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ON A ELLIPTIC PROBLEM INVOLVING VARIABLE-ORDER FRACTIONAL $p(\cdot)-$ LAPLACIAN AND LOGARITHMIC NONLINEARITY

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Abstract. This paper investigates the existence of weak solutions for a fractional elliptic problem with variable exponent and variable order, using Ekeland's variational principle. The equation studied involves the generalised fractional Laplacian operator, denoted $(-\Delta)^{s(\cdot)}_{p(\cdot)}$, specialised in modelling complex real or physical phenomena, where p and s are continuous functions of real variables with values in $(0, \infty)$ and $(0, 1)$, respectively. The method is based on the variational formulation associated with the fractional elliptic equation. We consider a functional for which a minimizer is sought in a fractional Sobolev space. Under certain assumptions on the exponents and the order of derivation, we have shown that this functional admits a minimizer. This minimizer is a weak solution of the elliptic equation. This approach makes it possible to treat non-local problems with variable exponents and order of derivation, thus offering an extension of the classical results to more complex cases. The functional setting involves Lebesgue and Sobolev spaces with variable exponent and variable-order.

1. Introduction

The fractional variable order derivatives suggested by Lorenzo and Hartley in [15] have become indispensable in the mathematical description or modeling of complex phenomena, where traditional operators have shown their limitations. Physics, biology, finance, electromagnetism, nuclear (strong) interactions, epidemics, and others are among the fields of application of these operators, see [1, 2, 6, 9]. This situation has attracted many researchers to study problems involving these operators, see [17, 19] . In 2017, U.Kaufman *et al.* [11] introduced the variable exponent

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fractional Laplacian $(-\Delta)^s_{p(\cdot)}$ defined by:

$$(-\Delta)^s_{p(\cdot)}u(x) = P.V \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s.p(x,y)}}dy,\ x \in \mathbb{R}^N,$$

where P.V. is a commonly used abbreviation for the Cauchy principal value. It is a fractional version of the $p(x)$-Laplacian operator given by $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, associated with the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$.
They also presented very interesting results on Sobolev embeddings in variable exponent fractional Sobolev spaces and they proved the existence and uniqueness of weak solution for the following problem

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(\cdot)-2}u(x) & = \lambda f(x) \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

with $f \in L^{\alpha(x)}$ for some $\alpha(x) > 1$.
Other authors, such as M. Hsini *et al.* [10], S. Korbeogo *et al.* [12] and A. Sabri *et al.* [17] , have been interested in problems involving the fractional Laplacian with variable exponent. In particular, M. Hsini *et al.* proved the existence of weak solution of the following problem via Ekeland's variational principle:

$$\begin{cases} (-\Delta)^s_{p(.)}u(x) + |u(x)|^{q(\cdot)-2}u(x) & = \lambda \dfrac{\partial F}{\partial u}(x,u) \quad \text{in } \Omega, \\ u & = 0 \quad \text{in } \partial\Omega, \end{cases} \tag{2}$$

where $F \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, locally Lipschitz and λ a positive parameter.
On the other hand, M. Xiang *et al.* [19] introduced the variable-order fractional p-Laplacian, defined by,

$$(-\Delta)^{s(\cdot)}_p u(z) = P.V \int_{\mathbb{R}^N} \frac{|u(z)-u(\xi)|^{p-2}(u(z)-u(\xi))}{|z-\xi|^{N+p.s(z,\xi)}}d\xi,\ z \in \mathbb{R}^N,$$

also demonstrating embedding results. When $s(\cdot) = s(constant) \in (0,1)$, the operator $(-\Delta)^{s(\cdot)}_p$ reduces to the usual fractional p-Laplacian.

Later, researchers quickly turned to a more general operator, namely the variable-order fractional Laplacian with a variable exponent, denoted by $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ and defined by:

$$(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) = P.V \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s(x,y).p(x,y)}}dy,\ x \in \mathbb{R}^N.$$

In [18] A. Sabri proved the existence and uniqueness of the weak solution to the problem formulated as follows:

$$\begin{cases} u_t + (-\Delta)^{s(\cdot)}_{p(\cdot)}u & = f \quad \text{in } Q_T := \Omega \times (0,T), \\ u & = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(\cdot,0) & = u_0(\cdot) \quad \text{in } \Omega. \end{cases}$$

Where $f \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega)$.
First of its kind, this paper aims to demonstrate the existence of weak solution to an elliptic problem $(P)_{s(\cdot),p(\cdot)}$ involving a fractional operator of variable order and exponent, using Ekeland's variational principle.

$$(P)_{s(\cdot),p(\cdot)} \begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)}u + |u|^{q(\cdot)-2}u & = \lambda |u|^{q(\cdot)-2}u\log(|u|) \quad \text{in} \quad \Omega, \\ u(.,0) & = u_0(.) \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N > 2$ is a bounded smooth domain and λ is a positive parameter. The problem addressed in this work generalises the work of M. Hsini *et al.* in [10], and therefore represents a significant advancement in the study of problems involving fractional derivatives.
We start by defining $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (0, \infty)$ and $s : \overline{\Omega} \times \overline{\Omega} \longrightarrow (0, 1)$ be two continuous and symmetric functions such that

$$1 < p^- = \min_{(y,z)\in\overline{\Omega}\times\overline{\Omega}} p(y, z) \leq p(y, z) \leq p^+ = \max_{(y,z)\in\overline{\Omega}\times\overline{\Omega}} p(y, z) < \infty$$

and

$$0 < s^- = \min_{(y,z)\in\overline{\Omega}\times\overline{\Omega}} s(y, z) \leq s(y, z) \leq s^+ = \max_{(y,z)\in\overline{\Omega}\times\overline{\Omega}} s(y, z) < 1,$$

with

$$N > p(\cdot)s(\cdot).$$

Moreover, the function $q : \overline{\Omega} \longrightarrow (0, \infty)$ is continuous satisfying

$$1 < q^- = \min_{y\in\overline{\Omega}} q(y) \leq q(y) \leq q^+ = \max_{y\in\overline{\Omega}} q(y) < \infty.$$

