Lifetime distributions transformation: On the DUS-exponentiated gamma distribution with application

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Abstract

Recent research has increasingly concentrated on the transformation of lifetime distributions employing the Dinesh–Umesh–Sanjay (DUS) technique. This study presents an innovative model that is based on the exponentiation of the DUS transformation, resulting in a distribution derived from the exponentiated gamma (EG) distribution. We conducted a rigorous examination of the statistical properties of the DUS-EG distribution. Utilizing the maximum likelihood estimation approach, we estimated the model parameters and performed a simulation study to evaluate the efficacy of the estimators. The proposed model was validated through a comprehensive analysis of empirical data.

Keywords: DUS transformation; exponentiated gamma distribution; Moments; Residual analysis; Entropy; Maximum likelihood estimation; Monte-Carlo simulation.

1. Introduction

Modeling and analyzing lifetime distributions are essential in various domains, including engineering and statistics, making the selection of suitable distributions

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critical for effective data analysis. The gamma distribution is a widely recognized lifetime distribution frequently employed to model waiting times, product lifespans, and applications in quality engineering and image analysis. Nonetheless, traditional hazard rate forms of this distribution often fail to adequately represent the bathtub curves commonly observed in real-world systems. Recent advancements have enhanced model flexibility by introducing additional parameters to existing distributions or by integrating multiple distributions, thereby improving goodness-of-fit and augmenting the efficacy of data analysis. The exponentiated gamma (EG) distribution has been introduced by Gupta et al. (1998), which has a versatile model that includes the gamma distribution as a particular case. Shawky and Bakoban (2008) conducted a comprehensive study on the EG distribution as a model for lifetime data, formulating both Bayesian and non-Bayesian estimators for the shape parameter, reliability, and failure rate functions in contexts of complete and type II censored samples, they further investigated order statistics derived from the EG distribution and their implications for statistical inference. Ghanizadeh, et al. (2011) focused on parameter estimation of the EG distribution while accounting for the presence of outliers, they employed moment and maximum likelihood estimation techniques to derive the estimates of distribution parameters. A Bayesian analysis of the EG distribution utilizing type II censored data has been studied by Singh et al. (2011). Nasiri et al. (2013) formulated Bayesian estimators for the parameters of the EG distribution based on censored samples, applying a general entropy loss framework. Hussian (2014) introduced a transmuted EG distribution as a novel lifetime model. Feroze and Elbatal (2016) generated Beta-EG as a four parameters lifetime model. Recently, Ragab et al. (2024) leveraged the type II Topp-Leone-G class of distributions to establish a flexible lifespan distribution (TIITL-EG).

A multitude of methodologies has been proposed for the derivation of new classes of distributions from existing ones, as evidenced by the research conducted by Eugene et al. (2002), Nadarajah and Kotz (2004), Cordeiro and Castro (2011), and Cordeiro et al. (2013). Additionally, Kumar et al. (2015) presented the Dinesh–Umesh–Sanjay (DUS) transformation, which contributed to the creation of a new and efficient class of distributions. The DUS transformation associated with an exponential distribution is analyzed in Maurya et al. (2017). Building upon this foundation, Deepthi and Chacko (2020) introduced the DUS-Lomax distribution, while the DUS-Kumaraswamy is studied by Karakaya et al. (2021). the DUS-inverse Weibull distributions is derived by Gauthami et al. (2021). Recently, Nayana et al. (2022) employ a similar methodology to introduce presented the DUS-Weibull distribution.

This study employs the DUS transformation applied to the exponentiated gamma distribution for evaluating the DUS-EG distribution, which is a novel lifetime model. The structure of the paper is organized as follows: Section 2 describes the formulation of the DUS-EG distribution. Section 3 explores the main statistical Properties the of the DUS-EG distribution as: ordinary moments, inverse moments, moment generating function, incomplete moments, mean deviation, conditional moments, mean residual life, mean inactivity time and Rényi entropy associated with the distribution. Section 4 presents a comprehensive overview of the maximum likelihood estimation methodology employed in the analysis, while Section 5 delineates a Monte-Carlo simulation study designed to assess the performance of the proposed distribution. Section 6 provides a numerical example that illustrates the practical applicability of the distribution to aircraft windshields data. Finally, Section 7 concludes with a summary of the key findings derived from the research.

2. Model Formulation

If $H(y;\tau)$ is the baseline cumulative distribution function (cdf), the DUS transformation generates a new cdf, $T(y;\tau)$, as follows:

$$T(y;\tau)=\frac{1}{e-1}\left\{\exp\left[H(y;\tau)\right]-1\right\}\ ;\ y>0.$$

The appropriate probability density function (pdf) is as follows:

$$t(y;\tau)=\frac{1}{e-1}\left\{h\left(y;\tau\right)\exp\left[H\left(y;\tau\right)\right]\right\}$$

This section applies the DUS transformation to the exponentiated gamma distribution, defined by its cdf and pdf:

$$H\left(y;\delta,\lambda\right)=\left[1-(1+\delta y)\,\mathrm{e}^{-\delta y}\right]^{\lambda}\quad;\quad y\geq 0,\delta,\lambda>0\ ,$$

and

$$h\left(y;\delta,\lambda\right)=\lambda\delta^{2}\,y\,\mathrm{e}^{-\delta y}\left[1-(1+\delta y)\,\mathrm{e}^{-\delta y}\right]^{\lambda-1}\quad;\quad y\geq 0,\delta,\lambda>0\ ,$$

Equations (3) and (4), combined with Equations (1) and (2), yield the established cdf and pdf of the DUS-EG distribution:

$$T(y;\delta,\lambda)=\frac{1}{e-1}\left\{\exp\left[\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\right]-1\right\};\ y\geq 0,\delta,\lambda>0\ ,$$

and

$$t(y;\delta,\lambda)=\frac{\lambda\delta^{2}y\,\mathrm{e}^{-\delta y}}{e-1}\left\{\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda-1}\exp\left[\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\right]\right\}$$

(6)The hazard rate (hr) function, also called the failure rate or hazard function, is a key concept in reliability theory and survival analysis. It represents the instantaneous failure rate at a specific time, given survival until that time. Mathematically, it is defined as:

$$z(y;\delta,\lambda)=\frac{h(y;\delta,\lambda)}{1-H(y;\delta,\lambda)}$$

$$=\frac{\lambda\delta^{2}y\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\exp\left[\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\right]}{\left(1+\delta x-\mathrm{e}^{\delta x}\right)\left(\exp\left[\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\right]-\mathrm{e}\right)}\ .$$

Figure 1 (a) presents the DUS-EG density function for various attribute values of the model. The pdf of the proposed model exhibits a rightward slope, demonstrates a monotonically decreasing nature, and is characterized as unimodal.

Figure 1 (b) illustrates the hazard rate function patterns for various distribution parameters. The DUS-EG distribution's hazard rate resembles a bathtub or an upside-down bathtub shape.

