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A Novel Robust M-Estimator for the Random-Coefficients Regression Model: Simulation and Its Application in Energy Management Systems

Amr R. Kamel[1,2,*], **Mohamed R. Abonazel**[1], **Ahmed H. Youssef**[1]

[1] Department of Applied Statistics and Econometrics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, 12613, Egypt.

[2] Department of Basic Sciences, El-Gazeera High Institute for Computers and Information Systems, Ministry of Higher Education, El-Mokattam 11571, Egypt.

* **Correspondence:** amr_ragab@pg.cu.edu.eg

Abstract: A random coefficient regression (RCR) model refers to a statistical model that incorporates random coefficients into degradation models, typically assumed to be normally distributed. An RCR model is a special type of panel data model. The RCR model provides a wide range of consequences for situations involving decision-making difficulties. The classical estimation methods for the RCR model perform well without outliers, but their performance degrades in the presence of outliers. To this end, this paper proposes a novel robust M-estimator with different objective functions and compares these with the non-robust (classical) estimators. The proposed robust M-estimators provide stable and reliable results even when outliers are present. A Monte Carlo simulation study and an empirical application to energy management systems were conducted to evaluate the performance of the non-robust RCR classical pooling (RCRCP) estimator, RCR mean group (RCRMG) estimator, and RCR Swamy's (RCRSW) estimator, with the proposed robust M-estimators: RCR Huber (RCRHU), RCR Hampel (RCRHM), and RCR Bisquare (RCRBI). The findings from the simulation and application indicate that the proposed robust M-estimators outperform the non-robust estimators in the presence of outliers in the RCR model. Furthermore, the RCRBI estimator is more efficient than the other proposed robust M-estimators.

Keywords: Mean group estimator, Monte Carlo simulation, Non-robust estimators, Outliers, Panel data models, RCR model, Novel robust M-estimator, Swamy's Estimator.

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1. Introduction

Modeling the linear relationship between a response variable and specific explanatory variables has garnered considerable attention in statistics lately. For this reason, econometrics often uses time series data that represent a single entity, usually an economy or market. Another category of data is referred to as ”panel data” in the literature on econometrics; in the biological sciences, it is also occasionally referred to as longitudinal data. Cross-sectional measurements are tracked over time in panel data, a type of two-dimensional data. As noted by Hsiao [1], this kind of data usually allows us to account for both intra-individual dynamics and the unobserved individual-specific variability. As a result, these estimates typically offer a more thorough source of variance that is more educational, allowing for more accurate model parameter estimation and reliable testing of more intricate behavioral models with less constrictive assumptions.

Panel data models encompass information in two dimensions, including time series (T) and cross-sectional (N). Generally speaking, the panel's time dimension (T) it is short and its cross-sectional dimension (N) it is very large. We search for estimation consistency along the dimension (N) in that case. This is the case because the research's primary focus is variability between units, and panel data are frequently used for cross-sectional analysis. In panel data research, three primary sources of variability are usually considered: (*i*) within variation, which is the variance across both dimensions; (*ii*) between variation; and (*iii*) within variation, which is the fluctuation from observation to observation in each cross-sectional unit. The statistical appeal of panel data models is frequently derived from their capacity to control individual variability while focusing particularly on explaining time-varying variations. Additionally, Baltagi [2] and Hsiao [3] provide a list of some advantages and disadvantages of using a panel dataset.

The best linear unbiased estimator (BLUE) for panel data models and pooled cross-sectional data is the classical pooling (CP) estimator, based on the standard assumptions of the general linear regression model. Panel data models are based on the fundamental idea that the people in our dataset are drawn from a population that shares a regression coefficient vector. This means that the coefficients in panel data models should be fixed. This assumption is not met by the majority of economic models, as Livingston et al. [4] and Alcácer et al. [5] have shown. This model is referred to as the ”random coefficients panel data (RCPD) model” if this assumption is loosened. Swamy has done a lot of research on this RCPD model [6, 7, 8]. Furthermore, this model is also known as the random coefficient regression (RCR) model or Swamy's model in a number of statistical and economic papers; see, e.g., [9, 10, 11, 12, 13].