The rest of the paper is organized as follows: In section 2, we recall some basic proprieties of Lebesgue and Sobolev spaces with variable exponent and variable-order and in section 3, we state and prove our main result.

2. Preliminaries

To start, we define the space

$$C_+(\overline{\Omega}) = \left\{h; h \in C(\overline{\Omega}), h(y) > 1 \text{ for any } y \in \overline{\Omega}\right\}.$$

For $q \in C_+(\overline{\Omega})$, we consider the function space

$$L^{q(\cdot)}(\Omega) = \left\{ f(measurable) : \Omega \longrightarrow \mathbb{R} : \exists\lambda > 0 : \int_\Omega \left|\frac{f(y)}{\lambda}\right|^{q(y)} dy < \infty \right\}$$

$L^{q(\cdot)}(\Omega)$ is separable, uniformly convex Banach space with variable exponents emdowed with the norm:

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left|\frac{f(y)}{\lambda}\right|^{q(y)} dy \leq 1 \right\}$$

$(L^{q(\cdot)}(\Omega), \|.\|_{L^{q(\cdot)}})$ is generalized Lebesgue space.
If $\dfrac{1}{q(y)} + \dfrac{1}{q'(y)} = 1$ then $L^{q(\cdot)}$ and $L^{q'(\cdot)}$ are conjugate.

Hölder-type inequality : if $u(y) \in L^{q(\cdot)}(\Omega)$ and $v(y) \in L^{q'(\cdot)}(\Omega)$ then the following inequality holds:

$$\left|\int_\Omega u(y)v(y)dy\right| \leq \left(\frac{1}{q-} + \frac{1}{q'-}\right) \|u(y)\|_{q(\cdot)} \|v(y)\|_{q'(\cdot)}.$$

In the absence of any ambiguity, we use $|\cdot|_{q(\cdot)}$ instead of $\|\cdot\|_{L^{q(\cdot)}}$.

Lemma 2.1 (see [12]). *If (u_n), $u \in L^{q(\cdot)}(\Omega)$ and $q^+ < \infty$, then we have the following relations:*

(i) $|u|_{q(\cdot)} > 1 \Rightarrow |u|^{q^-}_{q(\cdot)} \leq \int_\Omega |u|^{q(x)}dx \leq |u|^{q^+}_{q(\cdot)}$;

(ii) $|u|_{q(\cdot)} < 1 \Rightarrow |u|_{q(\cdot)}^{q^+} \leq \int_\Omega |u|^{q(x)}dx \leq |u|_{q(\cdot)}^{q^-}$;

(iii) $|u_n - u|_{q(\cdot)} \longrightarrow 0$ *if and only if* $\int_\Omega |u_n - u|^{q(x)}dx \longrightarrow 0.$

Proposition 2.1 (see [10]). *Let p and q be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and*
$1 \leq p(y)q(y) \leq \infty$ for any $y \in \mathbb{R}^N$. Let $u \in L^{q(\cdot)}(\mathbb{R}^N), u \neq 0$. Then

$$\min\left(|u|_{p(\cdot)q(\cdot)}^{p^+}, |u|_{p(\cdot)q(\cdot)}^{p^-}\right) \leq ||u|^{p(y)}|_{q(\cdot)} \leq \max\left(|u|_{p(\cdot)q(\cdot)}^{p^-}, |u|_{p(\cdot)q(\cdot)}^{p^+}\right). \tag{3}$$

If k is a positive integer number and $q \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by:

$$W^{k,q(\cdot)}(\Omega) = \Big\{u \in L^{q(\cdot)}(\Omega) : D^\alpha u \in L^{q(\cdot)}(\Omega), \text{ for all } |\alpha| \leq k\Big\}.$$

Here $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}...\partial^{\alpha_N}}. \tag{4}$$

On $W^{k,q(\cdot)}(\Omega)$ we consider the following norm

$$||u||_{k,q(\cdot)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{q(y)}. \tag{5}$$

We denote by $W_0^{k,q(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,q(\cdot)}(\Omega)$.

