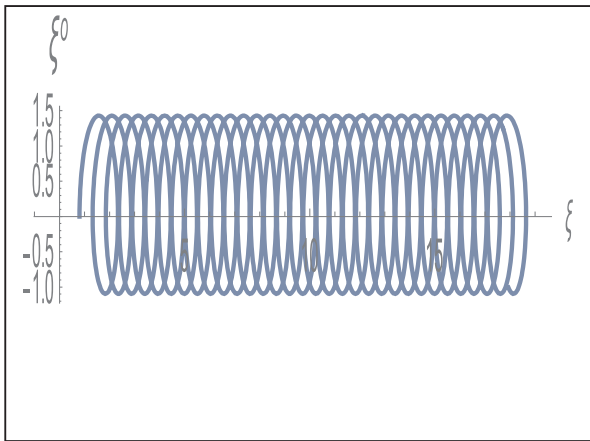

Fig. 8. The general solution about L4.

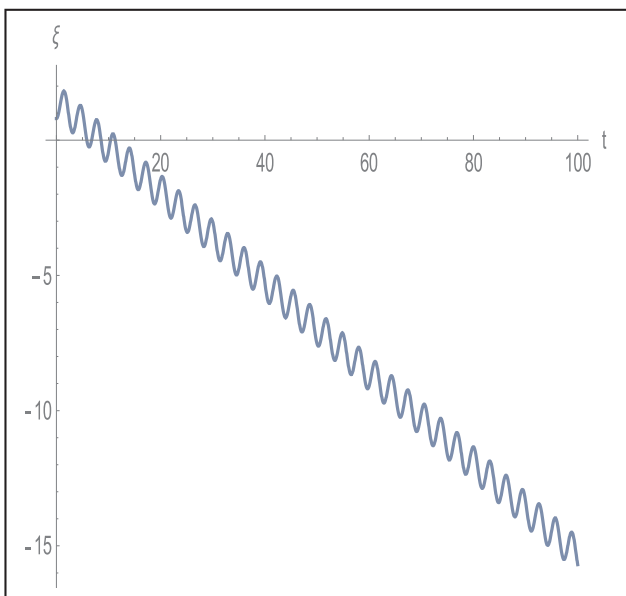**Fig. 9.** The phase space of the body about L4.

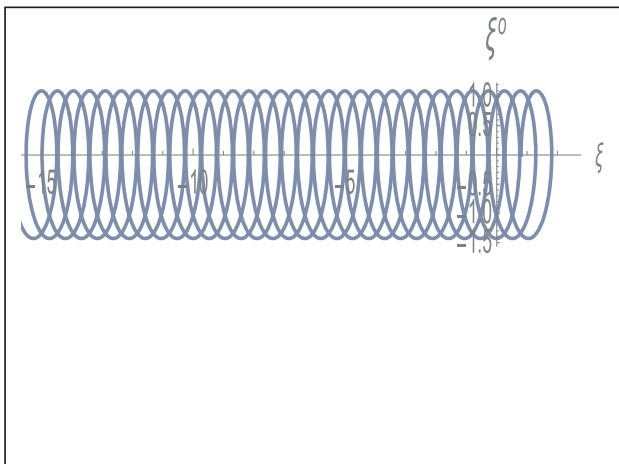**Fig. 10.** The general solution about L5.

Fig. 11. The phase space of the body about L5.

5.2. Trajectory around $L_4(0.8, \frac{\sqrt{3}}{2})$ and $L_2(1.2, 0)$

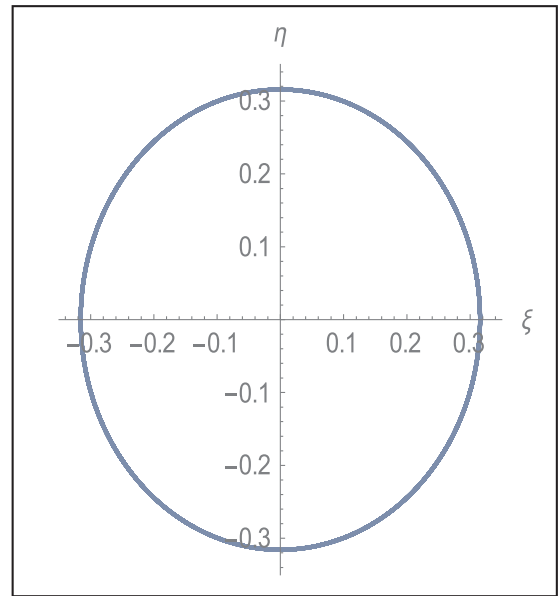The frequency of the out of plane motion is given by solving the equation

Fig. 12. The trajectory around L4.

$$s^4 + 4s^2 = 0$$

Where

$$s_{1,2} = 0, \; s_{3,4} = \pm i2$$

it is shown that more stable, then the periodic time $T = \frac{2\pi}{s_3} = \pi$, we get the eccentricity $e = (1-K^{-2})^{0.5} = 0.499$ while $k = 0.5\left(\frac{s_3 + u_{xx}}{2s_3}\right)$(Ibrahim, 2017); as shown in Figs. 12 and 13 the body moves in circular trajectory around L_4 and L_2.

6. Conclusion

Through this work the behavior of a body about the equilibrium points L_1, L_2, L_3, L_4 and L_5 is studied by Laplace transformation. The results obtained by Laplace transformations was a very interest to specify the behavior of the stability for each point. Also, an application has done for the motion of spacecraft near the equilibrium points of the Earth-Moon system and the results obtained was in a good agreement
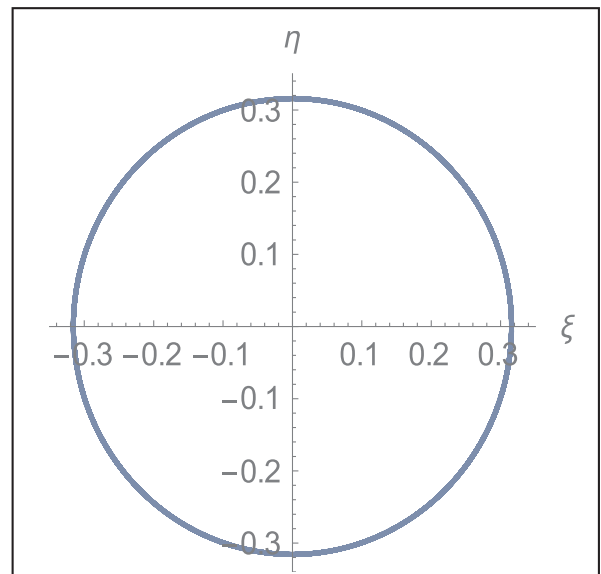

Fig. 13. The trajectory around L2.

with the previous work (Kumari and Kushvah, 2013).

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