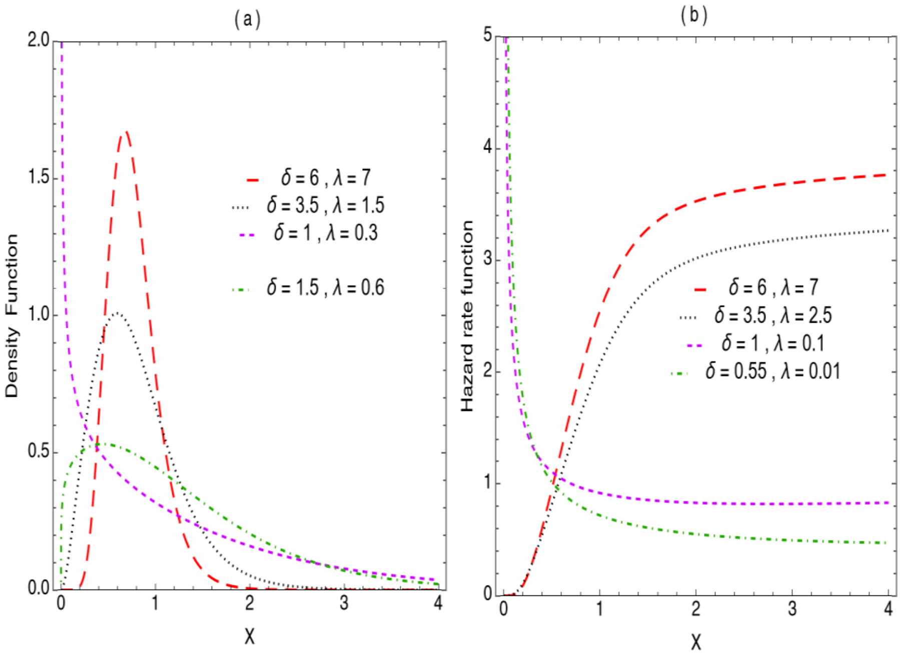


Figure 1: (a) The pdf plots of DUS-EG (b) The hr plots for DUS-EG distribution

2.1. Linear represnetation

This subsection enhances the DUS-EG distribution using an exponential series:

$\exp(w) = \sum_{k=0}^{\infty} w^k / k!$. Consequently, Equation (6) can be expressed as:

$$t(y;\delta,\lambda) = \frac{\lambda\delta^2 y\, e^{-\delta y}}{e-1} \sum_{k=1}^{\infty} \frac{\left(1-(1+\delta y)e^{-\delta y}\right)^{\lambda(k+1)-1}}{k!}.$$

By applying the generalized binomial theorem, we find that for any positive integer θ, we have:

$\left(1-\theta\right)^{\upsilon}=\sum_{m=0}^{\infty}(-1)^{m}\binom{\upsilon}{m}\theta^{k}$, for $|\theta|<1$. Hence, the pdf of DUS-EG is derived as:

$$t(y;\delta,\lambda)=\frac{\lambda\delta^{2}\,y\,\mathrm{e}^{-\delta y}}{e-1}\sum_{k,m=0}^{\infty}(-1)^{m}\binom{\lambda(k+1)-1}{m}\frac{(1+\delta y)^{m}\,\mathrm{e}^{-\delta m y}}{k\,!}\,.$$

Again, using the generalized binomial theorem for the term $(1+\delta y)^{m}$, yields:

$$t(y;\delta,\lambda)=\frac{\lambda\delta^{2}\,y\,\mathrm{e}^{-\delta y}}{e-1}\sum_{k,m=0}^{\infty}(-1)^{m}\binom{\lambda(k+1)-1}{m}\binom{m}{l}\frac{(\delta y)^{l}\,\mathrm{e}^{-\delta m y}}{k\,!}\,,$$

$$=\frac{\lambda}{e-1}\sum_{k,m,l=0}^{\infty}\frac{(-1)^{m}\,\delta^{l+2}}{k\,!}\binom{\lambda(k+1)-1}{m}\binom{m}{l}y^{\,l+1}\,\mathrm{e}^{-\delta(m+1)y}\,,$$

$$=\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\;y^{\,l+1}\,\mathrm{e}^{-\delta(m+1)y}$$

where,

$$\vartheta_{k,m,l}=\frac{(-1)^{m}\,\lambda\delta^{l+2}}{(e-1)k\,!}\binom{\lambda(k+1)-1}{m}\binom{m}{l}.$$

3. Statistical Properties

This section analyzed key statistical features of the DUS-EG distribution, such as the quantile function, median, moments, moment generating function, and conditional moment.

3.1. The ordinary moments

Moments are essential for understanding a distribution. This section highlights key moments and their implications:

- Mean (First Moment: $\mu_y = \mu_1$): Reflects the distribution's central tendency and indicates the average value of the dataset.
- Variance (Second Moment: $\sigma_y^2 = \mu_2 - (\mu_1)^2$): Assesses data dispersion; a higher variance signifies greater spread from the mean.
- Skewness (Third Moment: $\eta_1 = (\mu_3 - 3\,\mu_1\,\mu_2 + 2\,\mu_1^3)/\sigma_y^3$): Measures distribution asymmetry. A positive skew suggests a longer or fatter right tail, while a negative skew indicates the opposite.
- Kurtosis (Fourth Moment: $\eta_2 = (\mu_4 - 4\,\mu_1\,\mu_3 + 6\,\mu_1^2\,\mu_2 - 3\mu_1^4)/\sigma_y^4$): Evaluates the distribution's peakedness.

High kurtosis signifies heavy tails and a sharp peak, whereas low kurtosis indicates light tails and a rounded peak. These moments collectively offer a comprehensive understanding of a distribution's characteristics, making them essential in statistical analysis and probability theory. The n_{th} moment concerning the origin of the DUS-EG distribution can be calculated as:

$$
\begin{aligned}
v_n'(y) &= \int_0^\infty y^n t(y;\delta,\lambda)\,dy \\
&= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \int_0^\infty y^{n+l+1}\, \mathrm{e}^{-\delta(m+1)y}\, dy \\
&= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \int_0^\infty \frac{\left(y\,\delta(m+1)\right)^{n+l+1}}{\left(\delta(m+1)\right)^{n+l+2}}\, \mathrm{e}^{-\delta(m+1)y}\, d\delta(m+1)y \\
&= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma(n+l+2)}{\left(\delta(m+1)\right)^{n+l+2}}\ ;\ n = 1,2,3,\ldots
\end{aligned}
$$

where, $\Gamma(p) = \int_0^\infty y^{p-1} e^{-y} dy$ denotes the gamma function.