We consider the variable exponent Sobolev fractional space as follows:

$$W = W^{s(\cdot),p(\cdot)}(\Omega) = \Bigg\{u : \Omega \longrightarrow \mathbb{R} : u \in L^{\overline{p}(\cdot)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(y) - u(z)|^{p(y,z)}}{\lambda^{p(y,z)}|y - z|^{N+s(y,z)p(y,z)}}dydz < \infty, \forall\lambda > 0\Bigg\},$$

with $\overline{p}(y) = p(y, y), \forall y \in \Omega$.
Let

$$[u]_{s(\cdot),p(\cdot)} = \inf\left\{\lambda > 0, \int_\Omega \int_\Omega \frac{|u(y) - u(z)|^{p(y,z)}}{\lambda^{p(y,z)}|y - z|^{N+s(y,z)p(y,z)}}dydz < 1\right\} \tag{6}$$

be the variable exponent Gagliardo seminorm.
W is a separable reflexive banach space with the norm

$$\|u\|_W = [u]_{s(\cdot),p(\cdot)} + ||u||_{\overline{p}(\cdot)}. \tag{7}$$

We denote by $W_0 = W_0^{s(\cdot),p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in W, then W_0 is a Banach space with the norm $\|u\|_{W_0} = [u]_{s(\cdot),p(\cdot)}$.

Lemma 2.2 (see Proposition 1 in [17]). *Let $u \in W_0$ and $u_n \in W_0$, then*

(i) $[u]_{s(\cdot),p(\cdot)} < 1$ *(resp.* $= 1, > 1$*)* $\Longleftrightarrow$ $\displaystyle\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy < 1$ *(resp.* $= 1, > 1$*);*

(ii) *If* $1 \leq [u]_{s(\cdot),p(\cdot)} < \infty$, *then*

$$([u]_{s(\cdot),p(\cdot)})^{p-} \leq \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \leq ([u]_{s(\cdot),p(\cdot)})^{p+}; \quad (8)$$

(iii) *If* $[u]_{s(\cdot),p(\cdot)} \leq 1$, *then*

$$([u]_{s(\cdot),p(\cdot)})^{p+} \leq \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \leq ([u]_{s(\cdot),p(\cdot)})^{p-}; \quad (9)$$

(iv) $\lim_{n\longrightarrow\infty} [u_n]_{s(\cdot),p(\cdot)} = 0(\infty) \Longleftrightarrow$

$$\lim_{n\longrightarrow\infty} \int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy = 0(\infty);$$

(v) $\lim_{n\longrightarrow\infty} [u_n - u]_{s(\cdot),p(\cdot)} = 0 \Longleftrightarrow$

$$\lim_{n\longrightarrow\infty} \int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)-u(x)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy = 0$$

Lemma 2.3 (see Theorem 1 in [17])**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain and* $s \in (0,1)$. *Let* p *and* s *be two continuous variable exponents with* $s(y,z).p(y,z) < N$ *for* $(y,z) \in \overline{\Omega}\times\overline{\Omega}$. *Assume that* $r : \overline{\Omega} \longrightarrow (1,\infty)$ *is a continuous function such that*

$$p_s^*(y) = \frac{N\overline{p}(\cdot)}{N - \overline{s}(\cdot).\overline{p}(\cdot)} > r(y) \geq r^- = \min_{y\in\overline{\Omega}} r(y), \ \text{for } y \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N,s,p,r,\Omega)$ *such that for every* $u \in W_0$, *it holds that*

$$||u||_{r(\cdot)} \leq C\|u\|_{W_0}.$$

That is, the space W *is continuously embedded in* $L^{r(\cdot)}(\Omega)$. *Moreover, this embedding is compact.*

Lemma 2.4 (see [3])**.** *For all* u, $v \in W_0$, *we consider the following* $I : W_0 \longrightarrow W_0^*$ *such that*

$$\langle I(u), v\rangle = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y)(v(x)-v(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy.$$

Then

(i) I *is a bounded and strictly monotone operator;*

(ii) I *satisfy* (S_+) *condition, that is, if* $u_n \rightharpoonup u \in W_0$ *and* $\lim_{n\longrightarrow 0}\sup I(u_n)(u_n-u) \leq 0$, *then* $u_n \to u \in W_0$*;*

(iii) I *is a homeomorphism.*

Lemma 2.5 (see [5])**.** *If* $1 \leq p_0 \leq p_\theta \leq p_1 \leq \infty$, *then*

$$||u||_{p_\theta} \leq ||u||_{p_0}^{1-\theta}||u||_{p_1}^{\theta}, \quad (10)$$

for all $u \in L^{p_0}(\Omega)\cap L^{p_1}(\Omega)$ *with* $\theta \in (0,1)$ *defined by* $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

Lemma 2.6 (see [5]). *Let δ be a positive number. Then the following inequality holds*

$$|\log(\varrho)| \leq \frac{1}{\delta}|\varrho|^{\delta}, \tag{11}$$

for all $\varrho \in [1, \infty)$.

3. Main results

We define the weak solution to problem $(P)_{s(\cdot),p(\cdot)}$ as follows

Definition 3.1. *A function $u \in W_0$ is said to be a weak solution of $(P)_{s(\cdot),p(\cdot)}$ if*

$$\begin{aligned} &\int_{\Omega\times\Omega} \frac{|u(y)-u(z)|^{p(y,z)-2}(u(y)-u(z)(v(y)-v(z)}{|y-z|^{n+s(y,z)p(y,z)}}dydz \\ +& \int_{\Omega} |u|^{q(y)-2}uvdy = \lambda \int_{\Omega} |u|^{q(y)-2}u\log(|u|)vdy, \end{aligned} \tag{12}$$

for every $v \in W_0$.

