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BICOMPLEX BLOCH AND LITTLE BLOCH SPACES

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Abstract. In this paper, we study Bloch and little Bloch spaces in bicomplex setting. We also discuss the Möbius invariance of the bicomplex Bloch space and study the bicomplex Bergman projection onto the little Bloch space.

1. Introduction and Preliminaries

The Bloch space in complex analysis is interesting in its own right. In fact, the Bloch space was studied much earlier than the Bergman space, which has its own importance and stands as a very significant function space. For more details, see [13, 3].

Throughout this paper, we denote the set of bicomplex numbers by $\mathbb{BC}$. The theory of bicomplex holomorphic functions has seen substantial development; see [1, 2, 6, 8, 11, 12] and the references therein.

In the classical theory of holomorphic functions, one usually works on the unit disk, whereas in the bicomplex setting, we deal with the bidisk. Let $\mathbb{U}_{\mathbb{BC}} = \mathbb{U}_1 \times \mathbb{U}_2$ denote the bidisk in $\mathbb{BC}$. More generally, a bidisk $\mathbb{U}_{\mathbb{BC}}$ centered at (a_1, a_2) with associated radii (r_1, r_2) is defined as

$$\mathbb{U}_{\mathbb{BC}} = \left\{ Z \in \mathbb{BC} : Z = e\eta_1 + e^{\dagger}\eta_2,\ \|\eta_1 - a_1\|_k < r_1,\ \|\eta_2 - a_2\|_k < r_2 \right\}. \quad (1)$$

The bicomplex Bloch space was first introduced by Reséndis and Tovar in [10]. They studied the bicomplex Bergman projection onto the bicomplex Bloch space and also proved the decomposition

$$\mathfrak{B}_{\mathbb{BC}} = e\mathfrak{B} + e^{\dagger}\mathfrak{B}.$$

In this paper, we extend their work by defining the little Bloch space in the bicomplex setting. We denote the bicomplex Bloch and little Bloch spaces by $\mathfrak{B}_{\mathbb{BC}}$ and

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$\mathfrak{B}_{0,\mathbb{BC}}$, respectively. We also discuss the Möbius invariance of the bicomplex Bloch space, which follows directly from the idempotent decomposition of the space. Furthermore, we define the bicomplex little Bloch space $\mathfrak{B}_{0,\mathbb{BC}}$ on the bidisk $\mathbb{U}_{\mathbb{BC}}$ and show that $\mathfrak{B}_{0,\mathbb{BC}}$ can be decomposed into two classical little Bloch spaces on the unit disk. In addition, we study the bicomplex Bergman projection onto the little Bloch space.

We begin by recalling the definition of bicomplex numbers.

Definition 1.1. *The set of bicomplex numbers is defined as*

$$\mathbb{BC} = \{\eta_1 + j\eta_2 : \eta_1, \eta_2 \in \mathbb{C}(i)\},$$

where i and j are two imaginary units such that $ij = ji$ and $i^2 = j^2 = -1$, and $\mathbb{C}(i)$ denotes the set of complex numbers with imaginary unit i. The set $\mathbb{BC}$ forms a commutative ring with addition and multiplication defined by

$$Z + W = (\eta_1 + j\eta_2) + (w_1 + jw_2) = (\eta_1 + w_1) + j(\eta_2 + w_2),$$

and

$$ZW = (\eta_1 + j\eta_2)(w_1 + jw_2) = (\eta_1 w_1 - \eta_2 w_2) + j(\eta_1 w_2 + \eta_2 w_1).$$

Another important fact about bicomplex numbers is their *idempotent decomposition.* That is,

$$Z = e\gamma_1 + e^{\dagger}\gamma_2, \qquad \forall Z \in \mathbb{BC},$$

where $\gamma_1 = \eta_1 - i\eta_2$ and $\gamma_2 = \eta_1 + i\eta_2$, and where

$$e = \tfrac{1+ij}{2}, \qquad e^{\dagger} = \tfrac{1-ij}{2}$$

are mutually annihilating idempotents, i.e., $e + e^{\dagger} = 1$ and $e\,e^{\dagger} = 0$. The pair $\{e, e^{\dagger}\}$ forms the idempotent basis of $\mathbb{BC}$.

The representation of bicomplex numbers as pairs of complex numbers leads to three notions of conjugation: the *bar*-conjugation, the *dagger*-conjugation, and the *star*-conjugation (see [1]). Among these, we use the $*$-conjugation, defined by

$$Z^* = \overline{\eta_1} - j\overline{\eta_2} = e\,\overline{\gamma_1} + e^{\dagger}\,\overline{\gamma_2}.$$

A bicomplex number Z is invertible if $\|Z\|_k \neq 0$. In this context, we define the hyperbolic-valued modulus, also called the k-modulus, as

$$\|Z\|_k^2 = ZZ^*, \qquad Z^{-1} = \frac{Z^*}{\|Z\|_k^2}.$$

Taking the positive square root, we obtain

$$\|Z\|_k = e|\gamma_1| + e^{\dagger}|\gamma_2|.$$

For details, see [1]. The exponential and logarithmic representations of bicomplex functions yield the following remark.

Remark 1. For $\alpha \in \mathbb{R}$ and $\gamma_1, \gamma_2 > 0$, we have

$$\big(e\gamma_1 + e^{\dagger}\gamma_2\big)^{\alpha} = e\gamma_1^{\alpha} + e^{\dagger}\gamma_2^{\alpha}.$$

In particular,

$$\big(1 - \|Z\|_k^2\big)^{\alpha} = \big(e(1 - |\gamma_1|^2) + e^{\dagger}(1 - |\gamma_2|^2)\big)^{\alpha} = e(1 - |\gamma_1|^2)^{\alpha} + e^{\dagger}(1 - |\gamma_2|^2)^{\alpha}.$$

Next, we continue with the bicomplex holomorphic functions and their derivatives. Likewise the bicomplex numbers, the bicomplex functions also had its decompositions in terms of e and $e^\dagger$ and is unique in nature. That is, $F : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{BC}$, of one bicomplex variable Z is represented as follows:

$$F(Z) = eF_1(\beta_1) + e^\dagger F_2(\beta_2),$$

where $F_i's;\ i = 1, 2$ are usual complex valued functions.

