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STUDY ON P-TYPE ILC FOR HILFER-TYPE FRACTIONAL-ORDER QUATERNION-VALUED SYSTEMS WITH INITIAL STATE DEVIATION WITH APPLICATION IN SOFT ROBOTIC ACTUATORS

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Abstract. This paper examines a P-type iterative learning control law for linear quaternion-valued differential equations with respected to Hilfer fractional order. Convergence analysis is studied for both open-loop and closed-loop schemes, incorporating initial state deviations and random disturbances within the $(1-\gamma^*, \Lambda)$-norm concept. This study employs the properties of Mitta-Leffler functions to derive theoretical results, which are further validated through numerical examples that showcase the effectiveness of the proposed approach, with application in soft robotic actuators.

1. Introduction

Quaternions generalize complex numbers to four dimensions and are widely used for representing three-dimensional rotations and orientations. Quaternion-valued differential equations (QV-DEs) extend traditional differential equations by incorporating quaternions, making them highly effective for modeling systems involving both rotational and translational dynamics. These equations describe the relationship between changing quantities, typically with respect to time or space, and are particularly useful in applications such as robotics [24], aerospace engineering [32], and physics [5, 13], where objects experience simultaneous rotations and translations. Quaternions provide a concise and sophisticated concept for representing rotational motion and orientation changes. However, their algebraic complexity and the interactions between their components often require specialized mathematical techniques and computational tools for accurate solutions. Despite these

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challenges, QV-DEs remain indispensable for precisely simulating real-world 3D rotational dynamics across various scientific and engineering fields.

Over the past few years, there has been notable progress in the core theory of QV-DEs. Kou and Xia [14] studied solutions to linear QV-DEs and proposed two novel methods for determining the fundamental matrix, along with introducing the Wronskian and Liouville formula in the context of quaternions. Furthering this work, Kou et al. [15] proposed a technique for determining the fundamental matrix in linear systems characterized by multiple eigenvalues. Xia et al. [39] contributed to the field by deriving stability results specifically for quaternion periodic systems. They also formulated the variation of constants within the context of quaternions. Additionally, they developed an algorithm aimed at solving linear non-homogeneous QV-DEs. Additionally, Suo et al. [30] conducted an investigation into the solutions of linear quaternion-valued impulsive differential equations (QV-IDEs), addressing scenarios in both complex and quaternion settings.

Researchers have further explored periodic solutions for both homogeneous and non-homogeneous QV-IDEs. Chen et al. [3] applied Laplace transforms to establish the Hyers-Ulam(H-U) stability of linear QV-DEs and developed a novel approach to analyzing the controllability and observability of proposed linear systems. Under the permutation matrix hypothesis, Fu et al. [8] derived solutions for homogeneous and non-homogeneous linear QV-DEs using delayed quaternion matrix exponentials and the method of variation of constants. Feckan et al.[7] investigated the H-U stability of linear recurrence equations with constant coefficients in the quaternion concepts, whereas Lv et al.[23] used Fourier transforms to study the H-U stability of linear QV-DEs. Additionally, Huang et al. [11] utilized the second Lyapunov method to analyze the stability of QV-DEs, and Zahid et al. [43] computed the exponential matrix of QV-DEs.

Iterative learning control, or ILC, was first put forth by Uchiyama in 1978, with the original work published in Japanese. ILC refines tracking performance by using past control experiences, making it a valuable approach both theoretically and experimentally. While significant research has focused on P-type and D-type ILC for integer-order ordinary differential equations [1, 2, 4, 10, 20, 25, 31, 38, 41, 42, 44], including the linearization theorem of Fenner and Pinto [40] and various aspects of local integrability [28], studies on ILC for fractional-order differential equations (FODEs) remain relatively scarce [16, 17, 18, 19, 21]. Notably, for nonlinear FODEs [16] and non-instantaneous impulsive FODEs [21], researchers have established the robust convergence of tracking errors concerning initial positioning errors under the P-type ILC scheme. Meanwhile, D-type ILC has been investigated for linear time-delay FODEs [17]. D. Vivek et al.[34] were the first to study ILC for Hilfer-type QV-IDEs. Inspired by their research, we aim to examine P-type ILC while considering the effects of initial state deviations and random perturbations in the sense of Hilfer-type fractional-order quaternion-valued (QV) systems, formulated as follows:

$$\begin{cases} D^{\alpha,\beta}q(t) = \mathbb{A}q(t) + Bu(t), \quad t \in J = [0,b], \\ {}^{RL}I^{1-\gamma^*}q(0) = q_0. \end{cases} \tag{1}$$

To study the P-type ILC updating law, the output function considered by

$$r(t) = Cq(t), \tag{2}$$

where $D^{\alpha,\beta}$ denotes the Hilfer fractional derivative (HFD)of order α, and type β (Ref. [35], Definition. 2.3, Remark. 2.4, Page No. 851). Here $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma^* = \alpha + \beta - (\alpha\beta)$. $\mathbb{A}, B, C \in \mathbb{H}^{n\times n}$, $u(t)$ is a control vector. ${}^{RL}I^{1-\gamma^*}$ denotes the Riemann-Liouville (R-L) fractional integral.

According to (Ref. [27], Theorem. 1, Page No. 50, Theorem. 7, Page No. 59) and (Ref. [9], Lemma. 2.12, Page No. 4), the system (1) is equivalent to the integral equation given by

$$q(t) = t^{\gamma^*-1}\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)q_0 + \int_0^t (t-s)^{\alpha-1}\mathbb{E}^\star\alpha, \alpha(\mathbb{A}(t-s)^\alpha)Bu(s)ds, \qquad (3)$$

where $\mathbb{E}^\star_{\alpha,\gamma}(\cdot)$ denotes the Mittag-Leffler function $\mathbb{E}^\star_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\alpha\xi+\beta)}$.
From the initial condition

$$q_0 = {}^{RL} I^{1-\gamma^*}q(0) = \frac{1}{\gamma^*(1-\gamma^*)}\int_0^t (t-s)^{-\gamma^*}q(s)ds. \qquad (4)$$

Thus, we substitute this into the mild solution Eq.(3)

$$q(t) = t^{\gamma^*-1}\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)\Big(\frac{1}{\gamma^*(1-\gamma^*)}\int_0^t (t-s)^{-\gamma^*}q(s)ds\Big) + \int_0^t (t-s)^{\alpha-1}\mathbb{E}^\star\alpha, \alpha(\mathbb{A}(t-s)^\alpha)Bu(s)ds.$$