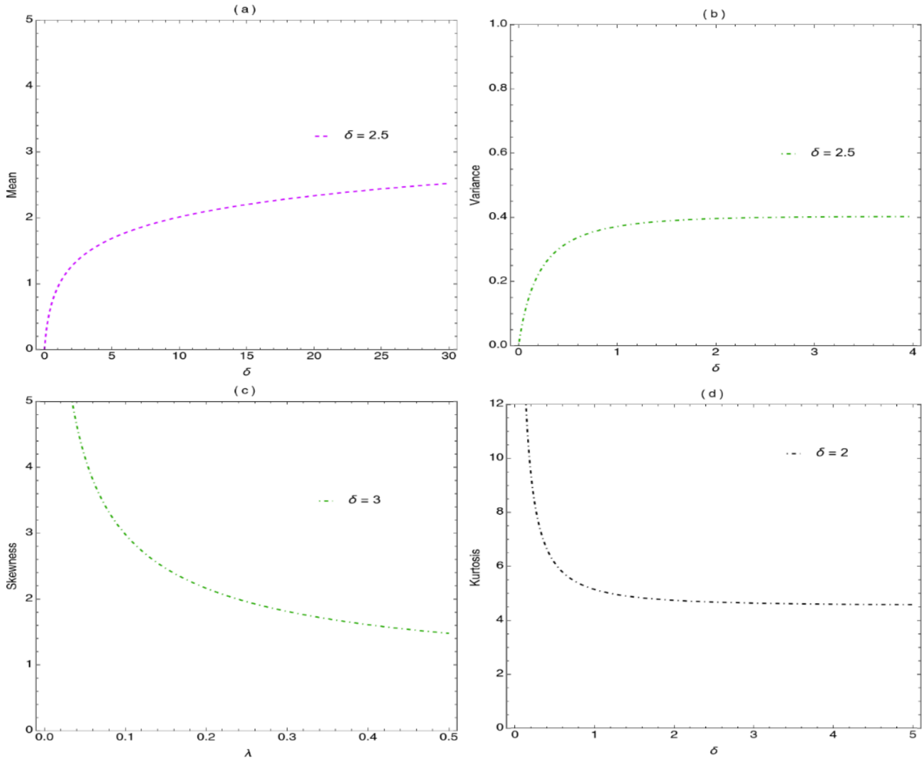


Figure 2: The mean, variance, skewness and kurtosis of DUS-EG Model

Table 1: displays the first four ordinary moments, variance, skewness, and kurtosis of the DUS-EG distribution at various parameter choices. As parameters δ and λ increase, the first four raw moments and variance rise, while skewness and kurtosis decrease. Figure 2 depicts these general patterns.

Table 1: displays the ordinary moments, variance, skewness, and kurtosis of the DUS-EG model

δ	λ	μ_1	μ_2	μ_3	μ_4	σ_y^2	η_1	η_2
0.5	0.5	3.36847	19.3715	152.943	1522.57	8.02493	1.47922	6.12422
0.5	1	4.76822	32.0224	274.713	2865.63	9.28654	1.18243	5.14446
0.5	1.5	5.65896	41.7389	378.666	4087.32	9.71501	1.07375	4.86476
0.5	2.5	6.82447	56.5608	553.443	6276.29	9.98739	0.986219	4.67777
0.5	3	7.24751	62.5554	629.397	7275.09	10.0291	0.96515	4.63951
0.5	5	8.44218	81.3101	885.421	10829.7	10.0397	0.926474	4.58091
1	0.5	1.68424	4.84289	19.1179	95.1605	2.00623	1.47922	6.12422
1	1	2.38411	8.00561	34.3391	179.102	2.32163	1.18243	5.14446
1	1.5	2.82948	10.4347	47.3333	255.458	2.42875	1.07375	4.86476
1	2.5	3.41224	14.1402	69.1804	392.268	2.49685	0.986219	4.67777
1	3	3.62375	15.6389	78.6746	454.693	2.50728	0.96515	4.63951
1	5	4.22109	20.3275	110.678	676.853	2.50993	0.926474	4.58091
1.5	0.5	1.12282	2.15239	5.66457	18.7971	0.891659	1.47922	6.12422
1.5	1	1.58941	3.55805	10.1745	35.3781	1.03184	1.18243	5.14446
1.5	1.5	1.88632	4.63765	14.0247	50.4608	1.07945	1.07375	4.86476
1.5	2.5	2.27482	6.28454	20.4979	77.4851	1.10971	0.986219	4.67777
1.5	3	2.41584	6.9506	23.311	89.8159	1.11435	0.96515	4.63951
1.5	5	2.81406	9.03446	32.7934	133.699	1.11552	0.926474	4.58091
2.5	0.5	0.67369	0.774862	1.22355	2.43611	0.320997	1.47922	6.12422
2.5	1	0.95364	1.2809	2.1977	4.585	0.371461	1.18243	5.14446
2.5	1.5	1.13179	1.66956	3.02933	6.53972	0.3886	1.07375	4.86476
2.5	2.5	1.36489	2.26243	4.42754	10.0421	0.399495	0.986219	4.67777
2.5	3	1.44950	2.50222	5.03518	11.6401	0.401165	0.96515	4.63951
2.5	5	1.68844	3.2524	7.08337	17.3274	0.401588	0.926474	4.58091
5	0.5	0.336847	0.193715	0.152943	0.152257	0.080249	1.47922	6.12422
5	1	0.476822	0.320224	0.274713	0.286563	0.092865	1.18243	5.14446
5	1.5	0.565896	0.417389	0.378666	0.408732	0.09715	1.07375	4.86476
5	2.5	0.682447	0.565608	0.553443	0.627629	0.099874	0.986219	4.67777
5	3	0.724751	0.625554	0.629397	0.727509	0.100291	0.96515	4.63951
5	5	0.844218	0.813101	0.885421	1.08297	0.100397	0.926474	4.58091

3.2. The inverse moments

Assume Y is a non-negative random variable following the DUS-EG distribution. The n_{th} inverse moment is expressed as follows:

$$\nu'_{n^{-1}}(y)=\int_0^\infty y^{-n}\ t(y;\delta,\lambda)\,dy$$

$$=\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\int_0^\infty y^{-n+l+1}\,e^{-\delta(m+1)y}\,dy$$

$$=\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\int_0^\infty y^{-n+l+1}\,e^{-\delta(m+1)y}\,dy$$

$$=\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\,\frac{\Gamma(l-n+2)}{(\delta(m+1))^{l-n+2}}\ ;\ n=1,2,3,\ldots$$

The harmonic mean is a measure of central tendency that is especially useful for rates or ratios. In contrast to the arithmetic mean, which sums values and divides by their count, the harmonic mean takes the reciprocal of each value. It is calculated using Equation (10) for $n=1$:

$$\zeta_H=\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\,\frac{\Gamma(l-1)}{(\delta(m+1))^{l-1}}\ .$$

3.3. Moment generating function

The moment generating function (mgf) is a valuable tool in probability theory and statistics as it encapsulates all moments of a probability distribution. For a random variable (Y), the mgf is defined as:

$$\kappa(z)=\int_0^\infty e^{zy}\ t(y;\delta,\lambda)\,dy$$

$$= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \int_{0}^{\infty} y^{l+1} \, e^{-\delta(m+1)y} \, dy$$

$$= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma(l+2)}{(\delta+\delta m - z)^{l+2}} \; ; \; z < \delta(m+1) \, . \qquad (11)$$

3.4. Incomplete moments

Incomplete moments provide insights into a random variable's distribution within a specific range, rather than its entire domain, making them particularly useful for analyzing distribution behavior in a defined scope. They can be categorized as follows:

- **Lower-Incomplete Moment (LIM):** This particular term refers to the moments of a statistical distribution that are computed only up to a certain defined threshold. The s_{th} LIM of random variable that follows DUS-EG is defined as:

$$\psi_s(z) = E\left(Y^s | Y < z\right) = \int_0^z y^s \, t(y;\delta,\lambda) \, dy \; ,$$

$$\psi_s(z) = \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \int_0^z y^{s+l+1} \, e^{-\delta(m+1)y} \, dy$$

$$= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\gamma\left(l+s+2, \delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+s+2}}.$$

where $\gamma(a,z) = \int_0^z x^{a-1} \exp(-x) \, dx$ is the lower incomplete gamma function.