Definition 1.2. *Let $F : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ and let $Z_0 \in \Omega$. The* derivative *of F at Z_0 is defined as*

$$F'_h(Z_0) = \lim_{\substack{H\to 0\\ H\notin \mathfrak{W}_0}} \frac{F(Z_0+H) - F(Z_0)}{H},$$

where $H = Z - Z_0$ is invertible and $\mathfrak{W}_0$ denotes the set of hyperbolic zero divisors (i.e., the null cone together with $0 \in \mathbb{BC}$).

If F is hyperbolically derivable at each $Z \in \Omega$, then F is called bicomplex holomorphic *in Ω.*

For further details, see [1, 9, 7].

Next, we define the bicomplex weighted Bergman space. For Bergman spaces with complex scalars, we refer to [1, 3, 4, 13].

Definition 1.3. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. The* bicomplex weighted Bergman space

$$A^p_\alpha(dV_\alpha)(\mathbb{U}_{\mathbb{BC}})$$

of the bidisk $\mathbb{U}_{\mathbb{BC}}$ is the space of bicomplex holomorphic functions $F : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{BC}$ that belong to the complete space $L^p_k(\mathbb{U}_{\mathbb{BC}}, dV_\alpha(Z))$, i.e.,

$$\int_{\mathbb{U}_{\mathbb{BC}}} \|F(Z)\|_k^p \, dV_\alpha(Z) < \infty,$$

where $\|F(Z)\|_k$ denotes the hyperbolic modulus of $F(Z)$, and the weighted measure $dV_\alpha(Z)$ is given by

$$dV_\alpha(Z) = \frac{\alpha+1}{4}\,(1-\|Z\|_k^2)^\alpha\, dx_1dy_1dx_2dy_2 = e\, dA_\alpha(\gamma_1)dA(\gamma_2) + e^\dagger\, dA(\gamma_1)dA_\alpha(\gamma_2),$$

with $e, e^\dagger$ being the idempotent components and dA_α the usual weighted area measure on the unit disk.

Lemma 1.1. [10] *Let $A \in \mathbb{U}_{\mathbb{BC}}$ and define the bicomplex Möbius transformation $S : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{U}_{\mathbb{BC}}$ by*

$$S(Z) = \zeta \frac{A - Z}{1 - A^* Z}, \qquad \text{with } \|\zeta\|_k = 1.$$

Then

$$(1 - \|Z\|_k^2)\|S'(Z)\|_k = 1 - \|S(Z)\|_k^2.$$

In particular, $\|S(Z)\|_k = 1$ if and only if $\|Z\|_k = 1$, i.e. Z belongs to the distinguished boundary of $\mathbb{U}_{\mathbb{BC}}$.

Recall that the Poincaré metric on $\mathbb{U}_{\mathbb{BC}}$ is denoted by ρ_k and defined by

$$\rho_k(Z, W) = \tfrac{1}{2} \log \frac{1 + \|\Upsilon_Z(W)\|_k}{1 - \|\Upsilon_Z(W)\|_k},$$

where $\Upsilon_Z : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{U}_{\mathbb{BC}}$ is a Möbius transformation and is given by

$$\Upsilon_Z(W) = \frac{Z - W}{1 - Z^* W}.$$

Definition 1.4. *The operator $P_{k,\alpha}$ denotes the weighted Bergman projection on $\mathbb{U}_{\mathbb{BC}}$. For any bicomplex-holomorphic function F,*

$$P_{k,\alpha}F(Z) = \int_{\mathbb{U}_{\mathbb{BC}}} \frac{F(W)}{\left(1 - ZW^*\right)^{2+\alpha}}\, dV_\alpha(W).$$

2. BICOMPLEX BLOCH SPACES

The bicomplex Bloch space was first introduced by Resendis and Tovar in [10]. The bicomplex Bloch space, denoted by $\mathfrak{B}_{\mathbb{BC}}$ in $\mathbb{U}_{\mathbb{BC}}$, is defined as the space of all holomorphic functions F on $\mathbb{U}_{\mathbb{BC}}$ such that

$$\|F\|_{\mathfrak{B}_{\mathbb{BC}}} = \sup\big\{(1 - \|Z\|_k^2)\, \|F'(Z)\|_k : Z \in \mathbb{U}_{\mathbb{BC}}\big\} < \infty,$$

and the hyperbolic norm is defined by

$$\|F\| = \|F(0)\|_k + \|F\|_{\mathfrak{B}_{\mathbb{BC}}}.$$

With this norm, $\mathfrak{B}_{\mathbb{BC}}$ is a Banach space.

Moreover, by Lemma 1.1, the Möbius invariance of $\|\cdot\|_{\mathfrak{B}_{\mathbb{BC}}}$ can be established. Indeed, let $F \in \mathfrak{B}_{\mathbb{BC}}$ and $\Phi \in \mathrm{Aut}(\mathbb{U}_{\mathbb{BC}})$. Then

$$\begin{aligned}
\|F \circ \Phi\|_{\mathfrak{B}_{\mathbb{BC}}} &= |F \circ \Phi(0)| + \sup\big\{(1 - \|Z\|_k^2)\, \|(F \circ \Phi)'(Z)\|_k : Z \in \mathbb{U}_{\mathbb{BC}}\big\} \\
&= |F(\Phi(0))| + \sup\big\{(1 - \|Z\|_k^2)\, \|F'(\Phi(Z))\|_k\, \|\Phi'(Z)\|_k : Z \in \mathbb{U}_{\mathbb{BC}}\big\} \\
&= |F(\Phi(0))| + \sup\big\{(1 - \|\Phi(Z)\|_k^2)\, \|F'(\Phi(Z))\|_k : \Phi(Z) \in \mathbb{U}_{\mathbb{BC}}\big\} \\
&= \|F\|_{\mathfrak{B}_{\mathbb{BC}}}.
\end{aligned}$$

