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AVERAGING PRINCIPLE FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY BOTH STANDARD AND FRACTIONAL BROWNIAN MOTIONS

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ABSTRACT. In this paper, we study the stochastic averaging principle for backward stochastic differential equations driven by both standard and fractional Brownian motions (SFrBSDEs in short). An averaged SFrBSDEs for the original SFrBSDEs is proposed, and their solutions are quantitatively compared. Under some appropriate assumptions, the solutions to original systems can be approximated by the solutions to averaged stochastic systems in the sense of mean square.

1. INTRODUCTION

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [11] with Lipschitz assumption under which they proved the celebrated existence and uniqueness result. This pioneer work was extensively used in many fields like stochastic interpretation of solutions of PDEs and financial mathematics. Few years later, several authors investigated BSDEs with respect to fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter H. This process is a self-similar, i.e. B_{at}^H has the same law as $a^H B_t^H$ for any $a > 0$, it has a long range dependence for $H > \frac{1}{2}$. For $H = \frac{1}{2}$ we obtain a standard Wiener process, but for $H \neq \frac{1}{2}$, this process is not a semimartingale. These properties make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Bender [3] gaves one of the earliest result on fractional BSDEs (FrBSDEs in short). The author established an explicit solution of a class of linear FrBSDEs with arbitrary Hurst parameter H. This is done essentially by means of solution of

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a specific linear parabolic PDE. There are two major obstacles depending on the properties of fractional Brownian motion: Firstly, the fractional Brownian motion is not a semimartingale except for the case of Brownian motion ($H = \frac{1}{2}$), hence the classical Itô calculus based on semimartingales cannot be transposed directly to the fractional case. Secondly, there is no martingale representation theorem with respect to the fractional Brownian motion. Studing nonlinear fractional BSDEs, Hu and Peng [7] overcame successfully the second obstacle in the case $H > \frac{1}{2}$ by means of the quasi-conditional expectation. The authors prove existence and uniqueness of the solution but with some restrictive assumptions on the generator. In this same spirit, Maticiuc and Nie [9] interesting in backward stochastic variational inequalities, improved this first result by weakening the required condition on the drift of the stochastic equation. Fei et al [5] introduced the following type of BSDEs driven by both standard and fractional Brownian motions (SFrBSDEs in short)

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds - \int_t^T Z_{1,s}dB_s - \int_t^T Z_{2,s}dB_s^H, \quad 0 \le t \le T, \tag{1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, $\left(B_t^H\right)_{t\geq 0}$ is a fractional Brownian motion and $\{\eta_t\}_{0\leq t\leq T}$ is a solution of a stochastic differential equation driven by both standard and fractional Brownian motions. In [5], the authors abtained the existence and uniqueness of the solution of SFrBSDEs under Lipschitz assumptions. Recently, new classes of BSDEs driven by two mutually independent fractional Brownian motions were introduced by Aidara and Sagna [1]. They established the existence and uniqueness of solutions.

Stochastic averaging principle, which is usually used to approximate dynamical systems under random fluctuations, has long and rich history in multiscale problems (see, e.g.,[10]). Recently, the averaging principle for BSDEs and one-barrier reflected BSDEs, with Lipschitz coefficients, were first studied by Jing and Li [8]. In the present paper, we study a stochastic averaging technique for a class of the SFrBSDEs (1). We present an averaging principle, and prove that the original SFrBSDEs can be approximated by an averaged SFrBSDEs in the sense of mean square convergence and convergence in probability, when a scaling parameter tends to zero.

The rest of the paper is arranged as follows. In Section 2, we recall some definitions and results about fractional stochastic integrals and the related Itô formula. In Section 3, we investigate the averaging principle for the SFrBSDEs under some proper conditions.

2. Fractional Stochastic calculus

Let Ω be a non-empty set, $\mathcal{F}$ a $\sigma-$algebra of sets Ω, $\mathbb{P}$ a probability measure defined on $\mathcal{F}$ and $\{\mathcal{F}_t,\ t \in [0,T]\}$ a $\sigma-$algebra generated by both standard and fractional Brownian motions. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space and $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$.

The fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with the covariance function

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \ge 0.$$

Suppose that the process $\left(B_t^H\right)_{t\geq 0}$ is independent of the standard Brownian motion $(B_t)_{t\geq 0}$. Throughout this paper it is assumed that $H \in (1/2, 1)$ is arbitrary but fixed.