- **Upper-Incomplete Moment (UIM):** Upper-Incomplete Moment (UIM): This measure assesses the moments of a distribution that begin at a specified threshold. The s_{th} UIM of random variable that follows DUS-EG is defined as:

$$\eta_s(z) = E\left(Y^s | Y > z\right) = \int_z^{\infty} y^s \, t(y;\delta,\lambda) \, dy \; ,$$

$$= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \int_z^{\infty} y^{s+l+1} \, e^{-\delta(m+1)y} \, dy$$

$$= \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma\left(l+s+2, \delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+s+2}}.$$

where $\gamma(a,z) = \int_0^z x^{a-1} \exp(-x)\, dx$ is the upper incomplete gamma function.
These moments are valuable for risk management and reliability analysis because they highlight distribution behavior relative to set criteria.

3.5. Mean deviation

Mean deviation (MD), or mean absolute deviation, measures the average distance between each data point in a set and the mean. It is useful for assessing variability: a small mean deviation indicates that data points are closely clustered around the mean, while a large mean deviation suggests they are more dispersed. The MD about the mean is derived as:

$$MD(\mu) = E\left[|y-\mu|\right] = \int_0^{\infty} |y-\mu|\, t(y;\delta,\lambda) dy$$

$$= \int_0^{\mu} [\mu - y]\, t(y;\delta,\lambda)\, dy + \int_{\mu}^{\infty} [y-\mu]\, t(y;\delta,\lambda)\, dy$$

$$= 2\mu T(\mu;\delta,\lambda) - 2\mu + 2\int_{\mu}^{\infty} y\ t(y;\delta,\lambda) dy$$

The first UIM can be calculated using the Equation (13) with $s=1$ as:

$$\eta_1(z) = \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma\left(l+3, \delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+3}}.$$

Using Equation (5) and (15) into Equation (14), yields:

$$MD(\mu) = \frac{2\mu}{e-1}\left\{\exp\left[\left(1-(1+\delta\mu)\mathrm{e}^{-\delta\mu}\right)^{\lambda}\right] - 1\right\} - 2\mu + 2\sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma\left(l+3, \delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+3}}.$$

(16)

Similarly, we can obtain the MD about the median (mo) as:

$$MD(m_o)=E\left[\left|y-m_o\right|\right]=\int_0^{\infty}\left|y-m_o\right|t(y;\delta,\lambda)\,dy$$

$$=-\mu+2\int_{m_0}^{\infty}y\,t(y;\delta,\lambda)dy=-\mu+2\eta_1(m_0)$$

$$=-\mu+2\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\frac{\Gamma\left(l+3,\delta(m+1)m_0\right)}{\left(\delta(m+1)\right)^{l+3}}.$$

3.6. Conditional moments

Conditional moments are key statistical measures that describe a distribution's characteristics under specified conditions or events. The DUS-EG distribution's conditional moments are as follows:

$$E\left(Y^s\mid Y>z\right)=\frac{\eta_s(z)}{1-T(z;\delta,\lambda)},$$

$$=\left(1-\frac{\left\{\exp\left[\left(1-(1+\delta z)e^{-\delta z}\right)^{\lambda}\right]-1\right\}}{e-1}\right)^{-1}\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\frac{\Gamma\left(l+s+2,\delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+s+2}}.$$

(18)

Conditional moments hold significant importance in the fields of finance, economics, and machine learning, as they facilitate a comprehensive analysis of a variable's behavior under designated conditions.

3.7. The mean residual life

The mean residual life (MRL) is a fundamental concept in probability theory and statistics, especially within the domains of survival analysis and reliability engineering. It quantifies the expected remaining lifespan of an entity that has already endured up to a designated point in time. The MRL of the DUS-EG distribution is determined as:

$$\rho(z)=\frac{1}{1-T(z;\delta,\lambda)}\int_z^{\infty}y\,t(y;\delta,\lambda)dy-z\quad;\quad z>0$$

$$= \frac{\eta_1(z)}{1-T(z;\delta,\lambda)} - z$$

Inserting Equations (5) and (15) into Equation (19) results in:

$$\rho(z) = \left(1 - \frac{\left\{ \exp\left[\left(1-(1+\delta z)\mathrm{e}^{-\delta z} \right)^{\lambda} \right] - 1 \right\}}{e-1} \right)^{-1} \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\Gamma\left(l+3, \delta(m+1)z \right)}{\left(\delta(m+1) \right)^{l+3}} - z \, . \quad (20)$$

The MRL function provides vital insights into a component or system's expected future performance, making it a valuable tool for preventive maintenance and risk assessment.

3.8. The mean inactivity time

Mean Inactivity Time (MIT) is a concept in reliability theory and survival analysis, akin to mean residual life, but it specifically measures the duration a system or component remains inactive or fails before resuming operation. The MIT of the DUS-EG distribution is derived as:

$$\sigma(z) = z - \frac{1}{T(z;\delta,\lambda)} \int_0^z y\, t(y;\delta,\lambda)\, dy \quad ; \quad z > 0$$

$$= z - \frac{\psi_1(z)}{T(z;\delta,\lambda)}$$

The first LIM can be calculated applying the Equation (12) for $s = 1$ as:

$$\psi_1(z) = \sum_{k,m,l=0}^{\infty} \vartheta_{k,m,l} \frac{\gamma\left(l+3, \delta(m+1)z \right)}{\left(\delta(m+1) \right)^{l+3}}.$$

Applying Equations (5) and (22) into Equation (21) produces:

$$\sigma(z)=z-\left(\frac{\left\{\exp\left[\left(1-(1+\delta z)e^{-\delta z}\right)^{\lambda}\right]-1\right\}}{e-1}\right)^{-1}\sum_{k,m,l=0}^{\infty}\vartheta_{k,m,l}\frac{\gamma\left(l+3,\delta(m+1)z\right)}{\left(\delta(m+1)\right)^{l+3}}.$$

(23)

The MIT plays a critical role in understanding system downtime and developing maintenance schedules, providing significant insights into projected outages and improving operational efficiency for systems with intermittent activity.

Figure 3: Illustrates how changes in distribution parameters affect MRL and MIT. The left panel reveals a notable decrease in MRL, while the right panel indicates a significant increase in MIT.