Also, the bicomplex Bloch space has idempotent decomposition $\mathfrak{B}_{\mathbb{BC}} = e\mathfrak{B} + e^\dagger\mathfrak{B}$, where $\mathfrak{B}$ is the one-dimensional complex Bloch-space. So, any $F \in \mathfrak{B}_{\mathbb{BC}}$ has idempotent decomposition $F(Z) = eG_1(\gamma_1) + e^\dagger G_2(\gamma_2)$. Then

$$\begin{aligned}
\|F\|_{\mathfrak{B}_{\mathbb{BC}}} &= \sup\{(1 - \|Z\|_k^2)\|F'(Z)\|_k \ ; Z \in \mathbb{U}_{\mathbb{BC}}\} \\
&= \sup\{(e(1 - |\gamma_1|^2) + e^\dagger(1 - |\gamma_2|^2))(e|G_1'(\gamma_1)| + e^\dagger|G_2'(\gamma_2)|); \gamma_1 \in \mathbb{U}_1, \gamma_2 \in \mathbb{U}_2\} \\
&= \sup\{e(1 - |\gamma_1|^2)|G_1'(\gamma_1)| + e^\dagger(1 - |\gamma_2|^2)|G_2'(\gamma_2)|; \gamma_1 \in \mathbb{U}_1, \gamma_2 \in \mathbb{U}_2\} \\
&= e\sup\{(1 - |\gamma_1|^2)|G_1'(\gamma_1) \ ; \gamma_1 \in \mathbb{U}_1\} + e^\dagger \sup\{(1 - |\gamma_2|^2)|G_2'(\gamma_2)| \ ; \ \gamma_2 \in \mathbb{U}_2\} \\
&= e\|G_1\|_{\mathfrak{B}} + e^\dagger\|G_2\|_{\mathfrak{B}}.
\end{aligned}$$

The next proposition shows that every bounded holomorphic function on $\mathbb{U}_{\mathbb{BC}}$ belongs to the bicomplex Bloch space.

[10] $H^\infty_{\mathbb{BC}} \subset \mathfrak{B}_{\mathbb{BC}}$. Moreover, for all $F \in H^\infty_{\mathbb{BC}}$,

$$\|F\|_{\mathfrak{B}_{\mathbb{BC}}} \le \|F\|_{k,\infty}.$$

Let $F \in \mathfrak{B}_{\mathbb{BC}}$ and $Z, W \in \mathbb{U}_{\mathbb{BC}}$. Then

$$\|F(Z) - F(W)\|_k \le \tfrac{1}{2}\, \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \log \frac{1 + \|\Upsilon_Z(W)\|_k}{1 - \|\Upsilon_Z(W)\|_k},$$

where $\Upsilon_Z : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{U}_{\mathbb{BC}}$ is the Möbius transformation

$$\Upsilon_Z(W) = \frac{Z - W}{1 - Z^* W}.$$

Proof. Since F is bicomplex holomorphic, we have

$$F(Z) - F(0) = \int_0^1 \frac{d}{dl} F(lZ)\, dl = \int_0^1 F'(lZ)\, Z\, dl.$$

Taking k-modulus,

$$\|F(Z) - F(0)\|_k \leq \|Z\|_k \int_0^1 \|F'(lZ)\|_k\, dl.$$

By the definition of the Bloch norm,

$$\|F(Z) - F(0)\|_k \leq \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \int_0^1 \frac{\|Z\|_k}{1 - l^2 \|Z\|_k^2}\, dl.$$

Evaluating the integral gives

$$\|F(Z) - F(0)\|_k \leq \tfrac{1}{2} \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \log \frac{1 + \|Z\|_k}{1 - \|Z\|_k},$$

for all $Z \in \mathbb{U}_{\mathbb{BC}}$ (with $W = 0$).

Now replacing F by $F \circ \Upsilon_Z$ and Z by $\Upsilon_Z(W)$, and using the Möbius invariance of the Bloch norm, we obtain

$$\|(F \circ \Upsilon_Z)(\Upsilon_Z(W)) - (F \circ \Upsilon_Z)(0)\|_k \leq \tfrac{1}{2} \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \log \frac{1 + \|\Upsilon_Z(W)\|_k}{1 - \|\Upsilon_Z(W)\|_k}.$$

This is equivalent to

$$\|F(W) - F(Z)\|_k \leq \tfrac{1}{2} \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \log \frac{1 + \|\Upsilon_Z(W)\|_k}{1 - \|\Upsilon_Z(W)\|_k}.$$

□

Theorem 2.1. *Let $F : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{BC}$ be a bicomplex holomorphic function such that $F \in \mathfrak{B}_{\mathbb{BC}}$. Then*

$$\|F\|_{\mathfrak{B}_{\mathbb{BC}}} = \sup_{\substack{Z, W \in \mathbb{U}_{\mathbb{BC}} \\ Z \neq W \\ Z - W \notin \mathfrak{W}_0}} \frac{\|F(Z) - F(W)\|_k}{\rho_k(Z, W)}.$$

Proof. Since $F : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{BC}$ is holomorphic, for any $Z \in \mathbb{U}_{\mathbb{BC}}$ we have

$$F(Z) - F(0) = Z \int_0^1 F'(tZ)\, dt.$$

Hence

$$\begin{aligned}
\|F(Z) - F(0)\|_k &\leq \|Z\|_k \int_0^1 \|F'(tZ)\|_k\, dt \\
&\leq \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \int_0^1 \frac{\|Z\|_k}{1 - t^2 \|Z\|_k^2}\, dt \\
&= \tfrac{1}{2} \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \log \frac{1 + \|Z\|_k}{1 - \|Z\|_k} \\
&= \|F\|_{\mathfrak{B}_{\mathbb{BC}}}\, \rho_k(Z, 0).
\end{aligned}$$

Now replace F by $F \circ \Upsilon_Z$ and Z by $\Upsilon_Z(W)$. Using the Möbius invariance of both the Bloch norm and the Poincaré metric, we obtain

$$\|F(W) - F(Z)\|_k \leq \|F\|_{\mathfrak{B}_{\mathbb{BC}}} \rho_k(Z,W), \qquad Z, W \in \mathbb{U}_{\mathbb{BC}}.$$

Thus

$$\sup_{\substack{Z,W \in \mathbb{U}_{\mathbb{BC}} \\ Z \neq W \\ Z-W \notin \mathfrak{W}_0}} \frac{\|F(Z) - F(W)\|_k}{\rho_k(Z,W)} \leq \|F\|_{\mathfrak{B}_{\mathbb{BC}}}.$$

Conversely, define

$$A = \sup_{\substack{Z,W \in \mathbb{U}_{\mathbb{BC}} \\ Z \neq W \\ Z-W \notin \mathfrak{W}_0}} \frac{\|F(Z) - F(W)\|_k}{\rho_k(Z,W)},$$

and suppose $A < \infty$. Then for each $Z \in \mathbb{U}_{\mathbb{BC}}$,

$$\lim_{W \to Z} \frac{\|F(Z) - F(W)\|_k}{\rho_k(Z,W)} = (1 - \|Z\|_k^2)\|F'(Z)\|_k \leq A.$$