To obtain a solution purely in terms of $q(0)$, we approximate $q(s)$ in the integral its Taylor series expansion (Ref. [33], Section. 1, Page No. 1) around $s = 0$,

$$q(s) = q(0) + \mathcal{O}(s^\alpha).$$

Substituting this into the integral

$$q_0 \approx \frac{q(0)}{\gamma^*(1-\gamma^*)}\int_0^t (t-s)^{-\gamma^*}ds.$$

Evaluate the Beta function integral, we get

$$q_0 = \frac{q(0)}{\gamma^*(1-\gamma^*)}\frac{t^{1-\gamma^*}}{1-\gamma^*}.$$

Rearranging for $q(0)$,

$$q(0) = q_0\frac{\gamma^*(1-\gamma^*)(1-\gamma^*)}{t^{1-\gamma^*}}.$$

Then, we have

$$q_0 = \frac{q(0)t^{1-\gamma^*}}{\gamma^*(2-\gamma^*)}.$$

Substituting into the mild solution Eq.(3), we get

$$q(t) = t^{\gamma^*-1}\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)\frac{q(0)t^{1-\gamma^*}}{\gamma^*(2-\gamma^*)} + \int_0^t (t-s)^{\alpha-1}\mathbb{E}^\star\alpha, \alpha(\mathbb{A}(t-s)^\alpha)Bu(s)ds$$

This simplifies

$$q(t) = \frac{\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)}q(0) + \int_0^t (t-s)^{\alpha-1}\mathbb{E}^\star\alpha, \alpha(\mathbb{A}(t-s)^\alpha)Bu(s)ds. \qquad (5)$$

2. Fundamental concepts

We denote the quaternion $q = q^0 + q^1 i + q^2 j + q^{12} k \in \mathbb{H}$, where $q^0, q^1, q^2, q^{12} \in \mathbb{R}$ and i, j, k are imaginary units satisfy the multiplication table formed by

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \\ ki = -ik = j. \end{cases}$$

If $q = q^0 + q^1 i + q^2 j + q^{12} k$, then its conjugate is $\bar{q} = q^0 - q^1 i - q^2 j - q^{12} k$ and norm $|q| = \sqrt{\bar{q}q} = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.
For any $p, h \in \mathbb{H}$, we have $\overline{qh} = \bar{h}\bar{q}$. The norm of a quaternion is:

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

and a *unit quaternion* satisfies $\|q\| = 1$. [Rotating a Vector Using a Quaternion] Let a 3D vector $\vec{v} = (1, 0, 0)$ be rotated by $\theta = 90^\circ$ around the z-axis. The axis of rotation is $\vec{u} = (0, 0, 1)$, a unit vector. The unit quaternion representing this rotation is:

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(0i + 0j + 1k) = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)k = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k$$

The vector $\vec{v}$ is treated as a pure quaternion: $v = 0 + 1i + 0j + 0k = i$. Compute the rotated vector using $v' = qvq^{-1}$ The inverse of a unit quaternion $q = a + \vec{v}$ is $q^{-1} = a - \vec{v}$ (conjugate). A calculation implies

$$q = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k, \quad q^{-1} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k, \quad v = i$$

Using quaternion multiplication:

$$v' = q \cdot i \cdot q^{-1} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right) i \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k\right)$$

After simplification (using $ki = -j$, $k^2 = -1$), we get:

$$v' = j$$

Thus, the rotated vector is $\vec{v}' = (0, 1, 0)$, which confirms a 90-degree rotation about the z-axis. Fig.1 illustrates the rotation of a 3D vector using quaternion algebra. The original vector (blue) points along the x-axis. After applying a quaternion-based rotation by 90° about the z-axis, the resulting vector (red) correctly aligns with the positive y-axis. This demonstrates one of the key advantages of quaternions: they provide a compact and robust representation of 3D rotations without the risk of gimbal lock, unlike Euler angles. The rotation is smooth, non-singular, and easily interpolated for animation or control systems.

Consider $a, b \in \mathbb{H}^n$,

$$a = (a_1, a_2, \dots, a_n)^T, \quad b = (b_1, b_2, \dots, b_n)^T, \quad a_i,\, b_i \in \mathbb{H},$$

where a^T is the transpose of a. Operations are given as

$$\begin{aligned} a + b &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)^T, \\ a\kappa &= (a_1\kappa, a_2\kappa, \dots, a_n\kappa)^T \neq (\kappa a_1, \kappa a_2, \dots, \kappa a_n)^T = \kappa a, \quad \kappa \in \mathbb{H}, \\ a\mu &= (a_1\mu, a_2\mu, \dots, a_n\mu)^T \neq (\mu a_1, \mu a_2, \dots, \mu a_n)^T = \mu a, \quad \mu \in \mathbb{R}, \end{aligned}$$

$\mathcal{B}(J,\mathbb{H})$ represents the Banach space of all continuous functions (CFs) from time interval J into $\mathbb{R}$ with the norm

$$||q_k^0(t)|| = \sup_{t\in J}|q_k^0(t)|, \quad ||q_k^1(t)|| = \sup_{t\in J}|q_k^1(t)|,$$
$$||q_k^2(t)|| = \sup_{t\in J}|q_k^2(t)|, \quad ||q_k^{12}(t)|| = \sup_{t\in J}|q_k^{12}(t)|,$$

where $q_k^0(t)$, $q_k^1(t)$, $q_k^2(t)$, $q_k^{12}(t)$ is the k^{th} component of $q^0(t), q^1(t), q^2(t), q^{12}(t) \in \mathbb{R}^n$ respectively.

The Banach space of all CFs from interval J into $\mathbb{R}^n$ with norm is indicated by $\mathcal{B}(J,\mathbb{R}^n)$ (Ref. [45], Section. 3, Page No. 599):

$$||q^0(t)||_{J\mathbb{R}} = \Big\{\sum_{k=1}^{n}||q_k^0(t)||^2\Big\}^{\frac{1}{2}},\ ||q^1(t)||_{J\mathbb{R}} = \Big\{\sum_{k=1}^{n}||q_k^1(t)||^2\Big\}^{\frac{1}{2}},$$
$$||q^2(t)||_{J\mathbb{R}} = \Big\{\sum_{k=1}^{n}||q_k^2(t)||^2\Big\}^{\frac{1}{2}},\ ||q^{12}(t)||_{J\mathbb{R}} = \Big\{\sum_{k=1}^{n}||q_k^{12}(t)||^2\Big\}^{\frac{1}{2}}.$$

The space of all QV functions defined by $\mathcal{B}(J,\mathbb{H}^n)$ is indicated by

$$\mathcal{B}(J,\mathbb{H}^n) = \{q(t)|q_k(t) = q_k^0(t) + q_k^1(t)i + q_k^2(t)j + q_k^{12}(t)k :$$
$$q_k^0(t), q_k^1(t), q_k^2(t), q_k^{12}(t) \in \mathcal{B}(J,\mathbb{R})\},$$

for $k = 1, 2, \ldots, n$, $q_k(t)$ denotes the k^{th} component of $q(t)$. For more details on QV space, (See. [29], Section. 2, Page No. 4-6).