Denote $\rho(t,s) = H(2H-1)|t-s|^{2H-2},\ (t,s) \in \mathbb{R}^2$. Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \rho(u,v)\xi(u)\eta(v)dudv \text{ and } \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0,T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. Let P_T^H be the set of all polynomials of fractional Brownian motion. Namely, P_T^H contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)$$

where f is a polynomial function of n variables. The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)\xi_i(s) \quad 0 \leq s \leq T.$$

Similarly, we can define the Malliavin derivative $D_t G$ of the Brownian functional

$$G(\omega) = f\left(\int_0^T \xi_1(t)dB_t, \int_0^T \xi_2(t)dB_t, \dots, \int_0^T \xi_n(t)dB_t\right).$$

The divergence operator D^H is closable from $L^2(\Omega, F, \mathbb{P})$ to $L^2(\Omega, F, \mathbb{P}, H)$. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of P_T^H with the norm

$$||F||_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}||D_s^H F||_T^2.$$

Now we introduce the Malliavin ρ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \rho(t,s) D_s^H F ds$$

and denote by $\mathbb{L}_H^{1,2}$ the space of all stochastic processes $F : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow H$ such that

$$\mathbb{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt\right) < +\infty.$$

We have the following (see[[6], Proposition 6.25]):

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathcal{H}$ be a stochastic processes such that*

$$\mathbb{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt\right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies

$$\mathbb{E}\left(\int_0^T F_s dB_s^H\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\int_0^T F_s dB_s^H\right)^2 = \mathbb{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt\right).$$

Let us recall the fractional Itô formula (see[[5], Theorem 3.1]).

Theorem 2.2. *Let $\sigma_1 \in L^2([0,T])$ and $\sigma_2 \in \mathcal{H}$ be deterministic continuous functions.*
Assume that $\|\sigma_2\|_t$ is continuously differentiable as a function of $t \in [0,T]$. Denote

$$X_t = X_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma_1(s)dB_s + \int_0^t \sigma_2(s)dB_s^H,$$

where X_0 is a constant, $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)|ds < +\infty$. Let $F(t,x)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)\left[\sigma_1^2(s) + \frac{d}{ds}\|\sigma_2\|_s^2\right]ds, \quad 0 \le t \le T. \end{aligned}$$

Let us finish this section by giving a fractional Itô chain rule (see[[5], Theorem 3.2]).

Theorem 2.3. *Assume that for $i = 1,2$, the processes μ_i, α_i and ϑ_i, satisfy*

$$\mathbb{E}\left[\int_0^T \mu_i^2(s)ds + \int_0^T \alpha_i^2(s)ds + \int_0^T \vartheta_i^2(s)ds\right] < \infty.$$

Suppose that $D_t\alpha_i(s)$ and $\mathbb{D}_t^H\vartheta_i(s)$ are continuously differentiable with respect to $(s,t) \in [0,T]^2$ for almost all $\omega \in \Omega$. Let X_t and Y_t be two processes satisfying

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_1(s)ds + \int_0^t \alpha_1(s)dB_s + \int_0^t \vartheta_1(s)dB_s^H, \qquad 0 \le t \le T, \\ Y_t &= Y_0 + \int_0^t \mu_2(s)ds + \int_0^t \alpha_2(s)dB_s + \int_0^t \vartheta_2(s)dB_s^H, \qquad 0 \le t \le T. \end{aligned}$$

If for $i = 1,2$, the following conditions hold:

$$\mathbb{E}\left[\int_0^T |D_t\alpha_i(s)|^2 dsdt\right] < +\infty, \qquad \mathbb{E}\left[\int_0^T |\mathbb{D}_t^H\vartheta_i(s)|^2 dsdt\right] < +\infty,$$

then

$$\begin{aligned} X_tY_t &= X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s \\ &\quad + \int_0^t \left[\alpha_1(s)D_sY_s + \alpha_2(s)D_sX_s + \vartheta_1(s)\mathbb{D}_s^H Y_s + \vartheta_2(s)\mathbb{D}_s^H X_s\right] ds, \end{aligned}$$

which may be written formally as

$$d\left(X_tY_t\right) = X_t dY_t + Y_t dX_t + \left[\alpha_1(t)D_tY_t + \alpha_2(t)D_tX_t + \vartheta_1(t)\mathbb{D}_t^H Y_t + \vartheta_2(t)\mathbb{D}_t^H X_t\right] dt.$$

In order to present a stochastic averaging principle, we need the following [12, Lemma 1].

Lemma 2.1. *Let B_t^H be a fractional Brownian motion with $\frac{1}{2} < H < 1$, and $u(s)$ be a stochastic process in $\mathbb{L}_H^{1,2}$. For every $T < +\infty$, there exists a constant $C_0(H,T) = HT^{2H-1}$ such that*

$$\mathbb{E}\left[\left(\int_0^T |u(s)|\, dB_s^H\right)^2\right] \leq C_0(H,T)\mathbb{E}\left[\int_0^T |u(s)|^2\, ds\right] + C_0T^2.$$

We are now in position to move on to study our main subject.