Therefore

$$\|F\|_{\mathfrak{B}_{\mathbb{BC}}} = \sup_{\substack{Z,W \in \mathbb{U}_{\mathbb{BC}} \\ Z \neq W \\ Z-W \notin \mathfrak{W}_0}} \frac{\|F(Z) - F(W)\|_k}{\rho_k(Z,W)}.$$

□

Corollary 2.1. [10, Theorem 4.3] *Let F be a bicomplex holomorphic function. Then $F \in \mathfrak{B}_{\mathbb{BC}}$ if and only if there exists a constant $C > 0$ such that*

$$\|F(Z) - F(W)\|_k \leq C\, \rho_k(Z,W), \qquad Z, W \in \mathbb{U}_{\mathbb{BC}}.$$

The bicomplex little Bloch space of $\mathbb{U}_{\mathbb{BC}}$ is denoted by $\mathfrak{B}_{o,\mathbb{BC}}$, and it is a closed subspace of $\mathfrak{B}_{\mathbb{BC}}$ consisting of holomorphic functions F such that

$$\lim_{\|Z\|_k \to 1^-} (1 - \|Z\|_k^2)\, \|F'(Z)\|_k = 0.$$

Moreover, $\mathfrak{B}_{o,\mathbb{BC}}$ is Möbius invariant, i.e., if $F \in \mathfrak{B}_{o,\mathbb{BC}}$ and $\Upsilon \in \mathrm{Aut}(\mathbb{U}_{\mathbb{BC}})$, then $F \circ \Upsilon \in \mathfrak{B}_{o,\mathbb{BC}}$.

Now we can prove the following theorem.

Theorem 2.2. *Let $F : \mathbb{U}_{\mathbb{BC}} \longrightarrow \mathbb{U}_{\mathbb{BC}}$ be a bicomplex holomorphic function with $F \in \mathfrak{B}_{\mathbb{BC}}$. Then $F \in \mathfrak{B}_{o,\mathbb{BC}}$ if and only if*

$$\|F_r - F\|_{\mathfrak{B}_{\mathbb{BC}}} \longrightarrow 0 \quad \text{as } r \to 1^-, \tag{2}$$

where $F_r(Z) = F(rZ)$ is the dilated function for all $Z \in \mathbb{U}_{\mathbb{BC}}$, and $r \in (0,1)_{\mathbb{D}}$.

Proof. Let $F_r = eF_{r_1,1} + e^{\dagger}F_{r_2,2}$ be the bicomplex dilated function in $\mathfrak{B}_{\mathbb{BC}}$, where $F_{r_1,1}$ and $F_{r_2,2}$ are dilations in the classical little Bloch space $\mathfrak{B}_o$. Also, let $F \in \mathfrak{B}_{\mathbb{BC}}$ have the decomposition

$$F = eF_1 + e^{\dagger}F_2, \tag{3}$$

with each $F_1, F_2 \in \mathfrak{B}$.

Suppose that equation (2) holds. Since each $F_{r_i,i} \in \mathfrak{B}_o$ for $i = 1, 2$, and because $\mathfrak{B}_{o,\mathbb{BC}}$ is closed in $\mathfrak{B}_{\mathbb{BC}}$, the convergence $\|F_r - F\|_{\mathfrak{B}_{\mathbb{BC}}} \to 0$ as $r \to 1^-$ implies that $F \in \mathfrak{B}_{o,\mathbb{BC}}$.

Conversely, suppose $F \in \mathfrak{B}_{o,\mathbb{BC}}$. We need to show that (2) holds. From the classical case, if $F_i \in \mathfrak{B}$, then $F_i \in \mathfrak{B}_o$ if and only if

$$\|F_{r_i,i} - F_i\|_{\mathfrak{B}} \longrightarrow 0 \quad \text{as } r \to 1^-.$$

Now,

$$\begin{aligned}\|F_r - F\|_{\mathfrak{B}_{\mathbb{BC}}} &= \|(eF_{r_1,1} + e^\dagger F_{r_2,2}) - (eF_1 + e^\dagger F_2)\|_{\mathfrak{B}_{\mathbb{BC}}}\\ &= \|e(F_{r_1,1} - F_1) + e^\dagger (F_{r_2,2} - F_2)\|_{\mathfrak{B}_{\mathbb{BC}}}\\ &\leq e\,\|F_{r_1,1} - F_1\|_{\mathfrak{B}} + e^\dagger\,\|F_{r_2,2} - F_2\|_{\mathfrak{B}}.\end{aligned}$$

As $\|F_{r_1,1} - F_1\|_{\mathfrak{B}} \to 0$ as $r \to 1^-$ and $\|F_{r_2,2} - F_2\|_{\mathfrak{B}} \to 0$ as $r \to 1^-$, we conclude that

$$\|F_r - F\|_{\mathfrak{B}_{\mathbb{BC}}} \longrightarrow 0 \quad \text{as } r \to 1^-,$$

for every $F \in \mathfrak{B}_{o,\mathbb{BC}}$. □

Lemma 2.2. *Let $F(Z,W)$ be a bounded and continuous bicomplex function on $\overline{\mathbb{U}}_{\mathbb{BC}} \times \mathbb{U}_{\mathbb{BC}}$. Then, for $\alpha > -1$ and $Z_0 \in \partial\mathbb{U}_{\mathbb{BC}}$, we have*

$$\lim_{Z \longrightarrow Z_0} \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - ZW^*}{(1 - Z^*W)^{2+\alpha}} F(Z,W)\,dV_\alpha(W) = \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - Z_0W^*}{(1 - Z_0^*W)^{2+\alpha}} F(Z_0,W)\,dV_\alpha(W). \tag{4}$$

Proof. Since $F(Z,W)$ is bicomplex holomorphic, we can write

$$F(Z,W) = eF_1(Z_1,W_1) + e^\dagger F_2(Z_2,W_2).$$

As $F(Z,W)$ is bounded and continuous on $\overline{\mathbb{U}}_{\mathbb{BC}} \times \mathbb{U}_{\mathbb{BC}}$, it follows that F_1 and F_2 are bounded and continuous on $\overline{\mathbb{U}}_1 \times \mathbb{U}_1$ and $\overline{\mathbb{U}}_2 \times \mathbb{U}_2$, respectively.