We apply some of the weighted space concepts of the CFs from the articles for solving our problem (Ref. [35], Section. 2, Page No. 851) and (Ref [6]. Section. 2, Page No.2).

Let $\mathcal{B}(J,\mathbb{H}^n)$ denote the Banach space of vector-valued CFs from the interval J to $\mathbb{H}^n$, equipped with the ∞-norm $\|q\|_\infty = \sup_{t\in J}\|q(t)\|$ or the weighted norm $\|q\|_\Lambda = \sup_{t\in J} e^{-\Lambda t}\|q(t)\|$.

We present the set

$$\mathcal{B}_{1-\gamma^*}(J,\mathbb{H}) = \{q \in \mathcal{B}(J,\mathbb{H}) : t^{1-\gamma^*}q(t) \in \mathcal{B}(J,\mathbb{H})\},$$

and define $(1-\gamma^*,\Lambda)$- weighted norm $||q||_{1-\gamma^*,\Lambda} = \sup_{t\in J} t^{1-\gamma^*}e^{-\Lambda t}||q(t)||$, $\gamma^* = \alpha + \beta - \alpha\beta$, $\alpha \in (0,1)$ and $\beta \in [0,1]$. Here, $(\mathcal{B}_{1-\gamma^*}(J,\mathbb{H}), ||\cdot||_{1-\gamma^*,\Lambda})$ is also being a Banach space, one can also refer to (Ref. [9], Section. 2, Page No. 2).

Several basic ideas are used to establish our main result: the R-L fractional integral and derivative (Ref. [26], Section. 2.3, Pge No. 62); the Caputo fractional derivative (Ref. [26], Eq. 2.142, Page No. 80); the HFD (Ref. [35], Definition. 2.3, Remark. 2.4, Page No. 851); the Mittag-Leffler functions (Ref. [27], Definition. 6, Lemma. 3, Page No. 48). A detailed discussion is omitted here, as it has already been comprehensively addressed in the literature. Readers are recommended to refer to the cited research publications for further deeper details.

Remark 1. *(Ref.* [36]*, Lemma. 2, Page No. 1862) and (Ref.* [37]*, Lemma. 2.7, Page No. 234) Let* $0 < \alpha < 2$ *and* $\beta > 0$ *be arbitrary. The functions* $\mathbb{E}^\star\alpha(\cdot)$, $\mathbb{E}^\star\alpha,\alpha(\cdot)$ *and* $\mathbb{E}^\star\alpha,\beta(\cdot)$ *are non-negative, and for all* $z < 0$*,*

$$\mathbb{E}^\star\alpha(z) := \mathbb{E}^\star\alpha, 1(z) \le 1, \quad \mathbb{E}^\star\alpha,\alpha(z) \le \frac{1}{\gamma^*(\alpha)}, \quad \mathbb{E}^\star\alpha,\beta(z) \le \frac{1}{\gamma^*(\beta)}.$$

3. Convergence of ILC

This section examines the Hilfer-type fractional-order QV system

$$\begin{cases} D^{\alpha,\beta} q_n(t) = \mathbb{A} q_n(t) + B u_n(t), \quad t \in J := [0,b], n = 0,1,2\ldots, \\ r_n(t) = C q_n(t), \end{cases} \tag{6}$$

for equation (6), the following P-type open and closed-loop ILC law is implemented:

$$u_{n+1}(t) = u_n(t) + L_1 e_n(t) + L_2 e_{n+1}(t), \tag{7}$$

where $e_n = r_d(t) - r_k(n)$, $r_d(t)$ are the provided functions, and L_1, L_2 are the parameters that will be specified. The starting point for every iterative learning process is

$$q_{n+1}(0) = q_n(0) + B L_1 e_n(t). \tag{8}$$

We enumerate the following assumptions:
Assumption (I): $1 - \Lambda^{-\alpha}||C||\,||BL_2|| < 0$,
Assumption (II): $\frac{I - \frac{C\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)} + \Lambda^{-\alpha}||C||\,||L_1 B||}{1 - \Lambda^{-\alpha}||C||\,||BL_2||} < 1$.
Assumption (III): $||w_n||_{1-\gamma^*,\Lambda} \le \epsilon_1, \quad ||v_n||_{1-\gamma^*,\Lambda} \le \epsilon_2$ for some positive constants ϵ_1, ϵ_2.
Assumption (IV):

$$\rho_1 = \left|\left| I + \frac{C\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)BL_2}{\gamma^*(2-\gamma^*)} \right|\right| - \Lambda^{-\alpha}||C||\,||BL_2|| > 0,$$
$$\rho_2 = \left|\left| I - \frac{C\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)} \right|\right| + \Lambda^{-\alpha}||C||\,||BL_1||.$$

Theorem 3.1. *If the initial state of each iterative learning satisfies Eq.* (8), $\lim_{n\to\infty} ||e_n||_\Lambda = 0, t \in J$, *then let* $r_n(\cdot)$ *be the output of Eq.* (6) *using the open and closed-loop* P*-type ILC law* (6).

Proof. To keep it concise, we set $\mathtt{P}_\alpha(t,s) := (t-s)^{\alpha-1}$.

$$\begin{aligned} q_{n+1}(t) &= \frac{\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)} q_{n+1}(0) + \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star \alpha, \alpha(\mathbb{A}(t-s)^\alpha) B u_{k+1}(s) ds \\ &= \frac{\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)} [q_n(0) + BL_1 e_n(t)] \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star \alpha, \alpha(\mathbb{A}(t-s)^\alpha) B[u_n(s) + L_1 e_n(s) + L_2 e_{n+1}(s)] ds \\ &= q_n(t) + \frac{\mathbb{E}^\star \alpha, \gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)} BL_1 e_n(t) + \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star \alpha, \alpha(\mathbb{A}(t-s)^\alpha) BL_1 e_n(s) ds \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star \alpha, \alpha(\mathbb{A}(t-s)^\alpha) BL_2 e_{n+1}(s) ds. \end{aligned}$$