3. Averaging Principle for SFrBSDEs

3.1. **SFrBSDEs.** Let us consider the following process

$$\eta_t = \eta_0 + b(t) + \int_0^t \sigma_1(s)dB_s + \int_0^t \sigma_2(s)dB_s^H, \quad 0 \leq t \leq T,$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant,
- $b, \sigma_1, \sigma_2 : [0,T] \to \mathbf{R}$ are deterministic continuous functions, σ_1 and σ_2 are differentiable and $\sigma_1(t) \neq 0,\ \sigma_2(t) \neq 0$ such that

$$|\sigma|_t^2 = \int_0^t \sigma_1^2(s)ds + \|\sigma_2\|_t^2, \quad 0 \leq t \leq T, \tag{2}$$

where $$\|\sigma_2\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma_2(u)\sigma_2(v)dudv.$$

Let $$\widehat{\sigma}_2(t) = \int_0^t \rho(t,v)\sigma_2(v)dv, \quad 0 \leq t \leq T.$$

The next remark will be useful in the sequel.

Remark 1. *The function $|\sigma|_t^2$ defined by eq.(2) is continuously differentiable with respect to t on $[0,T]$, and*

a) $\frac{d}{dt}|\sigma|_t^2 = \sigma_1^2(t) + \frac{d}{dt}\|\sigma_2\|_t^2 = \sigma_1^2(t) + \sigma_2(t)\widehat{\sigma}_2(t) > 0, \quad 0 \leq t \leq T.$

b) *for a suitable constant $C_1 > 0$, $\inf_{0\leq t\leq T} \frac{\widehat{\sigma}_2(t)}{\sigma_2(t)} \geq C_1$.*

Given ξ a measurable real valued random variable and the function

$$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

we consider the BSDEs driven by both standard and fractional Brownian motion (FrBSDEs)

$$Y_t = \xi + \int_t^T f(s,\eta_s,Y_s,Z_{1,s},Z_{2,s})ds - \int_t^T Z_{1,s}dB_s - \int_t^T Z_{2,s}dB_s^H, \quad 0 \leq t \leq T. \tag{3}$$

We introduce the following sets (where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$) :

- $C_{\text{pol}}^{1,2}([0,T] \times \mathbb{R})$ is the space of all $C^{1,2}$-functions over $[0,T] \times \mathbb{R}$, which together with their derivatives are of polynomial growth,
- $V_{[0,T]} = \left\{Y = \psi(\cdot,\eta) :\ \psi \in C_{\text{pol}}^{1,2}([0,T] \times \mathbb{R}), \frac{\partial \psi}{\partial t} \text{ is bounded},\ t \in [0,T]\right\},$

- $\widetilde{V}_{[0,T]}$ the completion of $V_{[0,T]}$ under the following norm
$$\|Y\| = \left(\int_0^T \mathbb{E}|Y_t|^2 dt\right)^{1/2} = \left(\int_0^T \mathbb{E}|\psi(t,\eta_t)|^2 dt\right)^{1/2}.$$

Definition 3.1. *A triplet of processes $(Y_t, Z_{1,t}, Z_{2,t})_{0\leq t\leq T}$ is called a solution to SFrBSDE* (3)*, if $(Y_t, Z_{1,t}, Z_{2,t})_{0\leq t\leq T} \in \widetilde{V}_{[0,T]} \times \widetilde{V}_{[0,T]} \times \widetilde{V}_{[0,T]}$ and satisfies eq.*(3)*.*

We have the following (see [[5], Theorem 5.3]).

Theorem 3.4. *Assume that σ_1 and σ_2 are continuous and $|\sigma|_t^2$ defined by eq.*(2) *is a strictly increasing function of t. Let the SFrBSDE* (3) *has a solution of the form $(Y_t = \psi(t,\eta_t),\ Z_{1,t} = -\varphi_1(t,\eta_t),\ Z_{2,t} = -\varphi_2(t,\eta_t))$, where $\psi \in C^{1,2}([0,T]\times\mathbb{R})$. Then*
$$\varphi_1(t,x) = \sigma_1(t)\psi'_x(t,x), \quad \varphi_2(t,x) = \sigma_2(t)\psi'_x(t,x).$$

The next proposition will be useful in the sequel.

Proposition 1. *Let $(Y_t, Z_{1,t}, Z_{2,t})_{0\leq t\leq T}$ be a solution of the SFrBSDE* (3)*. Then for almost $t \in [0,T]$,*
$$D_t Y_t = Z_{1,t}, \quad \textit{and} \quad \mathbb{D}_t^H Y_t = \frac{\widehat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.$$

Proof. Since $(Y_t, Z_{1,t}, Z_{2,t})$ satisfies the SFrBSDE (3) then we have $Y = \psi(\cdot,\eta)$ where
$\psi \in C^{1,2}([0,T]\times\mathbb{R})$. From Theorem 3.4, we have
$$Z_{1,t} = \sigma_1(t)\psi'_x(t,x), \quad Z_{2,t} = \sigma_2(t)\psi'_x(t,x).$$
Then we can write $D_t Y_t = \sigma_1(t)\psi'_x(t,x) = Z_{1,t}$ and
$$\begin{aligned}\mathbb{D}_t^H Y_t = \int_0^T \phi(t,s) D_s^H \psi(t,\eta_t) ds &= \psi'_x(t,\eta_t)\int_0^T \phi(t,s)\sigma_2(s) ds \\ &= \widehat{\sigma}_2(t)\psi'_x(t,\eta_t) = \frac{\widehat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.\end{aligned}$$
□