Let $Z_0 = eZ_{0,1} + e^\dagger Z_{0,2} \in \partial\mathbb{U}_{\mathbb{BC}}$. From the classical case (see [13]), we know that for $i = 1, 2$,

$$\lim_{Z_i \to Z_{0,i}} \int_{\mathbb{U}_i} \frac{1 - Z_i\overline{W_i}}{(1 - \overline{Z_i}W_i)^{2+\alpha}} F_i(Z_i,W_i)\,dA_\alpha(W_i) = \int_{\mathbb{U}_i} \frac{1 - Z_{0,i}\overline{W_i}}{(1 - \overline{Z_{0,i}}W_i)^{2+\alpha}} F_i(Z_{0,i},W_i)\,dA_\alpha(W_i). \tag{5}$$

Therefore,

$$\begin{aligned}&\lim_{Z \to Z_0} \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - ZW^*}{(1 - Z^*W)^{2+\alpha}} F(Z,W)\,dV_\alpha(W)\\ &= e \lim_{Z_1 \to Z_{0,1}} \int_{\mathbb{U}_1} \frac{1 - Z_1\overline{W}_1}{(1 - \overline{Z}_1W_1)^{2+\alpha}} F_1(Z_1,W_1)\,dA_\alpha(W_1)\\ &\quad + e^\dagger \lim_{Z_2 \to Z_{0,2}} \int_{\mathbb{U}_2} \frac{1 - Z_2\overline{W}_2}{(1 - \overline{Z}_2W_2)^{2+\alpha}} F_2(Z_2,W_2)\,dA_\alpha(W_2)\\ &= e \int_{\mathbb{U}_1} \frac{1 - Z_{0,1}\overline{W}_1}{(1 - \overline{Z}_{0,1}W_1)^{2+\alpha}} F_1(Z_{0,1},W_1)\,dA_\alpha(W_1)\\ &\quad + e^\dagger \int_{\mathbb{U}_2} \frac{1 - Z_{0,2}\overline{W}_2}{(1 - \overline{Z}_{0,2}W_2)^{2+\alpha}} F_2(Z_{0,2},W_2)\,dA_\alpha(W_2)\\ &= \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - Z_0W^*}{(1 - Z_0^*W)^{2+\alpha}} F(Z_0,W)\,dV_\alpha(W).\end{aligned}$$

□

For any $Z_o \in \partial\mathbb{U}_{\mathbb{BC}}$, the above lemma together with the reproducing property of the Bergman kernel gives

$$\int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - Z_o W^*}{(1 - Z_o^* W)^{2+\alpha}}\, dV_\alpha(W) = \lim_{Z \to Z_o} \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1 - ZW^*}{(1 - Z^* W)^{2+\alpha}}\, dV_\alpha(W) = \lim_{Z \to Z_o} \left(1 - \|Z\|_k^2\right) = 0. \tag{6}$$

Let $\mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$ denote the algebra of bicomplex-valued continuous functions on $\overline{\mathbb{U}}_{\mathbb{BC}}$, the Euclidean closure of $\mathbb{U}_{\mathbb{BC}}$. Furthermore, let $\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$ be the subalgebra of $\mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$ consisting of functions F such that $F(Z) \to 0$ as $\|Z\|_k \to 1^-$.

Theorem 2.3. *For every $F \in \mathfrak{B}_{\mathbb{BC}}$ and $\alpha > -1$, the following conditions are equivalent:*

(a) $F \in \mathfrak{B}_{o,\mathbb{BC}}$;
(b) $F = P_{k,\alpha}\Phi$ *for some* $\Phi \in \mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$;
(c) $F = P_{k,\alpha}\Phi$ *for some* $\Phi \in \mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$.

Proof. $(a) \Rightarrow (c)$: First suppose $F \in \mathfrak{B}_{o,\mathbb{BC}}$ and write

$$F(Z) = \sum_{n=0}^{\infty} C_n Z^n, \qquad Z \in \mathbb{U}_{\mathbb{BC}}.$$

Define

$$\Phi(Z) = \sum_{n=0}^{2m+1} \frac{\Gamma(n+\alpha+3)}{(\alpha+1)\Gamma(n+\alpha+2)} (1 - \|Z\|_k^2)\, C_n Z^n + \frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+m+1)} \sum_{n=2m+1}^{\infty} C_n\, n(n-1)\cdots(n-m+1) Z^{n-m}.$$

For $m = 1$ this becomes

$$\begin{aligned}\Phi(Z) =& \sum_{n=0}^{3} \frac{\Gamma(n+\alpha+3)}{(\alpha+1)\Gamma(n+\alpha+2)} (1 - \|Z\|_k^2) C_n Z^n \\ &+ \frac{(1 - \|Z\|_k^2)}{(\alpha+1)Z^*} \sum_{n=3}^{\infty} n C_n Z^{n-1} \\ =& (1 - \|Z\|_k^2)\frac{\alpha+2}{\alpha+1} C_0 + (1 - \|Z\|_k^2)\frac{\alpha+3}{\alpha+1} C_1 Z \\ &+ (1 - \|Z\|_k^2)\frac{\alpha+4}{\alpha+1} C_2 Z^2 + \frac{1}{\alpha+1} \sum_{n=3}^{\infty} \frac{n C_n Z^{n-1}}{Z^*} (1 - \|Z\|_k^2).\end{aligned} \tag{7}$$

Thus $\Phi \in \mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$ and $P_{k,\alpha}\Phi(Z) = F(Z)$, which gives (c), see [10].

$(b) \Rightarrow (a)$: Suppose that (b) holds, i.e., $F = P_{k,\alpha}\Phi$ for some $\Phi \in \mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$. Then

$$F(Z) = \int_{\mathbb{U}_{\mathbb{BC}}} \frac{\Phi(W)}{(1 - ZW^*)^{2+\alpha}}\, dV_\alpha(W), \qquad Z \in \mathbb{U}_{\mathbb{BC}}.$$

Differentiating under the integral gives

$$F'(Z) = (\alpha+2) \int_{\mathbb{U}_{\mathbb{BC}}} \frac{W^* \Phi(W)}{(1 - ZW^*)^{3+\alpha}}\, dV_\alpha(W), \qquad Z \in \mathbb{U}_{\mathbb{BC}}.$$

Hence

$$\|F'(Z)\|_k = (\alpha+2) \int_{\mathbb{U}_{\mathbb{BC}}} \frac{\|W^* \Phi(W)\|_k}{\|1 - ZW^*\|_k^{3+\alpha}}\, dV_\alpha(W).$$

Multiplying both sides by $(1-\|Z\|_k^2)$, we obtain

$$(1-\|Z\|_k^2)\|F'(Z)\|_k = (1-\|Z\|_k^2)\int_{\mathbb{U}_{\mathbb{BC}}} \frac{\|\Upsilon(W)\|_k}{\|1-ZW^*\|_k^{3+\alpha}}\, dV_\alpha(W),$$

where $\Upsilon(W) = (\alpha+2)W^*\Phi(W)$.