Thus, the iteration error $(n+1)^{th}$ is

$$\begin{aligned}
e_{n+1}(t) &= r_d(t) - Cq_{n+1}(t)\\
&= r_d(t) - C\Big(q_n(t) + \frac{\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1e_n(t) + \int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&+ \int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds\Big)\\
&= e_n(t) - C\Big(\frac{\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1e_n(t) + \int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&+ \int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds\Big)\\
&= \Big(I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1\Big)e_n(t) - C\int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&- C\int_0^t \mathtt{P}_{\alpha}(t,s)\mathbb{E}^{\star}\alpha, \alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds, \qquad (9)
\end{aligned}$$

take the norm of Eq.(9)

$$\begin{aligned}
||e_{n+1}(t)|| \leq & \Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n(t)||\\
&+ \frac{||C||}{\gamma^*(\alpha)}\int_0^t \mathtt{P}_{\alpha}(t,s)||BL_1||\,||e_n(s)||ds\\
&+ \frac{||C||}{\gamma^*(\alpha)}\int_0^t \mathtt{P}_{\alpha}(t,s)||BL_2||\,||e_{n+1}(s)||ds.
\end{aligned}$$

Then,

$$\begin{aligned}
t^{1-\gamma^*}||e_{n+1}(t)|| \leq & t^{1-\gamma^*}\Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n(t)||\\
&+ \frac{t^{1-\gamma^*}||C||}{\gamma^*(\alpha)}\int_0^t \mathtt{P}_{\alpha}(t,s)||BL_1||\,||e_n(s)||ds\\
&+ \frac{t^{1-\gamma^*}||C||}{\gamma^*(\alpha)}\int_0^t \mathtt{P}_{\alpha}(t,s)||BL_2||\,||e_{n+1}(s)||ds. \qquad (10)
\end{aligned}$$

Take the $(1-\gamma^*, \Lambda)$ norm of Eq.(10)

$$
\begin{aligned}
||e_{n+1}||_{1-\gamma^*,\Lambda} \leq & \Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n||_{1-\gamma^*,\Lambda} \\
& + \sup_{t\in J} e^{-\Lambda t} \frac{||CBL_1||}{\gamma^*(\alpha)} \int_0^t \mathtt{P}_\alpha(t,s) s^{1-\gamma^*} \, ||e_n(s)||ds \\
& + \sup_{t\in J} e^{-\Lambda t} \frac{||CBL_2||}{\gamma^*(\alpha)} \int_0^t \mathtt{P}_\alpha(t,s) s^{1-\gamma^*} \, ||e_{n+1}(s)||ds \\
\leq & \Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n||_{1-\gamma^*,\Lambda} \\
& + \sup_{t\in J} e^{-\Lambda t} \frac{||CBL_1||}{\gamma^*(\alpha)} \int_0^t \mathtt{P}_\alpha(t,s) e^{\Lambda s} ds \, ||e_n||_{1-\gamma^*,\Lambda} \\
& + \sup_{t\in J} e^{-\Lambda t} \frac{||CBL_2||}{\gamma^*(\alpha)} \int_0^t \mathtt{P}_\alpha(t,s) e^{\Lambda s} ds \, ||e_{n+1}||_{1-\gamma^*,\Lambda}. \qquad (11)
\end{aligned}
$$

Keep in mind that the fact,

$$
\begin{aligned}
\int_0^t \mathtt{P}_\alpha(t,s) e^{\Lambda s} ds &= \int_0^t w^{\alpha-1} e^{\Lambda(1-w)} dw = e^{\Lambda t} \int_0^t w^{\alpha-1} e^{-\Lambda w} dw \\
&= \frac{e^{\Lambda t}}{\Lambda^\alpha} \int_0^t \nu^{\alpha-1} e^{-\nu} d\nu \leq \frac{e^{\Lambda t}}{\Lambda^\alpha} \gamma^*(\alpha),
\end{aligned}
$$

next inequality (11), become

$$
\begin{aligned}
||e_{n+1}||_{1-\gamma^*,\Lambda} \leq & ||I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}|| \, ||e_n||_{1-\gamma^*,\Lambda} \\
& + \frac{||C|| \, ||BL_1||}{\Lambda^\alpha} ||e_n||_{1-\gamma^*,\Lambda} + \frac{||C|| \, ||BL_2||}{\Lambda^\alpha} \, ||e_{n+1}||_{1-\gamma^*,\Lambda},
\end{aligned}
$$

which implies

$$
\begin{aligned}
||e_{n+1}||_{1-\gamma^*,\Lambda} - \frac{||C|| \, ||BL_2||}{\Lambda^\alpha} ||e_{n+1}||_{1-\gamma^*,\Lambda} \leq & \Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| ||\mathbb{E}^\star n||_{1-\gamma^*,\Lambda} \\
& + \frac{||C|| \, ||BL_1||}{\Lambda^\alpha} \, ||e_n||_{1-\gamma^*,\Lambda} \\
\Big(1 - \Lambda^{-\alpha}||C|| \, ||BL_2||\Big) ||e_{n+1}||_{1-\gamma^*,\Lambda} \leq & \Big(\Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \\
& + \Lambda^{-\alpha}||C|| \, ||L_1 B||\Big) ||e_n||_{1-\gamma^*,\Lambda}.
\end{aligned}
$$

If $1 - \Lambda^{-\alpha}||C|| \, ||BL_2|| > 0$,

$$
||e_{n+1}||_{1-\gamma^*,\Lambda} \leq \Big(\frac{\Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| + \Lambda^{-\alpha}||C|| \, ||L_1 B||\Big)}{1 - \Lambda^{-\alpha}||C|| \, ||BL_2||} ||e_n||_{1-\gamma^*,\Lambda}, \qquad (12)
$$

let $\frac{\Big(\Big|\Big|I - \frac{C\mathbb{E}^\star\alpha, \gamma^*(\mathbb{A}t^\alpha)BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| + \Lambda^{-\alpha}||C|| \, ||L_1 B||\Big)}{1 - \Lambda^{-\alpha}||C|| \, ||BL_2||} < 1$, inequality (12) is contraction mapping, and it follows from the contraction mapping that $\lim_{n\to\infty} ||e_n||_{1-\gamma^*,\Lambda} = 0, t \in J$. This completes the proof. □

Next, we analyzes the P-type ILC for QV system with random disturbance via HFD

$$\begin{cases} D^{\alpha,\beta}q_n(t) = \mathbb{A}q_n(t) + Bu_n(t) + w_n(t), \quad t \in J := [0,b], n = 0,1,2,\dots, \\ r_n(t) = Cq_n(t) + v_n(t), \end{cases} \tag{13}$$

where $w_n(t), v_n(t)$ are random disturbance.

For Eq.(13), We opt for the following open and closed loop P-type ILC law:

$$u_{n+1}(t) = u_n(t) + L_1 e_n(t) + L_2 e_{n+1}(t) \tag{14}$$

where L_1, L_2 are the parameters which will be derived, and $e_n = r_d(t) - r_n(t)$, r_d are the values that are provided functions.