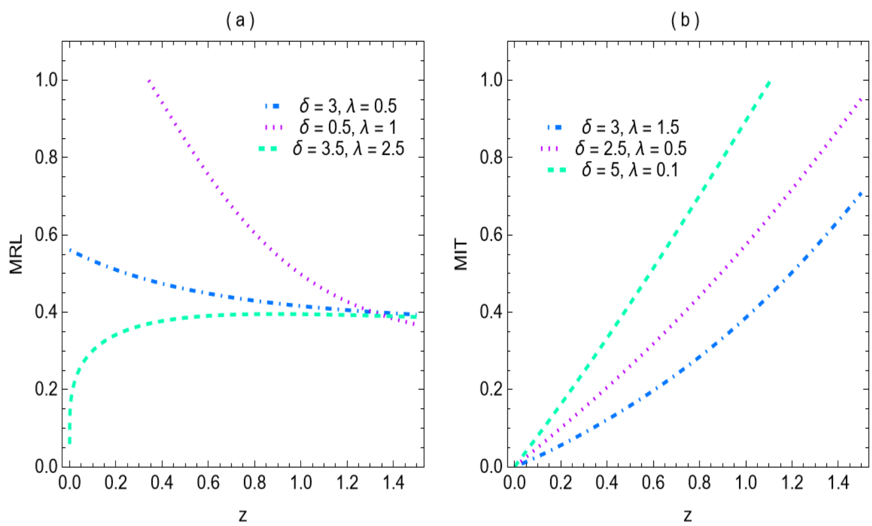


Figure 3: shows the MRL (a) and MIT (b) of the DUS-EG model

Table 2: shows the MRL and MIT of the DUS-EG distribution for various parameter settings. As the parameter values rise the MRL increases and the MIT goes down.

Table 2: The MRL and MIT functions of the TIIHLA distribution

$\delta\uparrow$	$\lambda\uparrow$	$MRL\uparrow$	$MIT\downarrow$
0.5	0.5	3.22613	0.25020
	1	4.33872	0.17282
	1.5	5.17231	0.13128
	2	5.81342	0.10571
	2.5	6.32490	0.08846
	3	6.74758	0.07606
1	0.5	1.55192	0.25154
	1	2.00397	0.17788
	1.5	2.36955	0.13733
	2	2.66872	0.11158
	2.5	2.91652	0.09389
	3	3.12512	0.08102
1.5	0.5	0.99954	0.25386
	1	1.24686	0.18239
	1.5	1.45597	0.14297
	2	1.63478	0.11741
	2.5	1.78866	0.09949
	3	1.92194	0.08626

3.9. The Rényi entropy

Rényi entropy provides a more flexible framework for assessing uncertainty and information content in diverse scenarios. Its versatility makes it valuable in fields like physics, information theory, and machine learning. The Rényi entropy of a random variable Y follows the DUS-EG distribution is given by:

$$R(\vartheta)=\frac{1}{1-\vartheta}\log\left[\int_0^{\infty} t^{\vartheta}(y;\delta,\lambda)\,dy\right];\vartheta>0,\ \vartheta\neq 1$$

The term $t^{\vartheta}(y;\delta,\lambda)$ can be derived as:

$$t^{\vartheta}(y;\delta,\lambda)=\left[\frac{\lambda\delta^2 y\,\mathrm{e}^{-\delta y}}{e-1}\right]^{\vartheta}\left\{\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\vartheta(\lambda-1)}\exp\left[\vartheta\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda}\right]\right\},$$

applying an exponential series: $\exp(w)=\sum_{k=0}^{\infty} w^{k}/k!$, yields:

$$t^{\vartheta}(y;\delta,\lambda)=\left[\frac{\lambda\delta^2 y\,\mathrm{e}^{-\delta y}}{e-1}\right]^{\vartheta}\sum_{k=1}^{\infty}\frac{\vartheta^k}{k!}\left(1-(1+\delta y)\mathrm{e}^{-\delta y}\right)^{\lambda(k+\vartheta)-\vartheta},$$

using the generalized binomial theorem, we have:

$$t^{\vartheta}(y;\delta,\lambda)=\left[\frac{\lambda\delta^2 y\,\mathrm{e}^{-\delta y}}{e-1}\right]^{\vartheta}\sum_{k,m=0}^{\infty}(-1)^m\frac{\vartheta^k}{k!}\binom{\lambda(k+\vartheta)-\vartheta}{m}(1+\delta y)^m\,\mathrm{e}^{-\delta m y}$$

$$=\sum_{k,m,l=0}^{\infty}\pi^{\vartheta}_{k,m,l}\;y^{l+\vartheta}\,\mathrm{e}^{-\delta(m+\vartheta)y}$$

where,

$$\pi^{\vartheta}_{k,m,l}=\frac{\lambda^{\vartheta}\,\delta^{l+2\vartheta}\,y^{\vartheta}\,\mathrm{e}^{-\vartheta\delta y}}{(e-1)^{\vartheta}}(-1)^m\frac{\vartheta^k}{k!}\binom{\lambda(k+\vartheta)-\vartheta}{m}\binom{m}{l}.$$

The integral of Equation (24) is derived as:

$$\int_0^{\infty} t^{\vartheta}(y;\delta,\lambda)\,dy=\sum_{k,m,l=0}^{\infty}\pi^{\vartheta}_{k,m,l}\int_0^{\infty} y^{l+\vartheta}\,\mathrm{e}^{-\delta(m+\vartheta)y}\,dy$$

$$=\sum_{k,m,l=0}^{\infty}\pi^{\vartheta}_{k,m,l}\frac{\Gamma(\vartheta+l+1)}{((\vartheta+m)\delta)^{\vartheta+l+1}}.$$

Substituting by Equation (25) in the Rényi entropy formula:

$$R(\vartheta)=\frac{1}{1-\vartheta}\log\left[\sum_{k,m,l=0}^{\infty}\pi^{\vartheta}_{k,m,l}\frac{\Gamma(\vartheta+l+1)}{((\vartheta+m)\delta)^{\vartheta+l+1}}\right];\vartheta>0,\ \vartheta\neq 1.$$

4. Parameter Estimation

The Maximum Likelihood Estimation (MLE) technique is commonly recognized as one of the most reliable methods for parameter estimation. The objective of MLE is to identify the parameter values that maximize the likelihood function, which assesses how well the model describes the observed data. MLE generally seeks the parameter configuration that maximizes the probability of the observed data. Let $y_1, y_2, \ldots, y_n$ be random samples of size n selected from the DUS-EG distribution with density function given in Equation (6). Therefore, the log-likelihood function of the proposed model states as:

$$\tau = \sum_{i=1}^{n}\left[1-(\delta y_i+1)e^{-\delta y_i}\right]^{\lambda} + (\lambda-1)\sum_{i=1}^{n}\log\left[1-(\delta y_i+1)e^{-\delta y_i}\right] + n\log(\lambda) - n\log(e-1)$$

$$+\sum_{i=1}^{n}\log\left[\delta(\delta y_i+1)\,e^{-\delta y_i} - \delta e^{-\delta y_i}\right]$$

Differentiating Eq. (26) with regard to parameters δ and λ respectively, allows one to derive the likelihood equations for the desired model as:

$$\frac{\partial\tau}{\partial\delta} = (\lambda-1)\sum_{i=1}^{n}\frac{y_i\left(\delta y_i+1\right)e^{-\delta y_i} - y_i e^{-\delta y_i}}{1-\left(\delta y_i+1\right)e^{-\delta y_i}} + \sum_{i=1}^{n}\delta\lambda y_i^2\left(1-(\delta y_i+1)\,e^{-\delta y_i}\right)^{\lambda-1} e^{-\delta y_i}$$

$$+\sum_{i=1}^{n}\frac{2\delta y_i\, e^{-\delta y_i} - \delta y_i\,(\delta y_i+1)\,e^{-\delta y_i} - e^{-\delta y_i} + (\delta y_i+1)e^{-\delta y_i}}{\delta(\delta y_i+1)\,e^{-\delta y_i} - \delta e^{-\delta y_i}}. \qquad (27)$$

$$\frac{\partial\tau}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\left[1-(\delta y_i+1)e^{-\delta y_i}\right]^{\lambda}\log\left[1-(\delta y_i+1)e^{-\delta y_i}\right] + \sum_{i=1}^{n}\log\left[1-(\delta y_i+1)e^{-\delta y_i}\right]. \qquad (28)$$

The MLEs of parameters δ and λ can be obtained by equating the system of nonlinear equations (27-28) to zero and solving them simultaneously. Due to these equations are nonlinear, we need to solve them analytically using Newton-Raphson method.