Theorem 3.1. *There exists $\lambda^* > 0$, such that for all $\lambda \in (0, \lambda^*)$, problem $(P)_{s(\cdot),p(\cdot)}$ has a weak solution.*

Proof. In order to formulate the variational approach, we introduce the energy function J_λ defined from W_0 to $\mathbb{R}$ by:

$$\begin{aligned} J_\lambda(u) = \ & \int_{\Omega}\int_{\Omega} \frac{|u(y)-u(z)|^{p(y,z)}}{p(y,z)|y-z|^{n+s(y,z)p(y,z)}}dydz + \int_{\Omega} \frac{|u(y)|^{q(y)}}{q(y)}dy \\ & -\lambda \int_{\Omega} \frac{|u|^{q(y)}\log(|u|)}{q(y)}dy + \lambda \int_{\Omega} \frac{|u|^{q(y)}}{(q(y))^2}dy. \end{aligned}$$

Note that J_λ is well-defined and Gateaux differentiable on W_0. Using standard arguments, we can demonstrate the equivalence between the minimizer of J_λ and the weak solution of the problem $(P)_{s(\cdot),p(\cdot)}$. Indeed, let $u \in W_0$ be a minimizer of J_λ. We will show that u satisfies problem $(P)_{s(\cdot),p(\cdot)}$. We have:

$$\begin{aligned} 0 &= \frac{d}{dt}J_\lambda(u+tv)\Big|_{t=0} \\ &= \int_{\Omega}\int_{\Omega} \frac{d}{dt}\frac{|u(y)-u(z)+t\big(v(y)-v(z)\big)|^{p(y,z)}}{p(y,z)|y-z|^{N+p(y,z)s(y,z)}}dydz\Bigg|_{t=0} - \int_{\Omega}\frac{d}{dt}\frac{|u(y)+tv(y)|^{q(y)}}{q(y)}dy\Bigg|_{t=0} \\ &\quad -\lambda\int_{\Omega}\frac{d}{dt}\frac{|u(y)+tv(y)|^{q(y)}\log(|u(y)+tv(y)|)}{q(y)}dy\Bigg|_{t=0} + \lambda\int_{\Omega}\frac{d}{dt}\frac{|u(y)+tv(y)|^{q(y)}}{(q(y))^2}dy\Bigg|_{t=0} \\ &= \int_{\Omega\times\Omega}\frac{|u(y)-u(z)|^{p(y,z)-2}(u(y)-u(z))(v(y)-v(z))}{|y-z|^{n+s(y,z)p(y,z)}}dydz + \int_{\Omega}|u|^{q(y)-2}uvdy \\ &\quad -\lambda\int_{\Omega}|u|^{q(y)-2}u\log(|u|)vdy. \end{aligned} \tag{13}$$

Thus, u is weak solution of $(P)_{s(\cdot),p(\cdot)}$, with v being a test function. Conversely, let us consider a weak solution u of problem $(P)_{s(\cdot),p(\cdot)}$ and show that it minimizes J_λ.

Let $v \in W_0$, then we have the following weak formulation:

$$\int_{\Omega\times\Omega} \frac{|u(y)-u(z)|^{p(y,z)-2}(u(y)-u(z)(v(y)-v(z)}{|y-z|^{n+s(y,z)p(y,z)}}dydz$$
$$+ \quad \int_\Omega |u|^{q(y)-2}uvdy - \lambda\int_\Omega |u|^{q(y)-2}u\log(|u|)vdy = 0, \tag{14}$$

which corresponds exactly to $J'_\lambda(u) = 0$, thus u minimizes J_λ.
The remainber of the proof of the Theorem 3.1 relies on the following lemmas:

Lemma 3.7. *For all $\epsilon > 0$, there exists $C(\epsilon)$ such that:*

$$\int_\Omega |u|^{q(y)}\log(|u|)dy \le C(\epsilon)\left([u]^{q^++\delta}_{s(\cdot),p(\cdot)} + ||u||^{p^{*+}_s}_{L^{\overline{p}(\cdot)}}\right). \tag{15}$$

Proof. Let us consider two disjoint subsets Ω_1 and Ω_2 of Ω defined as follows: $\Omega_1 = \{y \in \Omega : |u(y)| \le 1\}$ and $\Omega_2 = \{y \in \Omega : |u(y)| > 1\}$. We have:

$$\begin{aligned}\int_\Omega |u|^{q(y)}\log(|u|)dy &= \int_{\Omega_1} |u|^{q(y)}\log(|u|)dy + \int_{\Omega_2}|u|^{q(y)}\log(|u|)dy\\ &\le \frac{1}{\delta}\int_{\Omega_2}|u|^{q(y)+\delta}\\ &\le \frac{1}{\delta}||u||^{q^++\delta}_{L^{q(\cdot)+\delta}(\Omega_2)}.\end{aligned} \tag{16}$$