Assume that the starting condition for each iteration of learning is

$$q_{n+1}(0) = q_n(0) + BL_1 e_n(t) + BL_2 e_{n+1}(t), \tag{15}$$

where the parameters that will be defined are L_1, L_2.

Theorem 3.2. *Assume that (III) and (IV) are valid. Let $r_n(\cdot)$ be the result of Eq.(3). Assuming that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, $\rho_1 > \rho_2$, the open and closed-loop P-type ILC Eq.(14) guarantees that $\lim_{n\to\infty} ||e_n||_{1-\gamma^*,\Lambda} = 0, t \in J$.*

Proof. According to Eq.(3) and assumptions (II), (III), we know

$$\begin{aligned} q_{n+1}(t) &= \frac{\mathbb{E}^\star\alpha,\gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)} q_{n+1}(0) + \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)(Bu_{n+1}(s) + w_{n+1}(s))ds \\ &= \frac{\mathbb{E}^\star\alpha,\gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)}\big(q_n(0) + BL_1e_n(t) + BL_2e_{n+1}(t)\big) \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)(Bu_n(s) + L_1e_n(s) + L_2e_{n+1}(s))ds \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)w_{n+1}(s)ds \\ &= q_n(t) + \frac{\mathbb{E}^\star\alpha,\gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)}BL_1e_n(t) + \frac{\mathbb{E}^\star\alpha,\gamma^*(\mathbb{A}t^\alpha)}{\gamma^*(2-\gamma^*)}BL_2e_{n+1}(t) \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)BL_1e_n(s)ds \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)BL_2e_{n+1}(s)ds \\ &+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^\star\alpha,\alpha(\mathbb{A}(t-s)^\alpha)w_{n+1}(s)ds. \end{aligned}$$

The $(n+1)^{th}$ iterative error is

$$\begin{aligned}
e_{n+1}(t) &= r_d(t) - Cq_{n+1}(t) - v_{n+1}(t)\\
&= r_d(t) - C\Big(q_n(t) + \frac{\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1e_n(t) + \frac{\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_2e_{n+1}(t)\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})w_{n+1}(s)ds\Big) - v_{n+1}(t)\\
&= e_n(t) - C\Big(\frac{\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1e_n(t) + \frac{\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_2e_{n+1}(t)\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds\\
&+ \int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})w_{n+1}(s)ds\Big) - v_{n+1}(t)\\
&= \Big(I - \frac{C\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big)e_n(t) - C\frac{\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})}{\gamma^*(2-\gamma^*)}BL_1e_{n+1}(t)\\
&- C\int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_1e_n(s)ds\\
&- C\int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})BL_2e_{n+1}(s)ds\\
&- C\int_0^t \mathtt{P}_\alpha(t,s)\mathbb{E}^{\star}\alpha,\alpha(\mathbb{A}(t-s)^{\alpha})w_{n+1}(s)ds - v_{n+1}(t). \qquad (16)
\end{aligned}$$

By taking the norm of Eq. (16), we can easily obtain

$$\begin{aligned}
\Big|\Big|I + \frac{C\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})BL_2}{\gamma^*(2-\gamma^*)}\Big|\Big|||e_{n+1}(t)|| &\leq \Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big|||e_n(t)||\\
&+ \frac{||C||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)||BL_1||\,||e_n(s)||ds\\
&+ \frac{||C||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)||BL_2||\,||e_{n+1}(s)||ds\\
&+ \frac{||C||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)||w_{n+1}(s)||ds + ||v_{n+1}(t)||. \qquad (17)
\end{aligned}$$

Using the $(1-\gamma^*, \Lambda)$-norm, we get

$$\begin{aligned}
&\Big|\Big|I + \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_2}{\gamma^*(2-\gamma^*)}\Big|\Big| ||e_{n+1}||_{1-\gamma^*,\Lambda} \\
&\leq \Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n||_{1-\gamma^*,\Lambda} \\
&+ \sup_{t\in J} e^{-\Lambda t}\frac{||C||||BL_1||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)e^{\Lambda s}ds\, ||e_n||_{1-\gamma^*,\Lambda} \\
&+ \sup_{t\in J} e^{-\Lambda t}\frac{||C||||BL_2||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)e^{\Lambda s}ds\, ||e_{n+1}||_{1-\gamma^*,\Lambda} \\
&+ \sup_{t\in J} e^{-\Lambda t}\frac{||C||}{\gamma^*(\alpha)}\int_0^t P_\alpha(t,s)e^{\Lambda s}ds\, ||w_{n+1}||_{1-\gamma^*,\Lambda} \\
&+ \sup_{t\in J} e^{-\Lambda t}||v_{n+1}(t)||t^{1-\gamma^*} \\
&\leq \Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| \, ||e_n||_{1-\gamma^*,\Lambda} \\
&+ \frac{||C||||BL_1||}{\Lambda^\alpha}||e_n||_{1-\gamma^*,\Lambda} + \frac{||C||||BL_2||}{\Lambda^\alpha}||e_{n+1}||_{1-\gamma^*,\Lambda} \\
&+ \frac{||C||}{\Lambda^\alpha}\, ||w_{n+1}||_{1-\gamma^*,\Lambda} + ||v_{n+1}||_{1-\gamma^*,\Lambda},
\end{aligned}$$

This simplifies

$$\begin{aligned}
&\Big(I + \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_2}{\gamma^*(2-\gamma^*)} - \Lambda^{-\alpha}||C||\,||BL_2||\Big)||e_{n+1}||_{1-\gamma^*,\Lambda} \\
&\leq \Big(\Big|\Big|I - \frac{C\mathbb{E}^{\star}\alpha, \gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)}\Big|\Big| + ||C||\,||BL_1||\Big)\Lambda^{-\alpha}||e_n||_{1-\gamma^*,\Lambda} \\
&+ \Lambda^{-\alpha}||C||\,||w_{n+1}||_{1-\gamma^*,\Lambda} + ||v_{n+1}||_{1-\gamma^*,\Lambda}.
\end{aligned}$$

According to assumptions (III) and (IV), thus $\epsilon = \Lambda^{-\alpha}||C||\epsilon_1 + \epsilon_2$

$$\rho_1||e_{n+1}||_{1-\gamma^*,\Lambda} \leq \rho_2||e_n||_{1-\gamma^*,\Lambda} + \epsilon. \tag{18}$$

This suggests that

$$||e_n||_{1-\gamma^*,\Lambda} \leq \frac{\epsilon}{\rho_1-\rho_2},$$

if $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, then $\lim_{n\to\infty}||e_n||_{1-\gamma^*,\Lambda} \to 0, t \in J$, and the proof is concluded. □

4. Applications in Soft robotic actuators

Following the approach in (Ref.[12], Example 3.6, Page No. 1307 and Ref. [29], Section 6, Page No. 30), we examine the same example in the sense of HFD. Specifically, we analyze the P-type ILC while incorporating the effects of initial state error. A P-type Iterative Learning Control (ILC) law is a simple and widely used ILC strategy that applies a proportional (P) correction to the error from previous iterations.