5. Monte-Carlo Simulation

Monte Carlo simulation is a computational technique that employs random sampling to tackle complex mathematical problems. It generates random data points by periodically sampling a probability distribution, which are then used to estimate the distribution's parameters. This methodology is particularly advantageous in scenarios where deriving an accurate analytical solution is difficult or unfeasible. It is frequently utilized in finance, scientific research, and various other disciplines to evaluate intricate systems or models. In the present study, we intend to employ this approach to the DUS-EG distribution and leverage maximum likelihood estimation (MLE) to derive parameter estimates.

Main steps of a simulation study:

- The simulation examination included 1000 iterations, with Equation (5) generating sample sizes of 100, 150, 200, 300, 400, and 500.
- The simulated anticipates are based on actual parameter values:

Case I: $(\delta=0.5, \lambda=0.5)$, Case II: $(\delta=1, \delta=2)$ and Case III: $(\delta=3, \lambda=4)$.

- To evaluate the accuracy of MLE at a 95% confidence level, we use the mean of parameter estimates, mean squared error (MSE), average bias, coverage probability (CP), and average length of the confidence interval (ACI).
- The following formulas can be used to calculate the average MSEs and biases of the simulated estimates:

$$Bias = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\psi}_i - \psi\right) \quad \text{and} \quad MSE = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\psi}_i - \psi\right)^2$$

where, $\psi = (\delta, \lambda)$.

In **Tables (3-8),** the MLEs attain optimal dimensionality as the sample size n increases, while the mean squared errors (MSEs) and bias terms converge towards zero, as anticipated. Furthermore, the CP closely approximate the target assurance level of 95%. As the sample size n

expands, the approximate confidence intervals (ACI) for each parameter exhibit a tendency to stabilize.

Table 3: The simluation rsults of δ Case I

$\delta = 0.5$	$\hat{\delta}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n = 25$	0.54225	0.04225	0.01626	0.933	0.47190
$n = 50$	0.52138	0.02138	0.00715	0.937	0.32077
$n = 75$	0.51483	0.01483	0.00460	0.943	0.25949
$n = 100$	0.51155	0.01155	0.00321	0.945	0.21749
$n = 125$	0.50936	0.00936	0.00257	0.947	0.19556
$n = 150$	0.50766	0.00766	0.00201	0.947	0.17339
$n = 200$	0.50595	0.00595	0.00153	0.947	0.15163

Table 4: The simluation rsults of λ Case I

$\lambda = 0.5$	$\hat{\lambda}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n = 25$	0.57629	0.07629	0.04214	0.925	0.74747
$n = 50$	0.53860	0.03859	0.01497	0.944	0.45543
$n = 75$	0.52821	0.02821	0.00982	0.939	0.37251
$n = 100$	0.52106	0.02106	0.00680	0.938	0.31273
$n = 125$	0.51675	0.01675	0.00507	0.946	0.27136
$n = 150$	0.51459	0.01459	0.00424	0.948	0.24890
$n = 200$	0.51100	0.01100	0.00311	0.942	0.21444

Table 5: The simluation rsults of λ Case II

$\delta = 1$	$\hat{\delta}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n = 25$	1.05982	0.05982	0.03527	0.932	0.69818
$n = 50$	1.03107	0.03107	0.01653	0.936	0.48933
$n = 75$	1.02107	0.02107	0.01080	0.946	0.39921
$n = 100$	1.01644	0.01644	0.00757	0.947	0.33510
$n = 125$	1.01289	0.01289	0.00609	0.945	0.30180

$n=150$	1.01030	0.01030	0.00470	0.948	0.26594
$n=200$	1.00783	0.00783	0.00361	0.946	0.23360

Table 6: The simluation rsults of δ Case II

$\lambda=2$	$\hat{\lambda}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n=25$	2.48055	0.48055	1.37179	0.912	4.18908
$n=50$	2.23703	0.23703	0.48201	0.931	2.55930
$n=75$	2.16532	0.16532	0.30248	0.938	2.05724
$n=100$	2.12137	0.12137	0.19829	0.94	1.68034
$n=125$	2.09378	0.09378	0.14685	0.942	1.45722
$n=150$	2.07807	0.07807	0.11750	0.942	1.30904
$n=200$	2.05740	0.05740	0.08566	0.948	1.12561

Table 7: The simluation rsults of δ Case III

$\delta=3$	$\hat{\delta}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n=25$	3.07847	0.07847	0.15450	0.946	1.51056
$n=50$	3.05734	0.05734	0.09205	0.946	1.16847
$n=75$	3.04375	0.04375	0.06715	0.948	1.00176
$n=100$	3.03995	0.03995	0.05205	0.949	0.88094
$n=125$	3.03222	0.03222	0.04307	0.95	0.80410
$n=150$	3.02713	0.02713	0.03429	0.95	0.71840
$n=200$	3.02111	0.02111	0.02690	0.946	0.63791

Table 8: The simluation rsults of λ Case III

$\lambda=4$	$\hat{\lambda}$	*Bias*	*MSE*	95 % *CP*	95 % *ACI*
$n=25$	4.49569	0.49569	1.91461	0.985	5.06662
$n=50$	4.37867	0.37867	1.33373	0.999	4.27895
$n=75$	4.30344	0.30344	1.05154	0.956	3.84164
$n=100$	4.25975	0.25975	0.85924	0.915	3.48981
$n=125$	4.20802	0.20803	0.67496	0.922	3.11711
$n=150$	4.18134	0.18134	0.57672	0.924	2.89224
$n=200$	4.13688	0.13688	0.43240	0.941	2.52247

6. Application to aircraft windshields

The complexity of aircraft windshields is impressive. Their design allows them to endure extreme conditions, from high-altitude pressures to severe weather. A heated layer beneath the outer skin is essential for preventing ice formation, a significant flight hazard. Although damage to the outer ply or heating system failure may necessitate replacement, such issues do not compromise the aircraft's structural integrity. Aviation's stringent standards ensure the safety and reliability of these components. We analyze a real dataset on the service times of 63 aircraft windshields provided in Murthy et al. (2004).

To illustrate the DUS-EG distribution's flexibility, the goodness of fit for the proposed model is compared with several lifetime distributions: exponentiated gamma (EG), gamma (Gamma), exponentiated exponential (EE) by Gupta and Kundu (2001), weighted exponential (WE) by Gupta and Kundu (2009), and the DUS-exponential (DUS-E).