Using a change of variables and applying Lemma 2.2, for $Z_o \in \partial\mathbb{U}_{\mathbb{BC}}$, we get

$$\begin{aligned}\lim_{Z\to Z_o}(1-\|Z\|_k^2)\|F'(Z)\|_k &= \Upsilon(Z_o)\int_{\mathbb{U}_{\mathbb{BC}}} \frac{\|1-Z_oW^*\|_k}{\|1-Z_o^*W\|_k^{2+\alpha}}\, dV_\alpha(W)\\ &= 0,\end{aligned}$$

which implies $F \in \mathfrak{B}_{o,\mathbb{BC}}$.

$(c) \Rightarrow (a)$: This follows immediately. □

Theorem 2.4. *For any $F \in \mathfrak{B}_{\mathbb{BC}}$, an integer $n \geq 2$, and $\alpha > -1$, the following conditions are equivalent:*

(a) $F \in \mathfrak{B}_{o,\mathbb{BC}}$;
(b) $(1-\|Z\|_k^2)^n\|F^n(Z)\|_k \in \mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$;
(c) $(1-\|Z\|_k^2)^n\|F^n(Z)\|_k \in \mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$.

Proof. $(c) \Rightarrow (a)$: Consider $F(Z) = \sum_{n=0}^{\infty} C_n Z^n$ and define

$$\begin{aligned}T(Z) = &\sum_{n=0}^{2m+1} \frac{\Gamma(n+3+\alpha)}{\Gamma(n+2+\alpha)(\alpha+1)}(1-\|Z\|_k^2)^n C_n Z^n\\ &+ \frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(m+1+\alpha)}(1-\|Z\|_k^2)^{mn}\sum_{n=2m+1}^{\infty}\frac{C_n n(n-1)\cdots(n-m+1)Z^{n-m}}{(Z^*)^m}.\end{aligned}$$

If $(1-\|Z\|_k^2)^n\|F^n(Z)\|_k \in \mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$, then

$$\Phi(Z) = \frac{\Gamma(2+\alpha)}{(\alpha+1)\Gamma(m+1+\alpha)}(1-\|Z\|_k^2)^{mn}\sum_{n=2m+1}^{\infty}\frac{C_n n(n-1)\cdots(n-m+1)Z^{n-m}}{(Z^*)^m} \in \mathcal{C}(\overline{\mathbb{U}}_{\mathbb{BC}}).$$

Then, by Theorem 2.3, $F(Z) = P_{k,\alpha}T \in \mathfrak{B}_{o,\mathbb{BC}}$.

$(b) \Rightarrow (c)$ is trivial.

$(a) \Rightarrow (b)$: Let $F \in \mathfrak{B}_{o,\mathbb{BC}}$. By Theorem 2.3, there exists $\Phi \in \mathcal{C}_o(\overline{\mathbb{U}}_{\mathbb{BC}})$ such that $F = P_{k,\alpha}\Phi$, i.e.,

$$F(Z) = \int_{\mathbb{U}_{\mathbb{BC}}} \frac{\Phi(W)}{(1-ZW^*)^{2+\alpha}}\, dV_\alpha(W), \quad Z \in \mathbb{U}_{\mathbb{BC}}. \tag{8}$$

Taking the n-th derivative gives

$$\begin{aligned}F^n(Z) &= (n+1)!\int_{\mathbb{U}_{\mathbb{BC}}} \frac{W^{*n}\Phi(W)}{(1-ZW^*)^{n+2+\alpha}}\, dV_\alpha(W)\\ &= \int_{\mathbb{U}_{\mathbb{BC}}} \frac{\Upsilon(W)}{(1-ZW^*)^{n+2+\alpha}}\, dV_\alpha(W),\end{aligned}$$

where $\Upsilon(W) = (n+1)!W^{*n}\Phi(W) \in \mathcal{C}_o(\overline{\mathbb{U}}_{\mathbb{BC}})$. Then

$$\|F^n(Z)\|_k = \int_{\mathbb{U}_{\mathbb{BC}}} \frac{\|\Upsilon(W)\|_k}{\|1-ZW^*\|_k^{n+2+\alpha}}\, dV_\alpha(W).$$

Multiplying both sides by $(1-\|Z\|_k^2)^n$, we get

$$(1-\|Z\|_k^2)^n\|F^n(Z)\|_k = (1-\|Z\|_k^2)^n\int_{\mathbb{U}_{\mathbb{BC}}}\frac{\|\Upsilon(W)\|_k}{\|1-ZW^*\|_k^{n+2+\alpha}}\,dV_\alpha(W).$$

If $Z_o\in\partial\mathbb{U}_{\mathbb{BC}}$, then $\Upsilon(Z_o)=0$, and the bicomplex dominated convergence theorem [?] implies

$$\lim_{Z\to Z_o}(1-\|Z\|_k^2)^n\|F^n(Z)\|_k = 0.$$

Hence, $(1-\|Z\|_k^2)^n\|F^n(Z)\|_k\in\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$. □

Remark 2. From [10, Theorem 3.11], a simple construction shows that if m is a non-negative integer and $F(Z)=\sum_{n=2m+1}^{\infty}C_nZ^n$, then

$$P_{k,\alpha}\left(\frac{(1-\|Z\|_k^2)}{Z^{*m}}F^{(m)}(Z)\right)=(\alpha+1)\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+2)}\sum_{n=2m+1}^{\infty}C_nZ^n.$$