Key Properties:

- Convergence: If the learning gain is properly chosen, the system will gradually reduce tracking errors across iterations.

- Simplicity: The form of ILC is straightforward to implement.
  Limitations: Convergence speed and robustness depend on the choice of the parameters, and it may struggle with high-frequency noise or modeling errors.

It is widely used in repetitive or batch processes where the same task is performed multiple times. It is particularly beneficial when precise trajectory tracking or error minimization is required. In contrast to conventional rigid robots, soft robotic actuators use malleable, flexible materials (such as silicone, elastomers, and pneumatic/hydraulic systems) to enable safe and agile object contact. However, exact control might be difficult because of their very flexible and nonlinear character. During repeated tasks, P-type ILC helps them become more accurate in their motion.

For the system (13), we set $\alpha = 0.8$, $\beta = 0.5$, and choose $\gamma^* = 0.9$ with $J := [0, 1]$. Consider the following QV parameters:

$$\mathbb{A} = 1 + i + j + k, \qquad B = 2 + 2i - j + k,$$
$$C = 1, \qquad q_0 = 3 + i - 2j + k.$$

The control output is given by $u(t) = (1 - i + j + 2k)e^t$ under the P-type ILC law with an initial state error:

$$\begin{cases} u_{n+1}(t) = u_n(t) + L_1 e_n(t) + L_2 e_{n+1}(t), \\ q_{n+1}(t) = q_n(0) + BL_1 e_n(t), \end{cases} \tag{19}$$

where $L_1 = 1, L_2 = 4$.

The reference trajectory is defined as

$$r_d(t) = \begin{cases} t^{-0.1}\left(\frac{3+i-2j+k}{\gamma^*(0.9)} + \frac{(1+5i+j+k)t^{0.8}}{\gamma^*(1.7)}\right) \\ \frac{1}{\gamma^*(0.8)}(3 - 3j - 2jk + ji + ki - kj + 6k)t^{0.8}e^t. \end{cases}$$

According to (Ref. [29], Example 6.5, Page No. 37-39), we use the fundamental relations $jk = i$ (see Section 2), we rewrite $r_d(t)$ as follows:

- The real part: $\frac{3t^{-0.1}}{\gamma^*(0.9)} + \frac{t^{0.7}}{\gamma^*(1.7)} + \frac{3t^{0.8}e^t}{\gamma^*(0.8)}$.
- The coefficient of i: $\frac{t^{-0.1}}{\gamma^*(0.9)} + \frac{5t^{0.7}}{\gamma^*(1.7)} + \frac{3t^{0.8}e^t}{\gamma^*(0.8)}$.
- The coefficient of j: $\frac{-2t^{-0.1}}{\gamma^*(0.9)} + \frac{t^{0.7}}{\gamma^*(1.7)} - \frac{2t^{0.8}e^t}{\gamma^*(0.8)}$.
- The coefficient of k: $\frac{t^{-0.1}}{\gamma^*(0.9)} + \frac{t^{0.7}}{\gamma^*(1.7)} + \frac{5t^{0.8}e^t}{\gamma^*(0.8)}$.

Using Assumption (I), we compute

$$\|BL_2\| \approx 12.648.$$

Selecting $\Lambda = 2$, we obtain $\Lambda^{-\alpha}\|c\|\|BL_2\| \approx 7.263 > 1$. Thus, Assumption (I) holds.

From Assumption (II), we obtain

$$\Lambda^{-\alpha}\|C\|\|BL_2\| = 7.263,$$

$$\frac{I - \frac{C\mathbb{E}^{\star}\alpha,\gamma^*(\mathbb{A}t^{\alpha})BL_1}{\gamma^*(2-\gamma^*)} + \Lambda^{-\alpha}\|C\|\|L_1 B\|}{1 - \Lambda^{-\alpha}\|C\|\|BL_2\|} = \frac{-2.494}{-6.263} \approx 0.398 < 1.$$

Thus, Assumption (II) is satisfied.

The simulation results, illustrated in Figures 1 to 6, show that in both open-loop and closed-loop P-type ILC, the desired trajectory is gradually tracked as the number of iterations increases. We adopt the correction method from [22], which updates control input as:

$$u(n) = \begin{cases} u(n) - me(n), & \text{if } e(n) > 0, \\ u(n) + me(n), & \text{if } e(n) < 0. \end{cases}$$

By the 50th iteration, the tracking error approaches zero, demonstrating the feasibility and high efficiency of the iterative learning control algorithm.

Remark 2.

- *By tackling issues including nonlinearity, hysteresis, and compliance, P-Type Iterative Learning Control (ILC) significantly enhances the accuracy and versatility of soft robotic actuators. ILC improves actuator control over repetitive motions by learning from past mistakes, which results in more robust, energy-efficient, and trajectory-tracking applications.*
- *For soft robotic systems, which are challenging to simulate with conventional rigid-body dynamics, its model-free nature makes it especially appealing. However, issues like material variability, external disruptions, and time delays necessitate careful adjustment of learning gains and possible integration with AI-based adaptive control methods.*
- *P-type ILC, PID-type extensions, and hybrid learning techniques will continue to be useful tools for applications in grasping, exoskeletons, wearable robots, and bio-inspired locomotion as soft robotics develops.*

5. Conclusion

In this paper, we analysed a P-type ILC law for linear QV-DEs governed by the HFD. Through rigorous mathematical analysis, we established the convergence properties of the proposed ILC scheme under both open-loop and closed-loop frameworks, considering initial state deviations and random disturbances within the $(1-\gamma^*, \Lambda)$-norm concept. Theoretical insights were derived using the properties of the Mittag-Leffler function, ensuring a comprehensive understanding of system behavior. The numerical simulations validated the theoretical findings, demonstrating the effectiveness and robustness of the proposed ILC approach in handling quaternion-valued fractional systems. The results highlight the potential of Hilfer-type fractional-order iterative learning techniques in improving the performance of control systems, particularly in scenarios with uncertainties and disturbances.
Future work could extend the proposed ideas to nonlinear switched quaternion-valued systems, study adaptive learning strategies, and investigate the impact of HFD on system performance. Additionally, incorporating real-world applications such as robotics, signal processing, and control engineering could further validate the practical significance of this study.