By choosing δ such that $\overline{p}(\cdot) < q(\cdot)+\delta < p^*_s(\cdot)$ and using the interpolation inequality followed by the injection $W_0 \hookrightarrow L^{p^*_s(y)}$, we obtain:

$$\begin{aligned}\int_\Omega |u|^{q(y)}\log(|u|)dy &\le ||u||^{\theta(q^++\delta)}_{p^*_s(\cdot)}||u||^{(1-\theta)(q^++\delta)}_{\overline{p}(\cdot)}\\ &\le C[u]^{\theta(q^++\delta)}_{s(\cdot),p(\cdot)}||u||^{(1-\theta)(q^++\delta)}_{\overline{p}(\cdot)}\\ &\le C[u]^{\theta(q^++\delta)}_{s(\cdot),p(\cdot)}||u||^{(1-\theta)(q^++\delta)}_{\overline{p}(\cdot)},\end{aligned} \tag{17}$$

where $\theta = \dfrac{n\delta}{(p^*(\cdot)+\delta)sp^*(\cdot)} \in (0,1)$. Since $q(\cdot)+\delta < p^*_s(y)$ and $u \in \Omega_2$, then using Young inequality, we obtain:

$$\int_\Omega |u|^{q(y)}\log(|u|)dy \le C(\epsilon)\left([u]^{q^++\delta}_{s(\cdot),p(\cdot)} + ||u||^{p^{*+}_s}_{L^{\overline{p}(\cdot)}}\right). \tag{18}$$

□

Lemma 3.8. *Suppose we are under hypotheses of Theorem 3.1. Then for all $\rho \in (0,1)$, there exists $\lambda^* > 0$ and $\beta > 0$ such that for all $u \in W_0$ with $[u]_{s(\cdot),p(\cdot)} = \rho$*

$$J_\lambda(u) \ge \beta \textit{ for all } \lambda \in (0,\lambda^*). \tag{19}$$

Proof. Since the embedding $W_0 \hookrightarrow L^{q(\cdot)+\delta}(\Omega_2)$ is continuous, then

$$||u||_{L^{q(\cdot)+\delta}(\Omega)} \le C'[u]_{s(\cdot),p(\cdot)} \text{ with } C' > 0. \tag{20}$$

Are $\rho \in (0,1)$, we are allowed to assume that $[u]_{s(\cdot),p(\cdot)} < \min\{1, \frac{1}{C'}\}$. Then $||u||_{L^{q(\cdot)+\delta}(\Omega)} < 1$, and we have:

$$
\begin{aligned}
J_\lambda(u) &= \int_\Omega\int_\Omega \frac{|u(y)-u(z)|^{p(y,z)}}{p(y,z)|y-z|^{n+s(y,z)p(y,z)}}dydz + \int_\Omega \frac{|u(y)|^{q(y)}}{q(y)}dy \\
&- \lambda\int_\Omega \frac{|u|^{q(y)}\log(|u|)}{q(y)}dy + \lambda\int_\Omega \frac{|u|^{q(y)}}{(q(y))^2}dy \\
&\geq \int_\Omega\int_\Omega \frac{|u(y)-u(z)|^{p(y,z)}}{p(y,z)|y-z|^{n+s(y,z)p(y,z)}}dydz - \frac{\lambda}{\delta}||u||^{q^++\delta}_{L^{q(y)+\delta}(\Omega_2)} \\
&\geq \frac{1}{p^+}[u]^{p^+}_{s(\cdot),p(\cdot)} - \frac{\lambda}{\delta}||u||^{q^++\delta}_{L^{q(\cdot)+\delta}(\Omega_2)}. \qquad (21)
\end{aligned}
$$

Thus, by taking $\beta = \frac{1}{p^+}[u]^{p^+}_{s(\cdot),p(\cdot)} - \frac{\lambda}{\delta}||u||^{q^++\delta}_{L^{q(\cdot)+\delta}(\Omega_2)}$ and $\lambda^* = \frac{\delta[u]^{p^+}_{s(\cdot),p(\cdot)}}{p^+||u||^{q^++\delta}_{L^{q(\cdot)+\delta}(\Omega_2)}}$, we obtain that for all $\lambda \in (0,\lambda^*)$, $J_\lambda(u) \geq \beta$. $\square$

Lemma 3.9. *There exists $\varphi \in W_0$ such that $\varphi \geq 0, \varphi \neq 0$ and $J_\lambda(t\varphi) < 0$, for $t > 0$ small enough and $q(y) + \delta < p(y,z)$ $\forall y,z \in \Omega$.*

Proof. Observe that when $t \in (0,1)$ and $q < p$, it follows that $\frac{t^{p^-}}{p^-} \leq \frac{t^{q^-}}{q^-}$ and we have:

$$
\begin{aligned}
J_\lambda(t\varphi) &= \int_\Omega\int_\Omega \frac{|t\varphi(y)-t\varphi(z)|^{p(y,z)}}{p(y,z)|y-z|^{n+s(y,z)p(y,z)}}dydz + \int_\Omega \frac{|t\varphi|^{q(y)}}{q(y)}dy \\
&- \lambda\int_\Omega \frac{|t\varphi|^{q(y)}\log(|t\varphi|)}{q(y)}dy + \lambda\int_\Omega \frac{|t\varphi|^{q(y)}}{(q(y))^2}dy \\
&\leq \frac{t^{p^-}}{p^-}\int_\Omega\int_\Omega \frac{|\varphi(y)-\varphi(z)|^{p(y,z)}}{|y-z|^{n+s(y,z)p(y,z)}}dydz + \frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}dy \\
&\quad -\lambda\frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}log(|t|)dy - \lambda\frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy \\
&\quad +\frac{t^{q_0^-}}{(q_0^-)^2}\int_\Omega |\varphi|^{q(y)}dy. \qquad (22)
\end{aligned}
$$