We employ the MLE method to estimate the unknown parameters of each distribution, and subsequently apply these estimates to calculate the following information criterion (IC): Akaike IC (AIC), Bayesian IC (BIC), and Hannan-Quinn IC (HQIC). In general, the optimal model for fitting the data is the one with the lowest metric values.

We compared the models using three additional criteria:
- Kolmogorov-Smirnov test statistics (K-S): a lower value is better.
- Anderson-Darling (A*) and the Cramer-von Mises (W*) goodness of-fit statistics; a lower value is better for these measures.
- Negative log-likelihood; a lower value is desirable.

Tables 9: provides the maximum likelihood estimates (MLEs) and distributions for each dataset, with standard errors (SE) in parenthesis. The goodness of fit criteria for all fitted models are listed in **Tables (10 &11). Figures (4 & 5)** demonstrate the fitting of the aircraft windshields dataset to the DUS-EG model using estimated density, survival function and the QQ-plots for all fitted models. These graphics align with the numerical results, indicating a superior fit of the suggested model compared to other comparable distributions.

Table 9: The MLE and their SEs are in parentheses for the aircraft windshields data

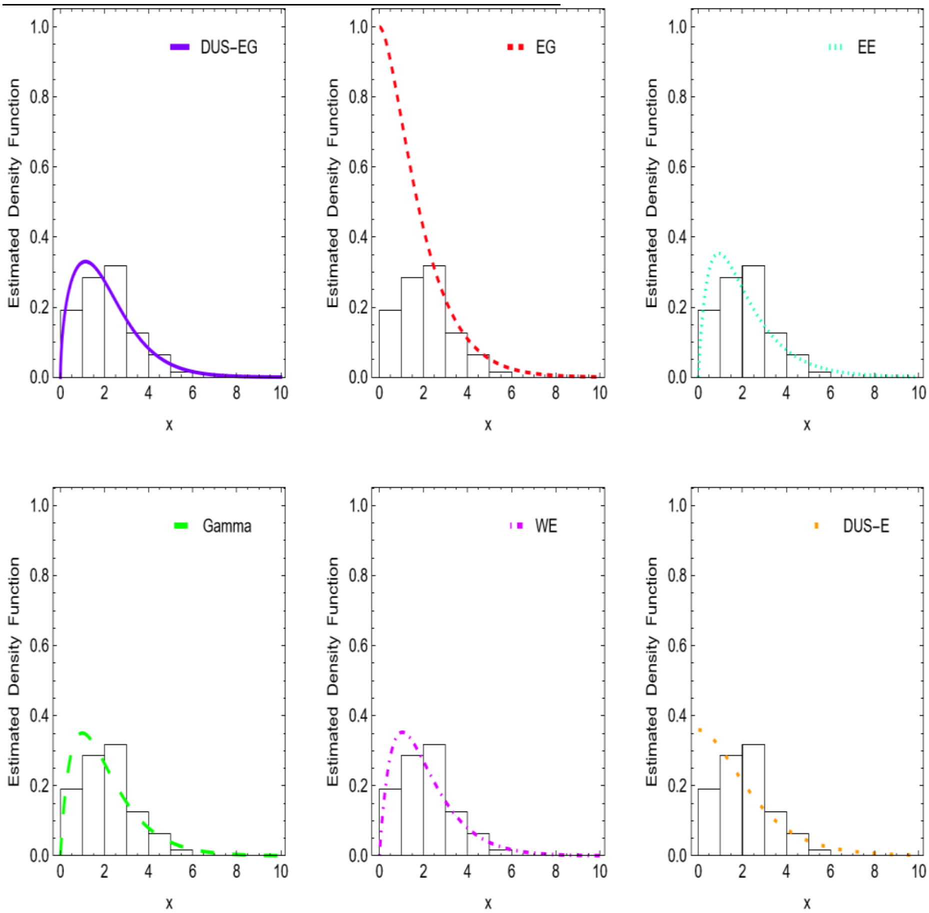
Model	Estimates	
DUS-EG(δ,λ)	$\hat{\delta}=0.810769$ (0.155137)	$\hat{\lambda}=1.04221$ (0.123583)
EG(δ,λ)	$\hat{\delta}=0.919072$ (0.16034)	$\hat{\lambda}=0.919244$ (0.117505)
Gamma(α,β)	$\hat{\alpha}=1.90846$ (0.314839)	$\hat{\beta}=1.09265$ (0.205964)
EE(η,γ)	$\hat{\eta}=0.692049$ (0.094202)	$\hat{\gamma}=1.89776$ (0.340169)
WE(θ,τ)	$\hat{\theta}=0.000612$ (0.68597)	$\hat{\tau}=0.958815$ (0.339858)
DUS-E(δ)	$\hat{\delta}=0.617824$ (0.066547)	- -

Table 10: The Log, AIC, BIC, and HQIC metrics for the aircraft windshields data

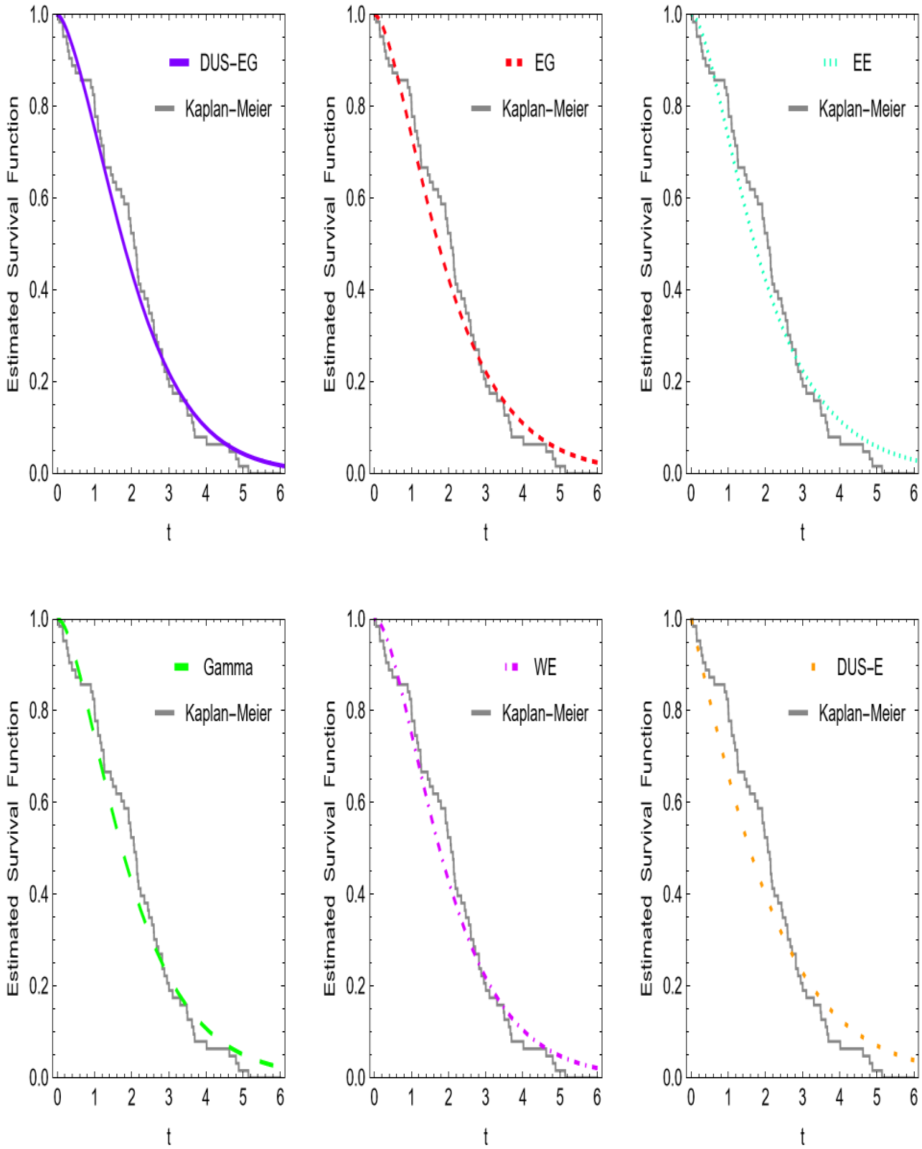
	Log	AIC	BIC	HQIC
DUS-EG	-101.045	206.289	210.375	207.775
EG	-102.755	209.509	213.796	211.195
Gamma	-102.833	209.665	213.951	211.351
EE	-103.547	211.093	215.38	212.779
WE	-102.873	209.747	214.033	211.433
DUS-E	-105.067	212.134	214.278	212.977