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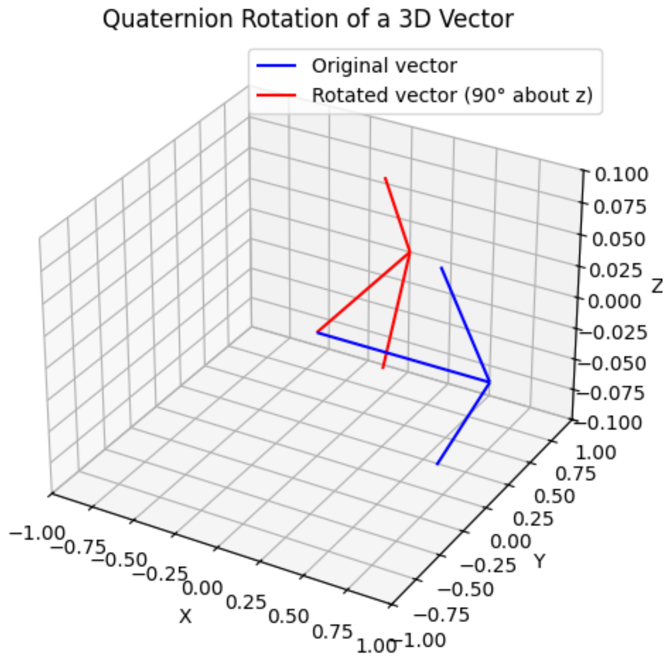

FIGURE 1. Quaternion rotation of vector $\vec{v} = (1, 0, 0)$ by 90° about the z-axis. The blue arrow represents the original vector, and the red arrow shows the rotated vector.