Since t is assumed to be quite small, then $log(|t|) < 0$, and we have
$-\int_\Omega |\varphi|^{q(y)}log(|t|)dy = \int_\Omega |\varphi|^{q(y)}|log(|t|)|dy.$
In short, (22) becomes:

$$\begin{aligned}
J_\lambda(t\varphi) &\leq \frac{t^{p^-}}{p^-}\int_\Omega\int_\Omega \frac{|\varphi(y)-\varphi(z)|^{p(y,z)}}{|y-z|^{n+s(y,z)p(y,z)}}dydz + \frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}dy \\
&\quad +\lambda\frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}|log(|t|)|dy - \lambda\frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy + \frac{t^{q^-}}{(q^-)^2}\int_\Omega |\varphi|^{q(y)}dy \\
&\leq \frac{t^{p^-}}{p^-}\int_\Omega\int_\Omega \frac{|\varphi(y)-\varphi(z)|^{p(y,z)}}{|y-z|^{n+s(y,z)p(y,z)}}dydz + \frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}dy \\
&\quad +\lambda\frac{t^{q^-}|log(|t|)|}{q^-}\int_\Omega |\varphi|^{q(y)}dy - \lambda\frac{t^{q^-}}{q^-}\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy + \frac{t^{q^-}}{(q^-)^2}\int_\Omega |\varphi|^{q(y)}dy \\
&\leq \frac{t^{q^-}}{q^-}\Big[\int_\Omega\int_\Omega \frac{|\varphi(y)-\varphi(z)|^{p(y,z)}}{|y-z|^{n+s(y,z)p(y,z)}}dydz + \int_\Omega |\varphi|^{q(y)}dy + \lambda|log(|t|)|\int_\Omega |\varphi|^{q(y)}dy \\
&\quad -\lambda\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy + \int_\Omega |\varphi|^{q(y)}dy\Big] \\
&\leq \frac{t^{q^-}}{q^-}\Big[\max\Big([\varphi]^{p^+}_{s(\cdot),p(\cdot)}, [\varphi]^{p^-}_{s(\cdot),p(\cdot)}\Big) + 2\max\Big(||\varphi||^{q^+}_{q(\cdot)}, ||\varphi||^{q^-}_{q(\cdot)}\Big) \\
&+ \lambda|log(|t|)|\max\Big(||\varphi||^{q^+}_{q(\cdot)}, ||\varphi||^{q^-}_{q(\cdot)}\Big) - \lambda\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy\Big].
\end{aligned} \tag{23}$$

Therefore $J_\lambda(t\varphi) < 0$, for

$$0 < t < \Bigg\{1, \exp\Bigg[\frac{\lambda\int_\Omega |\varphi|^{q(y)}log(|\varphi|)dy - \max\Big([\varphi]^{p^+}_{s(\cdot),p(\cdot)}, [\varphi]^{p^-}_{s(\cdot),p(\cdot)}\Big)}{\max\Big(||\varphi||^{q^+}_{q(\cdot)}, ||\varphi||^{q^-}_{q(\cdot)}\Big)} - \frac{2\max\Big(||\varphi||^{q^+}_{q(\cdot)}, ||\varphi||^{q^-}_{q(\cdot)}\Big)}{\max\Big(||\varphi||^{q^+}_{q(\cdot)}, ||\varphi||^{q^-}_{q(\cdot)}\Big)}\Bigg\} \tag{24}$$

□

Consider the boundary ball $B_\rho(0)$ with centered at the origin and radius ρ.
By the Lemma 3.8 we deduce that $\inf_{\partial B_\rho(0)} J_\lambda > 0$. Also, by Lemma 3.9, there exists $\varphi \in W_0$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. It follows that

$$-\infty < \varsigma := \inf_{\overline{B_\rho(0)}} J_\lambda < 0. \tag{25}$$

Let $0 < \eta < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{\overline{B_\rho(0)}} J_\lambda$. According to pevious informations, J_λ is lower bounded on $\overline{B_\rho(0)}$ and $J_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle, there exists $u_\eta \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \varsigma \leq J_\lambda(u_\eta) \leq \varsigma + \eta \\ 0 < J_\lambda(u) - J_\lambda(u_\eta) + \eta||u-u_\eta||_{W_0} \quad , \quad u \neq u_\eta. \end{cases} \tag{26}$$

□

Since $\inf\limits_{\overline{B_\rho(0)}} J_\lambda \leq \inf\limits_{B_\rho(0)} J_\lambda$, then

$$\begin{aligned} J_\lambda(u_\eta) &\leq \inf_{\overline{B_\rho(0)}} J_\lambda + \eta \\ &\leq \inf_{B_\rho(0)} J_\lambda + \eta \\ &\leq \inf_{\partial B_\rho(0)} J_\lambda, \end{aligned} \tag{27}$$

and we deduce that $u_\eta \in B_\rho(0)$.
We define $\chi_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $\chi_\lambda(u) = J_\lambda(u) + \eta||u - u_\eta||$.
It clear that u_η is a minimum point of χ_λ and thus

$$\frac{\chi_\lambda(u_\eta + t.v) - \chi_\lambda(u_\eta)}{t} \geq 0 \tag{28}$$

for small $t > 0$ and any $v \in B_\rho(0)$.
The above relation yields

$$\frac{J_\lambda(u_\eta + t.v) - J_\lambda(u_\eta)}{t} + \eta||v|| \geq 0. \tag{29}$$