Table 11: The A*, W*, KS and p-value of K-S statistics for the aircraft windshields data

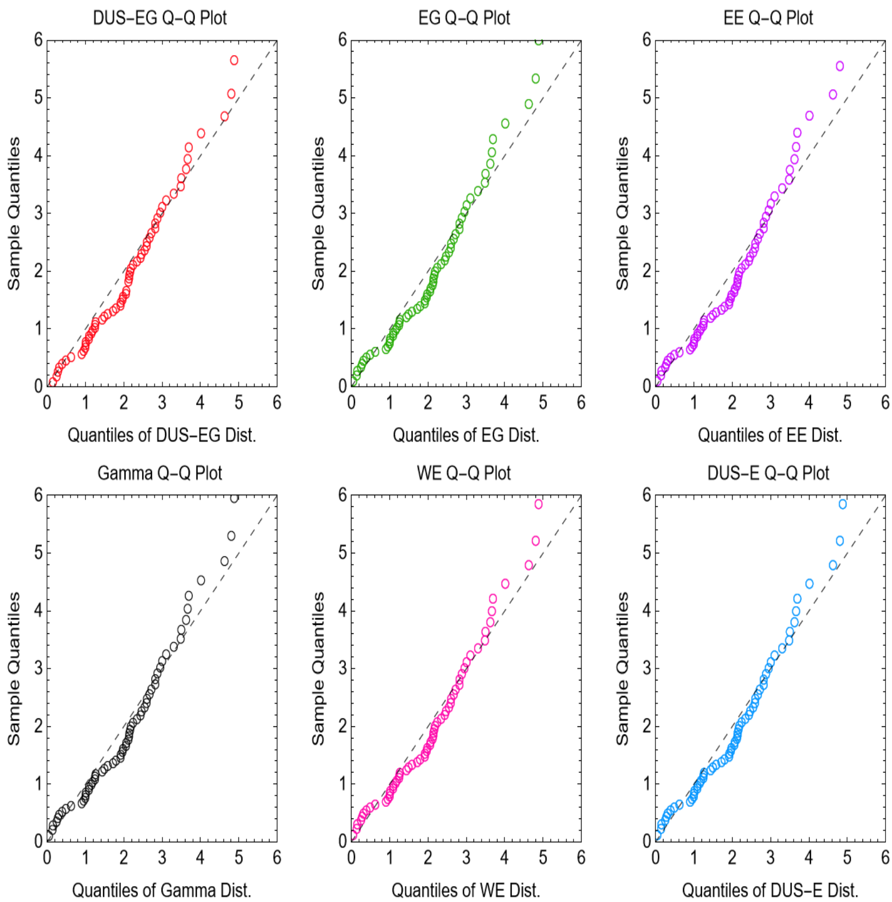
Model	A*	W*	K-S	P-Value
DUS-EG	0.829733	0.146264	0.125168	0.27704
EG	1.20763	0.215276	0.14066	0.165243
Gamma	1.17725	0.202315	0.138652	0.177317
EE	1.33158	0.234749	0.143753	0.147928
WE	1.13274	0.18327	0.135293	0.199056
DUS-E	2.56559	0.51421	0.169901	0.052653



Figures 4 (I): The estimated density for the aircraft windshields data



Figures 4 (II): The estimated survival function for the aircraft windshields data



Figures 5: The Q-Q plots for the aircraft windshields data

7. Conclusions and remarks

This study utilizes the DUS transformation on the exponentiated gamma distribution to evaluate the DUS-EG distribution, a novel lifetime model. The principal statistical characteristics of the DUS-EG distribution, encompassing ordinary moments, inverse moments, the moment generating function, incomplete moments, mean deviation, conditional moments, mean residual life, mean inactivity time, and Rényi entropy, are derived. We use maximum likelihood estimation to

estimate the two parameters of the DUS-EG distribution and perform a thorough numerical evaluation of its effectiveness. By analyzing aircraft windshield data, we assess the model's applicability and significance. Furthermore, we compare the DUS-EG distribution with several established statistical distributions, including: exponentiated gamma, exponentiated exponential, gamma, weighted exponential models, employing multiple metrics. The numerical and graphical analysis demonstrates that the DUS-EG model provides the best fit for the data in comparison to the competing models.

On the DUS-exponentiated gamma distribution with application

الملخص

ركزت الدراسات البحثية الحديثة بشكل متزايد على توليد نماذج احتمالية لتوزيعات الحياة باستخدام محول DUS لتقديم نموذج احتمالي جديد أكثر مرونة في نمذجة بيانات الحياة دون إضافة معالم جديدة لتوزيع الأصلي. في هذه الدراسة تم تقديم نموذجًا جديدًا باستخدام محول DUS ، فقد تم تطبيق محول DUSعلى توزيع جاما الاسي، مما أدى إلى توليد توزيع جديد DUS-EG ، التوزيع المولد أكثر مرونة من توزيع جاما الاسي (EG) في نمذجة بيانات الحياة. قمنا بدراسة الخصائص الإحصائية لنموذج الاحتمالي المقترح، كذلك تم تقدير معالم التوزيع للنموذج محل الدراسة باستخدام طريقة الإمكان الأكبر، وتم اجراء دراسة محاكاة لتقييم فعالية المقدرين. تم التحقق من فاعلية النموذج المقترح من خلال تحليل شامل للبيانات الحقيقة.

$$FDI_{it} = f(\,GNS_{it} + XPD_{it} + TAX_{it} + COE_{it} + EXP_{it} + AVI_{it} + EPI_{it} + CFW_{it} + (CDP_{it}) + \epsilon_{it} \qquad (1)$$

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