Econometric models that allow for various methods of parameter modification have been created in recent years. Swamy [7] has produced an asymptotically efficient approach for estimating the parameters in a broad RCR model after comparing all of these various schemes. These estimating techniques are only now being used for the examination of real-world data; for further information, see Fcige and Swamy [14]. The standard fixed-parameter regression model's applicability in analyzing cross-sectional data is limited by the diversity of individual decision units, which implies parameter change between units, as many econometricians have acknowledged. Micro panel data econometric analyses using RCR approaches have been demonstrated to yield more informative and fruitful results [15].

The RCR models have been applied across various domains and provide a unifying framework for addressing many statistical challenges. Practical applications encompass a wide range of disciplines, including behavioral and social sciences, economics, finance, and many others. A few consistent estimators for the coefficient mean and variances were proposed by Hildreth and Houck [16] and Swamy [6] from a parametric perspective, assuming that the covariances vanish. They also looked at appropriate linear models. Furthermore, as the works of Holzmann and Meister [17] and Lewbel and Pendakur [18] show, a great deal of research has been done in the last several decades in the areas of non-parametric, non-linear identification and estimation of the joint distribution of the coefficient. The CP and Swamy methods are typically used to draw statistical conclusions regarding the parameters of RCR models. In order to get reliable estimates of the RCR model parameters, traditional estimating methods, however, require specific assumptions that are rarely met in practice, such as stringent homogeneity and homoscedasticity of the error terms. Therefore, any departure from the model's assumptions or the presence of outliers may significantly affect the CP and RCR estimators. In addition, the CP and RCR estimators are highly sensitive to the leverage points due to the distortions caused by the outliers in the variables involved. Consequently, these estimators, along with popular estimators like the generalized least squares (GLS) estimator for random and fixed models based on several transformations, may yield imprecise and untrustworthy results.

In order to get over these problems, we looked into alternatives to conventional estimators in order to create extremely reliable techniques with a high breakdown point (BP). It is believed that robust methods are the only workable way to deal with outliers. When outliers are present in RCR models, robust estimates are necessary to detect them and generate reliable, resilient results. To our knowledge, very little research has been done on the resilience of conventional estimating methods when used with static linear panel data models, see Kamel [19]. Other noteworthy references on robust multivariate regression are the M-estimation of seemingly unrelated regression equations (SURE) model by Kamel [19] and the adaptation of S-estimator and MM-estimator to the SURE model by Youssef et al. [20, 21].

The main aim of this paper is to introduce a novel robust M-estimator that excels in performance across various contaminated datasets while maintaining high efficiency. This proposed robust M-estimator surpasses traditional methods in both efficiency and robustness, delivering impressive results. To accomplish this, we analyze the efficiency of three proposed robust M-estimators' objective functions in RCR models containing outliers, comparing them with the non-robust estimators. This is done through a Monte Carlo simulation study and a practical application on an energy management systems dataset.

This paper is structured as follows. In the beginning, we introduced the basic notation needed throughout the paper. In Section 2, we give a brief overview of the RCR model specifications and assumptions. Section 3 examines the classical estimation methods of the RCR model in our study. In Section 4, we introduce the outlier identification and repercussions in the RCR model. Section 5 offers a novel robust M-Estimator with an algorithm of the previously discussed RCR models that may be used to reduce the impact of outlier occurrences. The Monte Carlo simulation study and application of energy management systems are described in Sections 6 and 7, respectively. Finally, Section 8 provides some conclusions and recommendations.

2. The RCR Model and Assumptions

The RCR models provide a wide range of consequences for situations involving decision-making difficulties. A choice in the ordinary least squares model affects only the mean, whereas in the RCR model, it impacts both the mean and the variance. In this case, we loosen the typical regression assumptions by allowing the development factors to vary randomly. We consider the Hildreth and Houck [16] model, in which the response coefficients in a broad linear model are treated as random variables, and the mean of their distribution can be determined.

Consider the observations for N cross-sectional units across T periods. Assume the response variable y for the i-th unit at the time t is expressed as a linear function of K strictly explanatory variables, denoted as x_{kit}, $k = 1, ..., K$, in the following form:

$$y_{it} = \sum_{k=1}^{K} \beta_{ki} x_{kit} + \varepsilon_{it} = \mathbf{x}_{it}\boldsymbol{\beta}_i + \varepsilon_{it}, \quad i = 1, \ldots, N; \quad t = 1, \ldots, T \tag{2.1}$$

where β_i is the $K \times 1$ vector of coefficients, x_{it} is a $1 \times K$ vector of explanatory variables, and ε_{it} denotes the random error term. With time, the stacking Equation (2.1) model mentioned above yields