Moreover,

$$P_{k,\alpha}\left(\frac{(1-\|Z\|_k^2)}{Z^{*m}}F^{(m)}(Z)\right)=\int_{\mathbb{U}_{\mathbb{BC}}}\frac{(1-\|W\|_k^2)^mF^{(m)}(W)}{W^{*m}(1-ZW^*)^{2+\alpha}}\,dV_\alpha(W),$$

which gives the formula

$$F(Z)=F(0)+\int_{\mathbb{U}_{\mathbb{BC}}}\frac{(1-\|W\|_k^2)^mF^{(m)}(W)}{W^{*m}(1-ZW^*)^{2+\alpha}}\,dV_\alpha(W).$$

In particular, for $m=1$, we have

$$F(Z)=F(0)+\int_{\mathbb{U}_{\mathbb{BC}}}\frac{(1-\|W\|_k^2)F'(W)}{W^{*}(1-ZW^*)^{2+\alpha}}\,dV_\alpha(W).$$

Differentiating under the integral sign, we obtain

$$F''(0)=(2+\alpha)(3+\alpha)\int_{\mathbb{U}_{\mathbb{BC}}}W^*(1-\|W\|_k^2)F'(W)\,dV_\alpha(W),$$

i.e., $\|F''(0)\|_k\le(2+\alpha)(3+\alpha)\|F\|_{\mathfrak{B}_{\mathbb{BC}}}$.

Thus, the function defined in equation (7),

$$E(Z)=(1-\|Z\|_k^2)\left(\frac{\alpha+2}{\alpha+1}C_0+\frac{\alpha+3}{\alpha+1}C_1Z+\frac{\alpha+4}{\alpha+1}C_2Z^2+\frac{1}{\alpha+1}\sum_{n=3}^{\infty}\frac{nC_nZ^{n-1}}{Z^*}\right),$$

satisfies

$$\|E\|_{k,\infty}\le M\|F\|=M\big(\|F\|_{\mathfrak{B}_{\mathbb{BC}}}+\|F(0)\|_k\big),$$

where M is an absolute constant. Therefore, each $F\in\mathfrak{B}_{\mathbb{BC}}$ implies $F\in\mathfrak{B}_{o,\mathbb{BC}}$. We can choose $E\in L_k^\infty(\mathbb{U}_{\mathbb{BC}})$ (respectively in $\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$), and using the linearity of the Bergman projection, we have

$$P_{k,\alpha}E(Z)=F(Z),\quad\text{and}\quad\|E\|_{k,\infty}\le M\|F\|.$$

Theorem 2.5. [10] *Let $P_{k,\alpha}$ be the weighted Bergman projection with $-1<\alpha<\infty$. Then:*

1. $P_{k,\alpha}$ *maps* $L_k^\infty(\mathbb{U}_{\mathbb{BC}})$ *boundedly onto* $\mathfrak{B}_{\mathbb{BC}}$*;*
2. $P_{k,\alpha}$ *maps* $\mathbb{C}(\overline{\mathbb{U}}_{\mathbb{BC}})$ *boundedly onto* $\mathfrak{B}_{o,\mathbb{BC}}$*;*
3. $P_{k,\alpha}$ *maps* $\mathbb{C}(\mathbb{U}_{\mathbb{BC}})$ *boundedly onto* $\mathfrak{B}_{o,\mathbb{BC}}$*.*

Lemma 2.3. *The bicomplex operator* $T_k = T_{t,k,\alpha}$ *defined by*

$$T_kF(Z) = (1-\|Z\|_k^2)^t \int_{\mathbb{U}_{\mathbb{BC}}} \frac{(1-\|W\|_k^2)^\alpha}{(1-ZW^*)^{2+t+\alpha}} F(W)\,dV(W) \tag{9}$$

has the following properties:

(a) $T_kP_{k,\alpha} = T_k$;
(b) $T_k = (\alpha+t+1)T_k^2$;
(c) $P_{k,\alpha} = (\alpha+t+1)P_{k,\alpha}T_k$;
(d) T_k *is a bounded embedding of* $\mathfrak{B}_{\mathbb{BC}}$ *into* $L_k^\infty(\mathbb{U}_{\mathbb{BC}})$;
(e) T_k *is an embedding of* $\mathfrak{B}_{o,\mathbb{BC}}$ *into* $\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$.

Proof. We have

$$T_kF(Z) = (1-\|Z\|_k^2)^t \int_{\mathbb{U}_{\mathbb{BC}}} \frac{(1-\|W\|_k^2)^\alpha}{(1-ZW^*)^{2+t+\alpha}} F(W)\,dV(W). \tag{10}$$

For $Z = e\gamma_1 + e^\dagger\gamma_2$ and $W = eW_1 + e^\dagger W_2$, we have

$$\begin{aligned} T_kF(Z) &= e(1-|\gamma_1|^2)^t \int_{\mathbb{U}_1} \frac{F_1(W_1)}{(1-\gamma_1\overline{W_1})^{2+\alpha+t}} dA_\alpha(W_1) \\ &\quad + e^\dagger(1-|\gamma_2|^2)^t \int_{\mathbb{U}_2} \frac{F_2(W_2)}{(1-\gamma_2\overline{W_2})^{2+\alpha+t}} dA_\alpha(W_2) \\ &= eT_1F_1(\gamma_1) + e^\dagger T_2F_2(\gamma_2). \end{aligned} \tag{11}$$

Using the decomposition in (11), properties (a), (b), and (c) follow easily. The proof of (d) is analogous to the complex case.

For (e), let $F \in \mathfrak{B}_{o,\mathbb{BC}}$. Then $F = P_{k,\alpha}E$ for some $E \in \mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$. By Remark 2, we can choose E such that

$$\|E\|_{k,\infty} \le M\big(\|F\|_{\mathfrak{B}_{\mathbb{BC}}} + \|F(0)\|_k\big),$$

where M is an absolute constant. Define $H = E \circ \Upsilon_Z$. Then

$$T_kE(Z) = \int_{\mathbb{U}_{\mathbb{BC}}} E(\Upsilon_Z(W))\,dV_\alpha(W).$$

Since $E(\Upsilon_Z(W)) \to 0$ as $\|Z\|_k \to 1^-$ for each $W \in \mathbb{U}_{\mathbb{BC}}$, the bicomplex dominated convergence theorem implies $T_kE(Z) \to 0$ as $\|Z\|_k \to 1^-$. Hence $T_kE \in \mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$. But

$$T_kP_{k,\alpha}E = T_kE = T_kF,$$

so T_k maps $\mathfrak{B}_{o,\mathbb{BC}}$ into $\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$.