3.2. **An averaging principle.** In this section, we are going to investigate the averaging principle for the FrBSDEs under Lipschitz coefficients. Let us consider the standard form of equation (3): for all $t \in [0,T]$
$$Y_t^\varepsilon = \xi + \varepsilon^{2H}\int_t^T f\left(r, \eta_r^\varepsilon, Y_r^\varepsilon, Z_{1,r}^\varepsilon, Z_{2,r}^\varepsilon\right) dr - \varepsilon^H \int_t^T Z_{1,r}^\varepsilon dB_r - \varepsilon^H \int_t^T Z_{2,r}^\varepsilon dB_r^H, \quad (4)$$
where $\eta_t^\varepsilon = \eta_0 + \varepsilon^{2H}\int_0^t b(s)ds + \varepsilon^H \int_0^t \sigma_1(s)dB_s + \varepsilon^H\int_0^t \sigma_2(s)dB_s^H$.

According to the second part, equation (4) also has an adapted unique and square integrable solution. We will examine whether the solution Y_t^ε can be approximated to the solution process $\overline{Y}_t$ of the simplified equation: for all $t \in [0,T]$
$$\overline{Y}_t = \xi + \varepsilon^{2H}\int_t^T \overline{f}\left(\eta_r^\varepsilon, \overline{Y}_r, \overline{Z}_{1,r}, \overline{Z}_{2,r}\right) dr - \varepsilon^H \int_t^T \overline{Z}_{1,r} dB_r - \varepsilon^H \int_t^T \overline{Z}_{2,r} dB_r^H, \quad (5)$$
where $\left(\overline{Y}_t, \overline{Z}_{1,t}, \overline{Z}_{2,t}\right)$ has the same properties as $\left(Y_t^\varepsilon, Z_{1,t}^\varepsilon, Z_{2,t}^\varepsilon\right)$.

We assume that the coefficients f and $\overline{f}$ are continuous functions and satisfy the following assumption:

- **(A1)** There exists $L > 0$ such that, for all $(t, x, y, z_1, z_2, y', z_1', z_2') \in [0, T] \times \mathbb{R}^7$, we have

$$|f(t, x, y, z_1, z_2) - f(t, x, y', z_1', z_2')|^2 \le L\left(|y - y'|^2 + |z_1 - z_1'|^2 + |z_2 - z_2'|^2\right).$$

- **(A2)** For any $t \in [0, T_1] \subset [0, T]$ and for all $(x, y, z_1, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we have

$$\frac{1}{T_1 - t}\int_t^{T_1} \left|f(s, x, y, z_1, z_2) - \overline{f}(x, y, z_1, z_2)\right|^2 ds \le \phi(T_1 - t)\left(1 + |y|^2 + |z_1|^2 + |z_2|^2\right),$$

where ϕ is a bounded function.

In what follows, we establish the result which will be useful in the sequel.

Lemma 3.2. *Suppose that the original SFrBSDEs* (4) *and the averaged SFrBSDEs* (5) *both satisfy the assumptions* (**A1**) *and* (**A2**). *For a given arbitrarily small number* $u \in [0, t] \subset [0, T]$*, there exist* $L_1 > 0$ *and* $C_2 > 0$ *such that*

$$\mathbb{E}\left[\int_u^T \left[\left|Z_{1,s}^\varepsilon - \overline{Z}_{1,s}\right|^2 + \left|Z_{2,s}^\varepsilon - \overline{Z}_{2,s}\right|^2\right] ds\right] \le L_1 \mathbb{E}\left[\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds\right] + C_2\,(T - u). \tag{6}$$

Proof. Let us define $\overline{\Delta\delta}^\varepsilon = \delta^\varepsilon - \overline{\delta}$ for a process $\delta \in \{Y, Z_1, Z_2\}$.

It is easily seen that the pair of processes $\left(\overline{\Delta Y}_t^\varepsilon, \overline{\Delta Z}_{1,t}^\varepsilon, \overline{\Delta Z}_{2,t}^\varepsilon\right)_{0 \le t \le T}$ solves the SFrBSDE

$$\begin{aligned}\overline{\Delta Y}_t^\varepsilon &= \varepsilon^{2H}\textstyle\int_t^T \left(f(s, \eta_s^\varepsilon, Y_s^\varepsilon, Z_{1,s}^\varepsilon, Z_{2,s}^\varepsilon) - \overline{f}(\eta_s^\varepsilon, \overline{Y}_s, \overline{Z}_{1,s}, \overline{Z}_{2,s})\right) ds - \varepsilon^H \textstyle\int_t^T \overline{\Delta Z}_{1,s}^\varepsilon dB_s \\ &\quad - \varepsilon^H \textstyle\int_t^T \overline{\Delta Z}_{2,s}^\varepsilon dB_s^H.\end{aligned}$$