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i, \tag{2.2}$$

where $y_i = (y_{i1}, ..., y_{iT})'$, $X_i = (x'_{i1}, ..., x'_{iT})'$, $\beta_i = (\beta_{i1}, ..., \beta_{iK})'$, and $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{iT})'$. The ordinary least squares (OLS) approach can be used to estimate distinct equation regressions for each individual unit when the performance of a single individual from the panel data is of interest. The OLS estimator will be the BLUE under the following assumptions:

A 1. The mean of the random error term is equal to zero, i.e., $E(\varepsilon_i) = 0;\ \forall\ i = 1, ..., N.$

A 2. The random error term has the same variance:

$$E(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}'_j) = \begin{cases} \sigma_\varepsilon^2 \mathbf{I}_T & \text{for } i = j, \\ \mathbf{0} & \text{for } i \neq j, \end{cases} \quad i, j = 1, \ldots, N. \tag{2.3}$$

A 3. The explanatory variables (X_i) are fixed in repeated samples (non-stochastic) with full column rank, i.e., $\text{rank}(X_i) = K,\ N > K$ and $T > K;\ \forall\ i = 1, ..., N.$

These requirements don't have to be met in order for the OLS estimator to be optimum. Zellner [22] referred to this as the SURE model, in which the equations are coupled by the random errors' cross-equation correlation, if the covariance's between ε_i and ε_j $(i, j = 1, 2, \ldots, N)$ do not equal zero as in assumption **(A2)** above, then the contemporaneous correlation is existing. If contemporaneous correlation occurs and the X_i $(i = 1, 2, \ldots, N)$ matrices do not cover the same column space, the generalized least squares (GLS) estimation method applied to the full system of equation is a significantly more efficient estimator of β_i than the equation-by-equation OLS, as proved by Zellner [22]. Therefore, with a shared regression parameter $\bar{\beta}$, which is a constant component, and a random component γ_i, which will enable the coefficients to vary from unit to unit, we assume that the individuals in our panel data are drawn from a population.

A 4. We suppose that the vector of coefficients β_i is categorized as follows for the stationary random coefficient technique:

$$\boldsymbol{\beta}_i = \bar{\boldsymbol{\beta}} + \boldsymbol{\gamma}_i, \tag{2.4}$$

where $\bar{\beta}$ is the $K \times 1$ vector of coefficients, and γ_i is the $K \times 1$ vector of stationary random variables with zero means and constant variance-covariances as:

$$E(\boldsymbol{\gamma}_i) = \mathbf{0} \quad \text{and} \quad E(\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j') = \begin{cases} \boldsymbol{\Psi} & \text{for } i = j \\ \mathbf{0} & \text{for } i \neq j \end{cases}, \quad i, j = 1, \ldots, N,$$

where $\Psi = \text{diag}\{\phi_k^2\}$; for $k = 1, ..., K$, where $K < N$. Furthermore, $E(\gamma_i x_{jt}) = 0$ and $E(\gamma_i \varepsilon_{jt}) = 0 \; \forall \; i$ and j. The model in Equation (2.2) may be rewritten as follows with this assumption:

$$Y = X\bar{\beta} + e, \quad e = D\gamma + \varepsilon \tag{2.5}$$

where $Y = (y_1', ..., y_N')'$, $X = (X_1', ..., X_N')'$, $\bar{\beta} = (\bar{\beta}_1, ..., \bar{\beta}_K)'$, $\varepsilon = (\varepsilon_1', ..., \varepsilon_N')'$, while $\gamma = (\gamma_1', ..., \gamma_N')'$, and $D = \text{diag}\{X_i\}$; for $i = 1, ..., N$.

A 5. Furthermore, Swamy [6] assumed that the errors have different variances among individuals:

$$E(\varepsilon_i\varepsilon_j') = \begin{cases} \sigma_{ii} I_T & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}, \quad i, j = 1, ..., N.$$

Under all the above assumptions, the model in Equation (2.5) is called the RCR model.