To see that T_k is bounded, observe from (9) that

$$\|T_kE(Z)\|_k \le (1-\|Z\|_k^2)^t \|E\|_{k,\infty} \int_{\mathbb{U}_{\mathbb{BC}}} \frac{1}{\|1-W^*Z\|_k^{2+t+\alpha}}\,dV_\alpha(W),$$

and therefore

$$\|T_kF\|_{k,\infty} = \|T_kE\|_{k,\infty} \le \|E\|_{k,\infty} \le M\big(\|F\|_{\mathfrak{B}_{\mathbb{BC}}} + \|F(0)\|_k\big).$$

Finally, using part (a) and Theorem 2.5, we have for all $F \in \mathfrak{B}_{o,\mathbb{BC}}$:

$$\begin{aligned} \|F\|_{\mathfrak{B}_{\mathbb{BC}}} + \|F(0)\|_k &= \|P_{k,\alpha}F\| \\ &= (\alpha+t+1)\|P_{k,\alpha}T_kF\| \\ &\le (\alpha+t+1)\|P_{k,\alpha}\|\,\|T_kF\|_\infty. \end{aligned}$$

Hence T_k is bounded below and maps every function in $\mathfrak{B}_{o,\mathbb{BC}}$ into $\mathcal{C}_o(\mathbb{U}_{\mathbb{BC}})$. □

3. DUAL OF LITTLE BLOCH SPACE

In this section, we introduce the dual of the bicomplex little-Bloch space. The dual space of a Banach space X is denoted by X^* and consists of all bounded linear functionals on X. Each $\hat{F} \in X^*$ has the norm

$$\|\hat{F}\|_{k,\alpha} = \sup_{\|F\|_k=1} \|\hat{F}(F)\|_k.$$

We denote the space of all bounded linear functionals $\hat{F}$ on the bicomplex little-Bloch space by $\mathfrak{B}^*_{o,\mathbb{BC}}$. A linear functional $\hat{F}$ on $\mathfrak{B}_{o,\mathbb{BC}}$ is bounded if there exists a positive constant C such that

$$\|\hat{F}(F)\|_k \leq C\|F\|_{k,\alpha}, \quad \forall F \in \mathfrak{B}_{o,\mathbb{BC}}.$$

The next theorem shows that the dual of the little-Bloch space is the weighted bicomplex Bergman space $A^1_\alpha(dV_\alpha)(\mathbb{BC})$.

Theorem 3.6. *We have*

$$\mathfrak{B}^*_{o,\mathbb{BC}} \cong A^1_\alpha(dV_\alpha)(\mathbb{BC}), \quad \alpha > -1,$$

under the integral pairing

$$\langle F, G\rangle_{\alpha,\mathbb{BC}} = \int_{\mathbb{U}_{\mathbb{BC}}} F(Z)(G(Z))^* \, dV_\alpha(Z). \tag{12}$$

Proof. Let $F \in A^1_\alpha(dV_\alpha)(\mathbb{BC})$. Then, by (12),

$$\begin{aligned}\langle F, G\rangle_{\alpha,\mathbb{BC}} &= \int_{\mathbb{U}_{\mathbb{BC}}} F(Z)(G(Z))^* \, dV_\alpha(Z) \\ &= e\int_{\mathbb{U}_1} F_1(\gamma_1)\overline{G_1(\gamma_1)} \, dA_\alpha(\gamma_1) + e^\dagger \int_{\mathbb{U}_2} F_2(\gamma_2)\overline{G_2(\gamma_2)} \, dA_\alpha(\gamma_2),\end{aligned}$$

where $G_i \mapsto \int_{\mathbb{U}_i} F_i(\gamma_i)\overline{G_i(\gamma_i)} \, dA_\alpha(\gamma_i)$, $i = 1, 2$, defines a bounded linear functional on the classical little-Bloch space $\mathfrak{B}_o$. Since

$$\mathfrak{B}_{o,\mathbb{BC}} = e\mathfrak{B}_o + e^\dagger\mathfrak{B}_o,$$

it follows that $G \mapsto \int_{\mathbb{U}_{\mathbb{BC}}} F(Z)(G(Z))^* \, dV_\alpha(Z)$ also defines a bounded linear functional on $\mathfrak{B}_{o,\mathbb{BC}}$.

Conversely, let $\mathfrak{F} \in \mathfrak{B}^*_{o,\mathbb{BC}}$. Then we need to show that there exists $F \in A^1_\alpha(dV_\alpha)(\mathbb{BC})$ such that

$$\mathfrak{F}(G) = \int_{\mathbb{U}_{\mathbb{BC}}} G(Z)(F(Z))^* \, dV_\alpha(Z), \quad \forall G \in \mathfrak{B}_{o,\mathbb{BC}}.$$

Since $\mathfrak{F}$ is bicomplex linear, it admits the decomposition

$$\mathfrak{F} = e\mathfrak{F}_1 + e^\dagger\mathfrak{F}_2,$$

where each $\mathfrak{F}_i$ is a bounded linear functional on the classical little-Bloch space $\mathfrak{B}_o$. Therefore, for each $i = 1, 2$, there exists $F_i \in A^1_\alpha(dA_\alpha)$ such that

$$\mathfrak{F}_i(G_i) = \int_{\mathbb{U}_i} G_i(\gamma_i)\overline{F_i(\gamma_i)} \, dA_\alpha(\gamma_i), \quad \forall G_i \in \mathfrak{B}_o.$$

Define $F \in A^1_\alpha(dV_\alpha)(\mathbb{BC})$ by

$$F(Z) = eF_1(\gamma_1) + e^\dagger F_2(\gamma_2).$$

Then, for $G = eG_1 + e^\dagger G_2 \in \mathfrak{B}_{o,\mathbb{BC}}$, we have

$$\begin{aligned}\mathfrak{F}(G) &= e\mathfrak{F}_1(G_1) + e^\dagger\mathfrak{F}_2(G_2)\\ &= e\int_{\mathbb{U}_1} G_1(\gamma_1)\overline{F_1(\gamma_1)}\, dA_\alpha(\gamma_1) + e^\dagger\int_{\mathbb{U}_2} G_2(\gamma_2)\overline{F_2(\gamma_2)}\, dA_\alpha(\gamma_2)\\ &= \int_{\mathbb{U}_{\mathbb{BC}}} G(Z)(F(Z))^*\, dV_\alpha(Z),\end{aligned}$$

which completes the proof. □

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