Applying Itô's formula to $\left|\overline{\Delta Y}_t^\varepsilon\right|^2$, we obtain

$$\begin{aligned}&\left|\overline{\Delta Y}_t^\varepsilon\right|^2 + \varepsilon^H \int_u^T D_s \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{1,s}^\varepsilon ds + \varepsilon^H \int_u^T \mathbb{D}_s^H \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{2,s}^\varepsilon ds \\ &\quad = 2\varepsilon^{2H} \int_u^T \overline{\Delta Y}_s^\varepsilon \left(f(s, \eta_s^\varepsilon, Y_s^\varepsilon, Z_{1,s}^\varepsilon, Z_{2,s}^\varepsilon) - \overline{f}(\eta_s^\varepsilon, \overline{Y}_s, \overline{Z}_{1,s}, \overline{Z}_{2,s})\right) ds \\ &\qquad - 2\varepsilon^H \int_u^T \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{1,s}^\varepsilon dB_s - 2\varepsilon^H \int_u^T \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{2,s}^\varepsilon dB_s^H.\end{aligned} \tag{7}$$

Using the fact that $\left(\overline{\Delta Y}_s^\varepsilon, \overline{\Delta Z}_{1,s}^\varepsilon, \overline{\Delta Z}_{2,s}^\varepsilon\right)_{t \le s \le T} \in \widetilde{V}_{[0,T]} \times \widetilde{V}_{[0,T]} \times \widetilde{V}_{[0,T]}$ and $V_{[0,T]} \subset \mathbb{L}_H^{1,2}$ (see Lemma 8 in [9]) which implies in fact $F_{i,s} = \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{i,s}^\varepsilon \in \mathbb{L}_H^{1,2}$, (where $i = 1, 2$). Then by Theorem 2.1, we have

$$\mathbb{E}\left[\int_0^T \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{1,s}^\varepsilon dB_s + \int_0^T \overline{\Delta Y}_s^\varepsilon \overline{\Delta Z}_{2,s}^\varepsilon dB_s^H\right] = 0.$$

Hence we deduce from (7)

$$\begin{aligned}
&\mathbb{E}\left[\left|\overline{\Delta Y}_t^\varepsilon\right|^2\right]+\varepsilon^H\mathbb{E}\left[\int_u^T D_s\overline{\Delta Y}_s^\varepsilon\overline{\Delta Z}_{1,s}^\varepsilon ds\right]+\varepsilon^H\mathbb{E}\left[\int_u^T \mathbb{D}_s^H\overline{\Delta Y}_s^\varepsilon\overline{\Delta Z}_{2,s}^\varepsilon ds\right]\\
&\qquad=2\varepsilon^{2H}\mathbb{E}\left[\int_u^T\overline{\Delta Y}_s^\varepsilon\left(f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_{1,s}^\varepsilon,Z_{2,s}^\varepsilon)-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right)ds\right]\\
&\qquad\leq 2\varepsilon^{2H}\mathbb{E}\left[\int_u^T\overline{\Delta Y}_s^\varepsilon\left(f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_{1,s}^\varepsilon,Z_{2,s}^\varepsilon)-f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right)ds\right]\\
&\qquad\quad+2\varepsilon^{2H}\mathbb{E}\left[\int_u^T\overline{\Delta Y}_s^\varepsilon\left(f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right)ds\right] \qquad (8)\\
&\qquad=E_1+E_2,
\end{aligned}$$

where $E_1=2\varepsilon^{2H}\mathbb{E}\left[\int_u^T\overline{\Delta Y}_s^\varepsilon\left(f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_{1,s}^\varepsilon,Z_{2,s}^\varepsilon)-f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right)ds\right]$
and $E_2=2\varepsilon^{2H}\mathbb{E}\left[\int_u^T\overline{\Delta Y}_s^\varepsilon\left(f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right)ds\right]$.

For E_1, by using the condition (**A1**) and Holder's inequality, for any $\alpha>0$, $2ab\leq\alpha a^2+b^2/\alpha$, we deduce that

$$\begin{aligned}
E_1\leq&\alpha\varepsilon^{2H}\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right]+\frac{\varepsilon^{2H}}{\alpha}\mathbb{E}\left[\int_u^T\left|f(s,\eta_s^\varepsilon,Y_s^\varepsilon,Z_{1,s}^\varepsilon,Z_{2,s}^\varepsilon)-f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2ds\right]\\
\leq&\varepsilon^{2H}\left(\alpha+\frac{L}{\alpha}\right)\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right]+\frac{L\varepsilon^{2H}}{\alpha}\mathbb{E}\left[\int_u^T\left[\left|\overline{\Delta Z}_{1,s}^\varepsilon\right|^2+\left|\overline{\Delta Z}_{2,s}^\varepsilon\right|^2\right]ds\right]. \qquad (9)
\end{aligned}$$