3. The Classical Methods of Estimation

The RCR models extend the standard regression framework, sometimes called mixed-effects models or hierarchical linear models, which permit the regression coefficients to differ among individuals or groups. This recognizes that the independent and dependent variables may not have a consistent connection across the population. The conventional approaches of RCR model estimation are covered in this section. After estimating the model parameters, statistical inference can be carried out. This entails doing hypothesis tests to determine the significance of the effects and computing standard errors for the estimated coefficients and variance components. Wald tests and likelihood ratio tests are frequently employed when assessing hypotheses in RCR models. RCR model parameters can be estimated using a variety of techniques, the most popular as follows:

3.1. Classical Pooling Estimator

A simple technique for estimating a common link across several groups or periods in econometrics, especially when working with panel data, is the classical pooling (CP) estimator. In essence, it ignores the unique group or time-specific properties and does a standard regression on all the data as if it were a single, huge sample. The individuals in our database were selected from a population having a common regression vector of coefficients $\bar{\beta}$. When all of their coefficients are equal ($\beta_1 = \beta_2 = \cdots = \beta_N = \bar{\beta}$). In this case, a more efficient estimate of $\bar{\beta}$ can be obtained by pooling the observations for each individual and performing a single regression. The CP estimator of $\bar{\beta}$ is therefore provided by;

$$\hat{\bar{\beta}}_{RCRCP} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}, \tag{3.1}$$

The variance-covariance matrix of the estimated coefficients $\hat{\bar{\beta}}_{RCRCP}$ is given by:

$$\text{Var}(\hat{\hat{\boldsymbol{\beta}}}_{RCRCP}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\delta}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}, \tag{3.2}$$

where;

$$\delta = \begin{bmatrix} \sigma_1^2 I_T & 0 & \cdots & 0 \\ 0 & \sigma_2^2 I_T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 I_T \end{bmatrix}$$

where I is an $(NT \times NT)$ identity matrix, the unknown parameters σ_i^2 can be consistently estimated by:

$$\hat{\sigma}_i^2 = \frac{1}{(T-K)} \sum_{t=1}^{T} \hat{\varepsilon}_{it}^2 \quad \text{for } i = 1, ..., N; \ t = 1, ..., T,$$

where $\hat{\varepsilon}_{it}$ are the residuals derived from solving an equation i using OLS estimation. A straightforward but frequently unsuitable technique for panel data is the RCRCP estimator. It overlooks important facets of the data structure, which could produce skewed and inaccurate conclusions. Unless you have good evidence that your data fulfills the tight assumptions, it's advisable to utilize one of the more advanced panel data approaches [23].

3.2. Mean Group Estimator

A panel data analysis technique called the mean group (MG) estimator is employed when you have reason to believe that the relationship between your variables may differ for each individual or group in your dataset. It is especially helpful when you have a reasonable number of time periods (T) and a comparatively high number of individuals (N). The link is estimated independently for each person by the MG estimator. To provide a general picture of how independent variables affect dependent variables, it merely averages the individual estimates. A well-established concept that has been investigated in several contexts is estimating a common mean by averaging estimates of individual cross-sectional units in a panel. More general assumptions about β_i and the regressors can also yield a consistent estimator of $\bar{\beta}$. This estimator is known as the MG estimator. Pesaran and Smith [24] demonstrate its consistency when estimating long-term associations in panel data models. The average of these separate estimates is the MG estimator of the total impact of independent variables on dependent variables:

$$\hat{\hat{\boldsymbol{\beta}}}_{RCRMG} = \frac{1}{N} \sum_{i=1}^{N} \hat{\boldsymbol{\beta}}_i, \tag{3.3}$$

The MG estimator's variance-covariance matrix considers both the variances and the covariances among individual estimates of the coefficients. Standard errors and hypothesis testing on the average impact of your variable are made possible by this. The variance-covariance matrix for the MG estimator is given by:

$$\text{Cov}\left(\hat{\hat{\boldsymbol{\beta}}}_{RCRMG}\right) = \frac{1}{N}\left[\hat{\Psi}^* - \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2(X_i'X_i)^{-1}\right] + \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 E[(X_i'X_i)^{-1}] = \frac{1}{N}\hat{\Psi}^*, \tag{3.4}$$

where;

$$\hat{\Psi}^{*} = \frac{1}{N-1}\sum_{i=1}^{N}\sigma_{\hat{\beta}}.$$

The MG estimator is a useful tool for analyzing panel data when you suspect that the connection between your variables is heterogeneous. It offers a method to adjust for individual differences and is comparatively easy to utilize. Knowing its limitations is crucial, though, and when appropriate, taking into account other approaches. Methods that presume homogeneity will be more efficient than the MG estimator, which will have bigger standard errors if the true connection is the same for everyone. is susceptible to outliers: A small number of people with highly odd relationships may have an excessive impact on the average. Not taking time effects into account: Time-varying relationship changes are not taken into account by the basic MG estimator [25].