Letting $t \longrightarrow 0$, it follows that

$$\langle J'_\lambda(u_\eta), v\rangle + \eta||v||_{W_0} \geq 0. \tag{30}$$

For $v = -J'_\lambda(u_\eta)$, we have $||J'_\lambda(u_\eta)||_{W_0} \leq \eta$.
Let sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J(w_n) \longrightarrow m < 0 \text{ and } J'(w_n) \longrightarrow 0_{W_0^*}. \tag{31}$$

Since $\{w_n\} \subset B_\rho(0)$ then $||w_n||_{W_0} \leq \rho$, therefore w_n is bounded in W_0. We can therefore extract a subsequence again denoted $\{w_n\}$ such that $w_n \rightharpoonup w$ and since for all ϱ such that $1 \leq \varrho(\cdot) < p_s^*(y)$ the injection $W_0 \hookrightarrow L^{\varrho(\cdot)}$ is compact, we deduce that

$$w_n \longrightarrow w \text{ in } L^{\varrho(\cdot)} \text{ when } n \longrightarrow \infty. \tag{32}$$

In the following, we will need the next proposition:

Proposition 3.1. *If w_n converges weakly to w in W_0, then*

(i) $\lim\limits_{n\longrightarrow\infty} \int_\Omega |w_n|^{q(y)-2} w_n (w_n - w) dy = 0$;

(ii) $\lim\limits_{n\longrightarrow\infty} \int_\Omega |w_n|^{q(y)-2} w_n \log(|w_n|)(w_n - w) dy = 0$;

(iii) $\lim\limits_{n\longrightarrow\infty} \langle J'(w_n), w_n - w\rangle = 0$.

Proof. For (i), we will use the compact injection $W_0 \hookrightarrow L^{q(\cdot)}$. We have:

$$\begin{aligned} \int_\Omega |w_n|^{q(y)-2} w_n (w_n - w) dy &\leq \big||w_n|^{q(\cdot)-2} w_n\big|_{\frac{q(\cdot)}{q(\cdot)-1}} \big|w_n - w\big|_{q(\cdot)} \\ &\leq |w_n|_{q(\cdot)}^{q^+} \big|w_n - w\big|_{q(\cdot)} \end{aligned} \tag{33}$$

Therefore, for $\varrho(\cdot) = q(\cdot) \leq p_s^*(y)$ passing to the limit, we have the result thanks to (32).

For (ii),

$$\begin{aligned}\int_{\Omega}|w_n|^{q(y)-2}w_n\log(|w_n|)(w_n-w)dy &\leq \frac{1}{\delta}\int_{\Omega}|w_n|^{q(y)-2}w_n|w_n|^{\delta}(w_n-w)dy\\ &\leq \frac{1}{\delta}\int_{\Omega}|w_n|^{q(y)-2+\delta}w_n(w_n-w)dy\\ &\leq \frac{1}{\delta}\big||w_n|^{q(\cdot)-2+\delta}w_n\big|_{\frac{q(\cdot)+\delta}{q(\cdot)-1+\delta}}\big|w_n-w\big|_{q(\cdot)+\delta}\\ &\leq \frac{1}{\delta}|w_n|^{q^++\delta}_{q(\cdot)+\delta}\big|w_n-w\big|_{q(\cdot)+\delta}. \qquad (34)\end{aligned}$$

Since $\varrho(\cdot)=q(\cdot)+\delta\leq p_s^*(y)$ then, thanks to (32) and (34), we obtain (ii).
To prove (iii), start from $J'(w_n)\longrightarrow 0_{W_0^*}$. Since w_n is bounded in W_0, we have:

$$\langle J'(w_n),w_n-w\rangle\leq ||J'(w_n)||_{W_0^*}||w_n||_{W_0}+||J'(w_n)||_{W_0^*}||w||_{W_0} \qquad (35)$$

By passing to the limit in (35), we obtain $\lim\limits_{n\longrightarrow\infty}\langle J'(w_n),w_n-w\rangle=0$. Therefore,

$$\lim_{n\longrightarrow\infty}\int_{\Omega\times\Omega}\frac{|w_n(y)-w_n(z)|^{p(y,z)-2}(w_n(y)-w_n(z))((w_n-w)(y)-(w_n-w)(z))}{|y-z|^{n+s(y,z)p(y,z)}}dydz = 0. \qquad (36)$$

Thus, thanks to the Lemma 2.4 and the equation (36), we obtain strong convergence of w_n to w in W_0. Moreover, J_λ being in the space $C^1(W_0,\mathbb{R})$, it follows that:

$$J'_\lambda(w_n)\longrightarrow J'_\lambda(w),\ \text{when } n\longrightarrow\infty. \qquad (37)$$

From relations (31) and (37),we deduce that $J'_\lambda(w)=0$ and thus w is weak solution of problem $(P)_{s(\cdot),p(\cdot)}$. □

Theorem 3.2. *Assume that $q^++\delta<p^-$. Then, for any $\lambda>0$, problem $(P)_{s(\cdot),p(\cdot)}$ has a weak solution.*

Proof. Since $q(\cdot)+\delta<p_s^*(\cdot)$, thanks to the Lemma 2.2 and 2.3, for $[u]_{s(\cdot),p(\cdot)}>1$, we have:

$$\begin{aligned}J_\lambda(u) &\geq \frac{1}{p^+}[u]^{p^-}_{s(\cdot),p(\cdot)}-\frac{\lambda}{\delta}||u||^{q^++\delta}_{L^{q(\cdot)+\delta}(\Omega_2)}\\ &\geq \frac{1}{p^+}[u]^{p^-}_{s(\cdot),p(\cdot)}-C'\frac{\lambda}{\delta}[u]^{q^++\delta}_{s(\cdot),p(\cdot)}. \qquad (38)\end{aligned}$$

Given $q^++\delta<p^-$, it follows that the functional J_λ is coercive. Furthermore, since J_λ is weakly lower semicontinuous, it attains its infimum; thus, it admits a global minimizer, which corresponds to a weak solution of the problem $(P)_{s(\cdot),p(\cdot)}$. □

4. Conclusion

At the end of this study, we have shown the existence of a weak solution to an elliptic problem involving the operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$. This is due to Ekeland's variational principle. The study also highlighted the impact of the variable exponent.