For E_2, by using assumption (**A2**), Holder's inequality and Young's inequality, we have

$$\begin{aligned}
E_2\leq&\, 2\varepsilon^{2H}\mathbb{E}\left[\left(\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right)^{\frac12}\left(\int_t^T\left|f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2ds\right)^{\frac12}\right]\\
\leq&\,2\varepsilon^{2H}\mathbb{E}\left[\left((T-u)\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right)^{\frac12}\left(\frac{1}{T-u}\int_u^T\left|f(s,\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta_s^\varepsilon,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2ds\right)^{\frac12}\right]\\
\leq&\, 2\varepsilon^{2H}C_2\mathbb{E}\left[\left(\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right)^{\frac12}\right]\\
\leq&\, \varepsilon^{2H}C_2\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds+T-u\right]\\
\leq&\, \varepsilon^{2H}C_2\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}_s^\varepsilon\right|^2ds\right]+\varepsilon^{2H}C_2\left(T-u\right), \qquad (10)
\end{aligned}$$

where $C_2 = \sqrt{(T-u)\sup_{u\le s\le T}\phi(s-u)\left[1+\sup_{u\le s\le T}\mathbb{E}(\left|\overline{Y}_s\right|^2)+\sup_{u\le s\le T}\mathbb{E}(\left|\overline{Z}_{1,s}\right|^2)+\sup_{u\le s\le T}\mathbb{E}(\left|\overline{Z}_{2,s}\right|^2)\right]}$.

By the stochastic representation given in Proposition 1 and the Remark 1, we have

$$\mathbb{E}\left[\int_u^T D_s\overline{\Delta Y}^\varepsilon_s\overline{\Delta Z}^\varepsilon_{1,s}ds\right]=\mathbb{E}\left[\int_u^T\left|\overline{\Delta Z}^\varepsilon_{1,s}\right|^2ds\right] \text{ and } \mathbb{E}\left[\int_u^T\mathbb{D}^H_s\overline{\Delta Y}^\varepsilon_s\overline{\Delta Z}^\varepsilon_{2,s}ds\right]\ge C_1\mathbb{E}\left[\int_u^T\left|\overline{\Delta Z}^\varepsilon_{2,s}\right|^2ds\right].$$

Putting pieces together, we deduce from (8) that

$$\begin{aligned}\mathbb{E}\left[\left|\overline{\Delta Y}^\varepsilon_t\right|^2\right]&+\varepsilon^H\mathbb{E}\left[\int_u^T\left|\overline{\Delta Z}^\varepsilon_{1,s}\right|^2ds\right]+C_1\varepsilon^H\mathbb{E}\left[\int_u^T\left|\overline{\Delta Z}^\varepsilon_{2,s}\right|^2ds\right]\\&\le\varepsilon^{2H}\left(\alpha+\frac{L}{\alpha}+C_2\right)\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}^\varepsilon_s\right|^2ds\right]+\varepsilon^{2H}C_2\left(T-u\right)\\&\quad+\frac{L\varepsilon^{2H}}{\alpha}\mathbb{E}\left[\int_u^T\left[\left|\overline{\Delta Z}^\varepsilon_{1,s}\right|^2+\left|\overline{\Delta Z}^\varepsilon_{2,s}\right|^2\right]ds\right].\end{aligned}\tag{11}$$

Hence if we choose $\alpha=\alpha_0$ satisfying $\frac{\varepsilon^H}{\alpha_0}\min\left\{\alpha_0-L\varepsilon^H,\alpha_0C_1-L\varepsilon^H\right\}=\varepsilon^{2H}$, then we obtain

$$\varepsilon^{2H}\mathbb{E}\left[\int_u^T\left[\left|\overline{\Delta Z}^\varepsilon_{1,s}\right|^2+\left|\overline{\Delta Z}^\varepsilon_{2,s}\right|^2\right]ds\right]\le\varepsilon^{2H}\left(\alpha_0+\frac{L}{\alpha_0}+C_2\right)\mathbb{E}\left[\int_u^T\left|\overline{\Delta Y}^\varepsilon_s\right|^2ds\right]+\varepsilon^{2H}C_2\left(T-u\right).$$

Thus,

$$\mathbb{E}\left[\int_u^T\left[\left|Z^\varepsilon_{1,s}-\overline{Z}_{1,s}\right|^2+\left|Z^\varepsilon_{2,s}-\overline{Z}_{2,s}\right|^2\right]ds\right]\le L_1\mathbb{E}\int_u^T\left|Y^\varepsilon_s-\overline{Y}_s\right|^2ds+C_2(T-u),$$

where $L_1=\alpha_0+\frac{L}{\alpha_0}+C_2$. This completes the proof. □

Now, we claim the main theorem showing the relationship between solution processes Y^ε_t to the original (4) and $\overline{Y}_t$ to the averaged (5). It shows that the solution of the averaged (5) converges to that of the original (4) in mean square sense.