3.3. Swamy's Estimator

In econometrics, Swamy's estimator is a widely used method for analyzing RCR models. It's beneficial when you want to consider the heterogeneity in your analysis and have a suspicion that the relationship between your variables may differ for each person or group in your dataset. Swamy's estimator is a two-step procedure that estimates the population average coefficients ($\hat{\bar{\beta}}$) and the variance components. Swamy [6] suggested that Swamy's estimator of $\hat{\bar{\beta}}$ is given by;

$$\hat{\bar{\boldsymbol{\beta}}}_{RCRSW} = \left(\mathbf{X}'\mathbf{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Omega}^{-1}\mathbf{Y}, \tag{3.5}$$

where Ω is the variance-covariance matrix of e, accounting for heteroscedasticity across cross-sectional units. Explicitly, Ω is given by:

$$E(ee') = \Omega = \begin{bmatrix} X_1\Psi X_1' + \sigma_1^2 I_T & 0 & \cdots & 0 \\ 0 & X_2\Psi X_2' + \sigma_2^2 I_T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_N\Psi X_N' + \sigma_N^2 I_T \end{bmatrix};$$

which can be rewritten as:

$$\Omega = (\Sigma_{RCRSW}\otimes I_T) + D\,(I_N\otimes\Psi)\,D'; \tag{3.6}$$

where Σ_{RCRSW}= diag $\{\sigma_{ii}\}$, for $i = 1,\ldots,N$, and $\otimes$ is the Kronecker product. Swamy [6] showed that the $\hat{\bar{\beta}}_{RCRSW}$ estimator is rewriteable as:

$$\hat{\bar{\beta}}_{RCRSW} = \left[\sum_{i=1}^{N} X_i'\left(X_i\Psi X_i' + \sigma_{ii}I_T\right)^{-1}X_i\right]^{-1}\sum_{i=1}^{N} X_i'\left(X_i\Psi X_i' + \sigma_{ii}I_T\right)^{-1}y_i, \tag{3.7}$$

According to the RCR assumptions, the $\hat{\bar{\beta}}_{RCRSW}$ has a variance-covariance matrix as follows:

$$var\left(\hat{\bar{\beta}}_{RCRSW}\right) = \left(X'\Omega^{-1}X\right)^{-1} = \left\{\sum_{i=1}^{N}\left[\Psi + \sigma_{ii}\left(X_i'X_i\right)^{-1}\right]^{-1}\right\}^{-1}, \tag{3.8}$$

The $\hat{\bar{\beta}}_{RCRSW}$ estimator contains the unknown σ_{ii}'s and Ψ. The unknown σ_{ii}'s can be estimated by;

$$\hat{\sigma}_{ii} = \hat{\varepsilon}_i' \hat{\varepsilon}_i / (T - K), \tag{3.9}$$

where $\hat{\varepsilon}_i' = [I - X_i(X_i'X_i)^{-1}X_i']y_i = M_i y_i$. For the $\hat{\hat{\beta}}_{RCRSW}$ estimator to be practical, Swamy [8] proposed the following unbiased estimator for Ψ;

$$\hat{\Psi} = \left[\frac{1}{N-1}\left(\sum_{i=1}^{N} \hat{\beta}_i \hat{\beta}_i' - \frac{1}{N}\sum_{i=1}^{N} \hat{\beta}_i \sum_{i=1}^{N} \hat{\beta}_i'\right)\right] - \left[\frac{1}{N}\sum_{i=1}^{N} \hat{\sigma}_{ii}\left(X_i'X_i\right)^{-1}\right]. \tag{3.10}$$

When N and $T \to \infty$, the $\hat{\hat{\beta}}_{RCRSW}$ is consistent while $T \to \infty$, the $\hat{\hat{\beta}}_{RCRSW}$ is asymptotically efficient as proved by Swamy [7, 8, 26]. Swamy's estimator is more flexible; it allows for heterogeneity in the relationship between the independent and dependent variables across individuals. moreover, under certain assumptions, Swamy's estimator is more efficient than the MG and CP estimators, especially when the number of time periods (T) is small.