References

[1] N. F. Abdo, E. Ahmed and A. I. Elmahdy *Some real life apllictions of fractional calculus*, Journal of Fractional Calculus and Applications, Vol. 10(2), pp. 1-2.

[2] D. Applebaum, *Processes and Stochastic Calculus*, Camb.Stud.Adv.Math.116, Second Ed, Cambridge University Press, Cambridge, 2009.

[3] A. Bahrouni and V. D. Radulescu, On a new fractional Sobolev space and apllications to nonlocal variational problems with variable exponent, Discrete contin. Dyn. Syst. Ser. S. 11(3),pp.379-389, 2018.

[4] T. Boudjeriou *Global existence and blow-up of solutions for a parabolic equation involving the fractional p(x)-laplacian*, University of Bejaia, 6000, Algeria.

[5] T. Boudjeriou; *Stability of solution for a parabolic problem involving fractional p-Laplacian with logarithmic nonlinearity*, 2006. 11178v1.

[6] R. Cont and P. Tankov, *Financial Modelling with jump processes*, Chapman and Hall/CRC Financ. Math. Ser. Chapman and Hall/CRC, Boca Raton, FI, 2004.

[7] D. E. Edmunds and J. Ràkosnik, Sobolev embeddings with variable exponent, Studia Math, 143, pp. 267-293, 2000.

[8] J. Giacomoni, S. Tiwari, *Existence and global behavior of solutions to fractional p-Laplacien parabolic problems*, Electronic journal of differential equations, Vol. 2018 (2018), No. 44, pp. 1-20. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu.

[9] T. C.Halsey, *Electrorheological fluids*, Science 258(1992), 761-766.

[10] M. Hsini, K. Kefi and N. Irzi; *On a fractional problem with variable exponent*, Proceedings of the Romanian Academy, Series A, Volume 21, Number 2/2020, pp.105-114.

[11] U. Kaufmann, J. D. Rossi and R. Vidal *Fractional Sobolev spaces with variable exponents and fractional p(x)-Laplacian*, Electronic Journal of Qualitative Theory of Differential Equations 2017, No.76, pp.1-10; https://doi.org/10.14232/ejqtde.2017.1.76.

[12] S. Korbeogo , F. D. Y. Zongo and A. Ouedraogo *On a fractional parabolic problems with variable exponent*, Linear and Nonlinear analysis journal, volume 9, number 1, (2023), 65-81.

[13] S. Korbeogo , F. D. Y. Zongo and A. Ouedraogo, *On a variable-order fractional parabolic problems*, Proceeding of the 2nd International Conference on Nonlinear Dynamics and Applications (ICNDA 2024), Volume 2, Complex Systems, Fractals and Nonlinear Flows, pp. 204-219, 2024,https://doi.org/10.1007/978-3-031-69134-8-14.

[14] S. Liang, G. M. Bisci, B. Zhang *Sign-changing solutions for Kirchhoff-type problems involving variable-order fractional Laplacian and critical exponents*, Nonlinear Analysis: Modelling and Control, Vol. 27, No. 3, 556-575, 2022, https://doi.og/10.15388/namc. 2022.27.26575.

[15] C.F. Lorenzo, T.T. Hartley, *Initialized fractional calculus,* Int. J. Appl. Math. 3 (2000) 249–265.

[16] L. M. Pezzo and J.D. Rossi, *Traces for fractional Sobolev spaces with variable exponents*, Adv. Oper. Theory 2, pp. 435-446, 2017.

[17] A. Sabri, A. Jamea, H. T. Alaoui *Existence of entropy solutions to nonlinear degenerate parabolic problems with variable exponent and L^1-data*, Communications in Mathematics 28 (2020), no. 1, pp. 67-88, doi: 10.2478/cm-2020-0006, sciendo (2020), 67-88.

[18] A. Sabri *Weak Solution for Nonlinear Fractional $p(\cdot)-$Laplacian Problem with Variable Order via Rothe's Time-Discretization Method*, Mathematical Modelling and Analysis, Volume 27, Issue 4, 533-546, 2022 https://doi.org/10.3846/mma.2022.15740.

[19] M. Xiang, B. Zhang, D. Yang,, *Multiplicity results for variable-order fractional Laplacian equations with variable growth*, Nonlinear Anal., 178:190-204, 2019, https://doi.org/10.1016/j.na.2018.07.016.

[20] F. D. Y. Zongo, S. Korbeogo and A. Ouedraogo *On some fractional magnetic problems* Pan-American Journal of Mathematics 2 (2023), 17 https://doi.org/10.28919/cpr-pajm/ 2-17.

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