Theorem 3.5. *Under the assumption of Lemma 3.2 are satisfied. For a given arbitrarily small number $\delta_1>0$, there exists $\varepsilon_1\in[0,\varepsilon_0]$ and $\beta\in[0,1]$ such that for all $\varepsilon\in[0,\varepsilon_1]$ having*

$$\sup_{T\varepsilon^{1-\beta}\le t\le T}\mathbb{E}\left|Y^\varepsilon_t-\overline{Y}_t\right|^2\le\delta_1.$$

Proof. With the help of Lemma 3.2, now we can prove the Theorem 3.5. Using the elementary inequality and the isometry property, we derive that

$$\begin{aligned}
\mathbb{E}\left[\left|\overline{\Delta Y}^{\varepsilon}_s\right|^2\right] \leq & 2\varepsilon^{4H}\mathbb{E}\left[\left|\int_u^T \left[f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_{1,s},Z^{\varepsilon}_{2,s})-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right]ds\right|^2\right] \\
& + 2\mathbb{E}\left[\left|\varepsilon^H\int_u^T \overline{\Delta Z}^{\varepsilon}_{1,s}dB_s + \varepsilon^H\int_u^T \overline{\Delta Z}^{\varepsilon}_{2,s}dB^H_s\right|^2\right] \\
& \leq 4\varepsilon^{4H}\mathbb{E}\left[\left|\int_u^T \left[f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_{1,s},Z^{\varepsilon}_{2,s})-f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right]ds\right|^2\right] \\
& + 4\varepsilon^{4H}\mathbb{E}\left[\left|\int_u^T \left[f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right]ds\right|^2\right] \\
& + 4\varepsilon^{4H}\mathbb{E}\left[\left|\int_u^T \overline{\Delta Z}^{\varepsilon}_{1,s}dB_s\right|^2\right] + 4\varepsilon^{4H}\mathbb{E}\left[\left|\int_u^T \overline{\Delta Z}^{\varepsilon}_{2,s}dB^H_s\right|^2\right] \\
& = I_1 + I_2 + I_3 + I_4. \qquad (12)
\end{aligned}$$

Applying Holder's inequality and the assumption (**A1**), we obtain

$$\begin{aligned}
I_1 & \leq 4(T-u)\varepsilon^{4H}\mathbb{E}\left[\int_u^T \left|f(s,\eta^{\varepsilon}_s,Y^{\varepsilon}_s,Z^{\varepsilon}_{1,s},Z^{\varepsilon}_{2,s})-f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2 ds\right] \\
& \leq 4(T-u)L\varepsilon^{4H}\mathbb{E}\left[\int_u^T \left[\left|\overline{\Delta Y}^{\varepsilon}_s\right|^2 + \left|\overline{\Delta Z}^{\varepsilon}_{1,s}\right|^2 + \left|\overline{\Delta Z}^{\varepsilon}_{2,s}\right|^2\right]ds\right]. \qquad (13)
\end{aligned}$$

Then, together with Holder's inequality and the assumption (**A2**), we get

$$\begin{aligned}
I_2 \leq & 4(T-u)\varepsilon^{4H}\mathbb{E}\left[\int_u^T \left|f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2 ds\right] \\
\leq & 4(T-u)^2\varepsilon^{4H}\mathbb{E}\left[\frac{1}{T-u}\int_u^T \left|f(s,\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})-\overline{f}(\eta^{\varepsilon}_s,\overline{Y}_s,\overline{Z}_{1,s},\overline{Z}_{2,s})\right|^2 ds\right] \\
\leq & C_3(T-u)^2\varepsilon^{4H}, \qquad (14)
\end{aligned}$$

where $C_3 = 4\sup_{u\leq s\leq T}[\phi(s-u)]\left(1+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Y}_s\right|^2\right)+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Z}_{1,s}\right|^2\right)+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Z}_{2,s}\right|^2\right)\right)$.

By the Lemma 2.1, we obtain

$$I_3 + I_4 \leq 2\varepsilon^{2H}HT^{2H-1}\mathbb{E}\left[\int_u^T \left[\left|\overline{\Delta Z}^{\varepsilon}_{1,s}\right|^2 + \left|\overline{\Delta Z}^{\varepsilon}_{2,s}\right|^2\right]ds\right] + 4\varepsilon^{2H}C_0T^2. \qquad (15)$$

Using above inequalities, from (12), we deduce

$$\sup_{u\le t\le T}\mathbb{E}\left[\left|\overline{\Delta Y}^{\varepsilon}_t\right|^2\right]\le\left(4(T-u)L\varepsilon^{4H}+2\varepsilon^{2H}HT^{2H-1}\right)\sup_{u\le t\le T}\mathbb{E}\left[\int_u^T\left[\left|\overline{\Delta Z}^{\varepsilon}_{1,s}\right|^2+\left|\overline{\Delta Z}^{\varepsilon}_{2,s}\right|^2\right]ds\right]$$
$$+4(T-u)L\varepsilon^{4H}\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|\overline{\Delta Y}^{\varepsilon}_s\right|^2ds+C_3(T-u)^2\varepsilon^{4H}+4\varepsilon^{2H}C_0T^2.$$