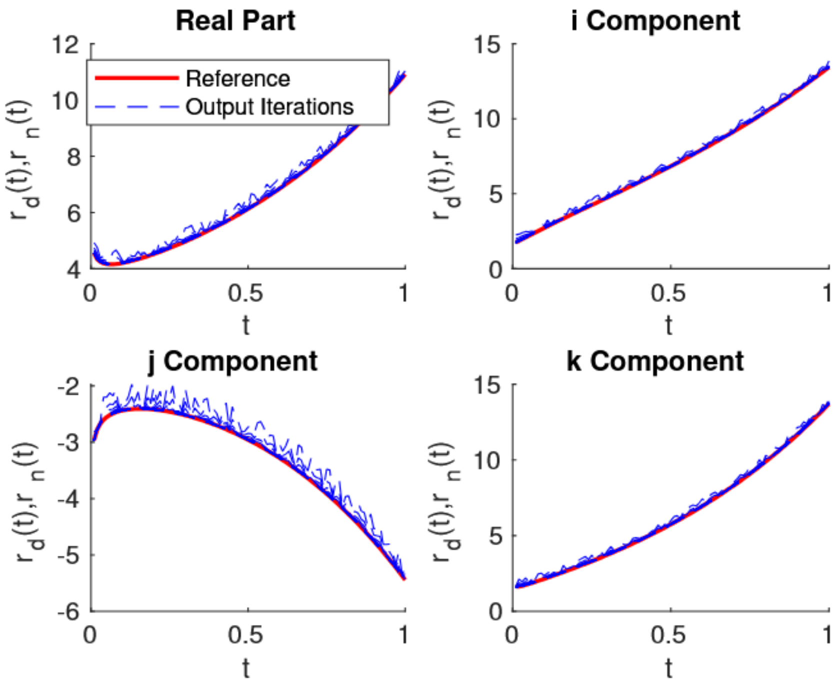

FIGURE 2. System output and reference trajectory for $m = 0.5$

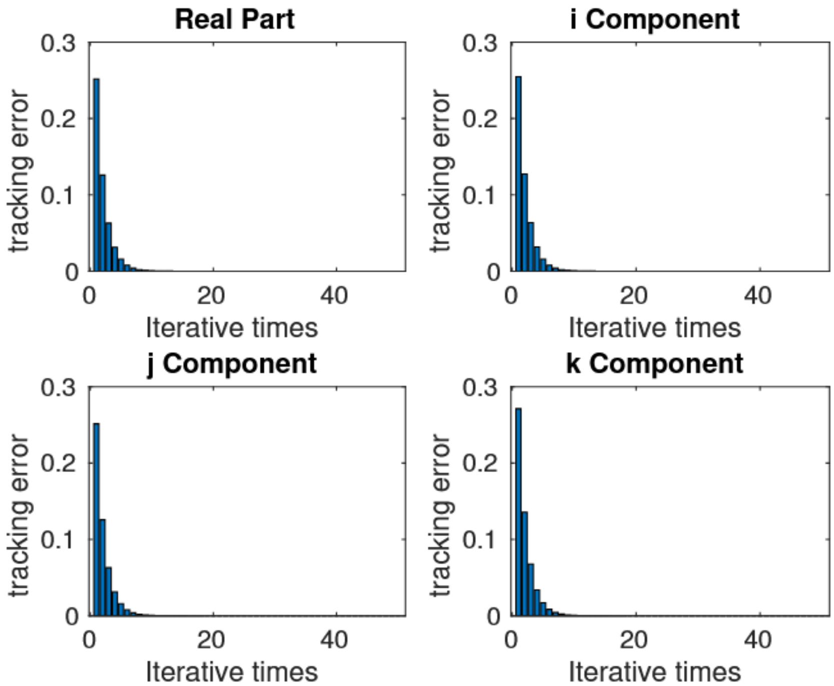

FIGURE 3. Tracking error for $m = 0.5$

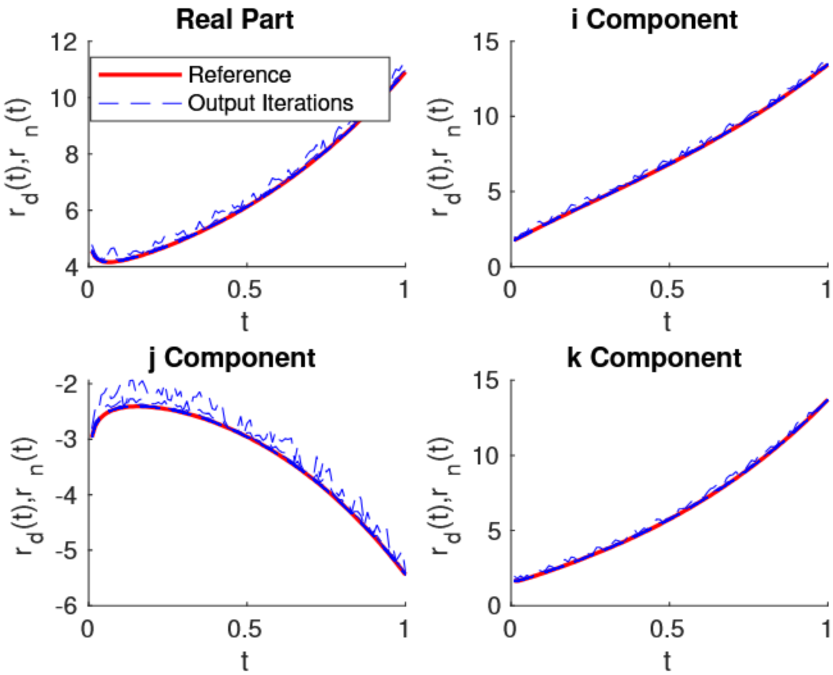

FIGURE 4. System output and reference trajectory for $m = 0.7$

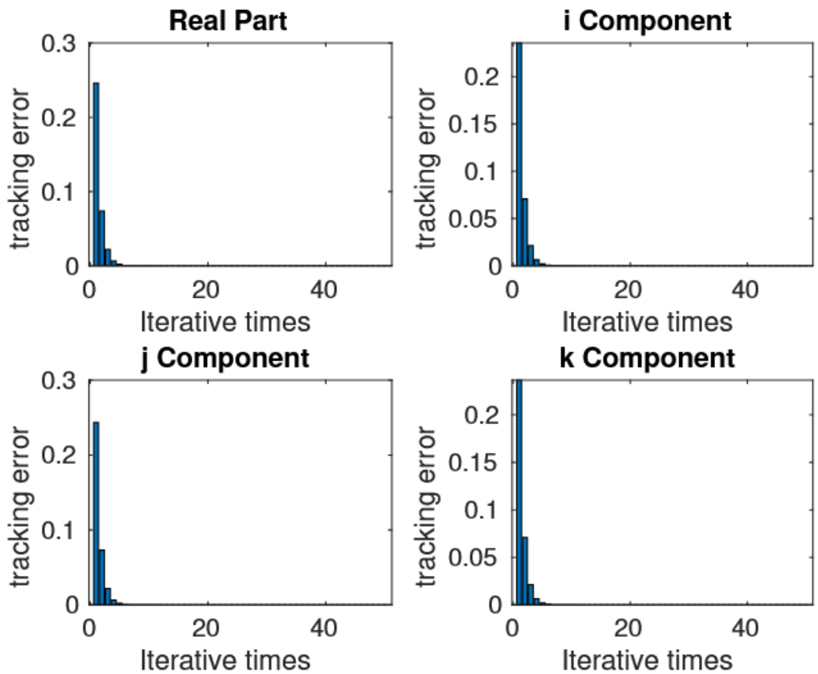

FIGURE 5. Tracking error for $m = 0.7$

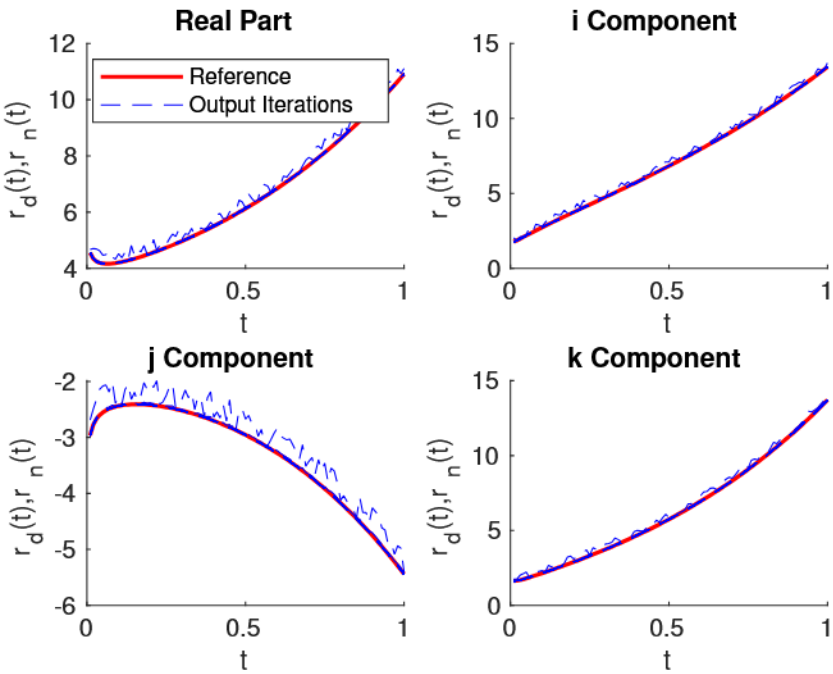

FIGURE 6. System output and reference trajectory for $m = 0.9$

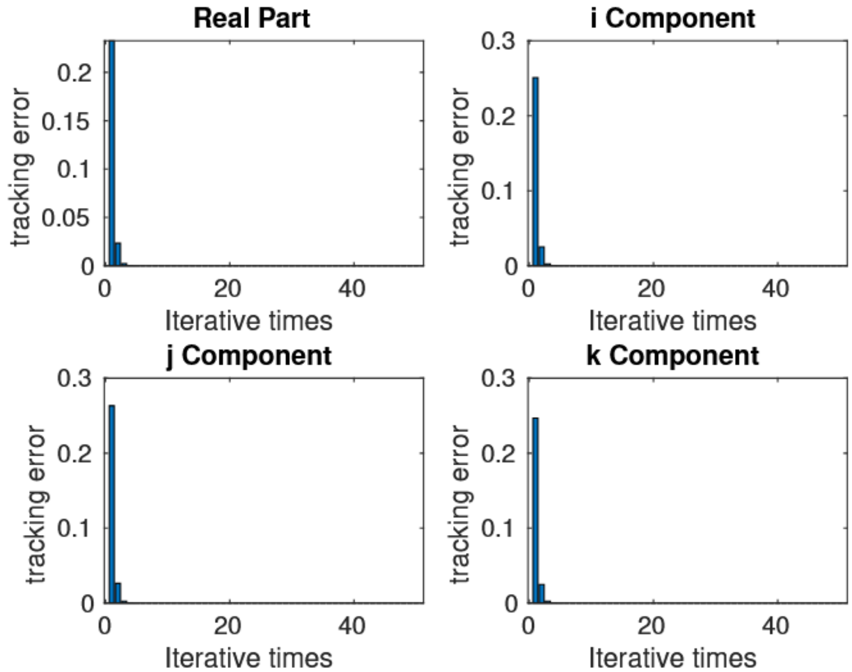

Figure 7. Tracking error for $m = 0.9$