Applying Lemma 3.2 to the above inequality we get

$$\sup_{u\le t\le T}\mathbb{E}\left[\left|\overline{\Delta Y}^{\varepsilon}_t\right|^2\right]\le\left[4(T-u)L\varepsilon^{4H}\left(L_1+1\right)+2L_1\varepsilon^{2H}HT^{2H-1}\right]\int_u^T\sup_{u\le s_1\le s}\mathbb{E}\left|\overline{\Delta Y}^{\varepsilon}_{s_1}\right|^2ds$$
$$+\varepsilon^{2H}\left[\left(4(T-u)L\varepsilon^{2H}+2HT^{2H-1}\right)C_2(T-u)+C_3(T-u)^2\varepsilon^{2H}+4C_0T^2\right].\tag{16}$$

Thanks to Gronwall's inequality, we obtain

$$\sup_{u\le t\le T}\mathbb{E}\left|\overline{\Delta Y}^{\varepsilon}_t\right|^2\le\varepsilon^{2H}\left[\left(4(T-u)L\varepsilon^{2H}+2HT^{2H-1}\right)C_2(T-u)+C_3(T-u)^2\varepsilon^{2H}+4C_0T^2\right]$$
$$\times e^{(T-u)\left[4(T-u)L\varepsilon^{4H}(L_1+1)+2L_1\varepsilon^{2H}HT^{2H-1}\right]}.$$

Obviously, the above estimate implies that there exist $\beta\in[0,1]$ and $K>0$ such that for evry $t\in(0,K\varepsilon^{-2H\beta}]\subseteq[0,T]$,

$$\sup_{K\varepsilon^{1-\beta}\le t\le T}\mathbb{E}\left|Y^{\varepsilon}_t-\overline{Y}_t\right|^2\le C_4\varepsilon^{1-2H\beta},\tag{17}$$

in which

$$C_4=\left[\left(4(T-K\varepsilon^{-2H\beta})L\varepsilon^{2H}+2HT^{2H-1}\right)C_2(T-K\varepsilon^{-2H\beta})+C_3(T-K\varepsilon^{-2H\beta})^2\varepsilon^{2H}+4C_0T^2\right]$$
$$\times\varepsilon^{2H(1+\beta)-1}e^{(T-K\varepsilon^{-2H\beta})\left[4(T-K\varepsilon^{-2H\beta})L\varepsilon^{4H}(L_1+1)+2L_1\varepsilon^{2H}HT^{2H-1}\right]}$$

is constant.

Consequently, for any number $\delta_1>0$, we can choose $\varepsilon_1\in[0,\varepsilon_0]$ such that for every $\varepsilon_1\in[0,\varepsilon_0]$ and for each $t\in(0,K\varepsilon^{-2H\beta}]$

$$\sup_{K\varepsilon^{-2H\beta}\le t\le T}\mathbb{E}\left|Y^{\varepsilon}_t-\overline{Y}_t\right|^2\le\delta_1.\tag{18}$$

This completes the proof. □

With Theorem 3.5, it is easy to show the convergence in probability between solution processes Y^{ε}_t to the original (4) and $\overline{Y}_t$ to the averaged (5).

Corollary 3.1. *Let the assumptions* (**A1**) *and* (**A2**) *hold. For a given arbitary small number $\delta_2>0$, there exists $\varepsilon_2\in[0,\varepsilon_0]$ such that for all $\varepsilon\in(0,\varepsilon_2]$, we have*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{K\varepsilon^{1-\beta}\le t\le T}\left|Y^{\varepsilon}_t-\overline{Y}_t\right|>\delta_2\right)=0,\tag{19}$$

where β defined by Theorem 3.5 such that $\beta<\frac{1}{2H}$.

Proof. By Theorem 3.5 and the Chebyshev inequality, for any given number $\delta_2>0$, we can obtain

$$\mathbb{P}\left(\sup_{K\varepsilon^{1-\beta}\le t\le T}\left|Y^{\varepsilon}_t-\overline{Y}_t\right|>\delta_2\right)\le\frac{1}{\delta_2^2}\mathbb{E}\left(\sup_{K\varepsilon^{1-\beta}\le t\le T}\left|Y^{\varepsilon}_t-\overline{Y}_t\right|^2\right)\le\frac{C_4\varepsilon^{1-2H\beta}}{\delta_2^2}.$$

Let $\varepsilon \to 0$ and the required result follows. □

Remark 2. *Corollary 3.1 means the convergence in probability between the original solution* $\left(Y_t^\varepsilon, Z_{1,t}^\varepsilon, Z_{2,t}^\varepsilon\right)$ *and the averaged solution* $\left(\overline{Y}_t, \overline{Z}_{1,t}, \overline{Z}_{2,t}\right)$.

4. Conclusion

Backward stochastic differential equations are widely used in finance and optimal control problems. In this paper, we compare a traditional and an averaged backward stochastic differential equations driven by both standard and fractional Brownian motions. Under some appropriate assumptions, the solution to original systems is approximated by the solutions to averaged stochastic systems in the sense of mean square.

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