4. Repercussions of Outliers in the RCR Model

Outliers are observations that do not belong to the same population or may be caused by an exceptional event (due to a catastrophe). Therefore, outliers can be described as points that do not follow the trend of the majority of the data, see Abonazel and Rabie [27]. Outliers can wreak havoc on RCR models, just as they do in standard regression. Because RCR models estimate coefficients for each individual or group and a population average, outliers can affect both levels of estimation, leading to a variety of problems. Here's a breakdown of the consequences:

- **Inflated Variance:** Outliers can inflate the estimated variance of the individual-specific random effects. This makes it appear as though there's more heterogeneity across individuals than exists. Essentially, the model tries to accommodate the outlier by saying, "This individual is different," even if the difference is spurious.
- **Biased Coefficient Estimates:** Outliers can directly bias the estimates of the individual coefficients. Imagine an individual with an extremely high value on the dependent variable. The model might fit a line with a steeper slope for that individual just to accommodate that one point, even if the true relationship is different.
- **Distorted Standard Errors:** The standard errors of the individual coefficient estimates can also be distorted, leading to incorrect t-statistics and p-values.
- **Biased Population Average:** Because the population average coefficients are calculated as a weighted average of the individual coefficients, outliers in the individual data can bias the overall average. A few extreme values can pull the average in the wrong direction.
- **Inflated Standard Errors:** Even if the population average isn't dramatically biased, the inflated variance from the individual level can still lead to larger standard errors for the population average estimates. This makes it harder to find statistically significant effects, even if they truly exist, see Lyu [28].
- **Biased Variance Component Estimates:** Outliers can bias the estimates of the variances and covariances of the random effects. This can distort our understanding of the true degree of heterogeneity in the population. We might overestimate or underestimate how much the coefficients vary across individuals.

- **Incorrect Functional Form:** In some cases, outliers might even lead the model to suggest an incorrect functional form for the relationship. For instance, a single outlier might make a linear relationship appear non-linear.

Outliers can substantially affect RCR models, impacting both individual and population-level estimates, as well as variance components. Detecting or diagnosing outliers is a very important process in RCR analysis; it helps to increase both robustness and efficiency in the estimated RCR model. It is a primary step in many dataset applications. Moreover, influential points cause many problems in regression analysis, so it is important to detect these points for more accurate results. Robust regression estimators are essential tools in dealing with this problem.

Also, it can often be used with Bayesian estimation and model selection to handle outliers [29, 30, 31, 32, 33, 34]. The main purpose of robust estimation is to provide resistant results in the presence of outliers. In order to achieve this stability, robust regression limits the influence of outliers, see [19]. It turns out that the outlier robust method constitutes a useful addition to the econometrician's toolkit. The estimator provides automatic protection against aberrant observations by replacing the standard moment conditions with observation-weighted moment conditions. Moreover, the observation weights produced by the robust estimator can be used as a diagnostic device to assess which observations are not described by the postulated model. In this way, our robust method can give useful (additional) guidance for possible directions of model re-specification, see, e.g., [35, 36, 37, 38, 39].

5. A Novel Robust M-Estimator for the RCR Model

The M-estimation method is a generalization of the maximum likelihood (ML) estimator. As the objective, the M-estimation method minimizes some function of the residual log $f(x, \vartheta)$ as in ML estimation, a more general function $\rho(x, \ \vartheta)$ is allowed. As in the case of M-estimation of location, the robustness of the estimator is determined by the choice of weight function, since the M-estimation is based on the residual scale of Swamy's estimator. If we assume linearity, homoscedasticity, and uncorrelated errors, the ML estimator of $\bar{\beta}$ is simply Swamy's estimator found by minimizing the sum of squares function;

$$\hat{\bar{\beta}}_M = \min_{\bar{\beta}} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(y_{it} - \sum_{k=1}^{K} \beta_{ki} x_{kit} \right)^2 = \min_{\bar{\beta}} \sum_{i=1}^{N} \sum_{t=1}^{T} (\varepsilon_{it})^2 \ , \tag{5.1}$$

Following from M-estimation of location, instead of minimizing the sum of squared residuals, a robust regression M-estimator minimizes the sum of a less rapidly increasing function of the residuals;

$$\hat{\bar{\beta}}_M = \min_{\bar{\beta}} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho \left(y_{it} - \sum_{k=1}^{K} \beta_{ki} x_{kit} \right) = \min_{\bar{\beta}} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho \left(\varepsilon_{it} \right) , \tag{5.2}$$

where the function $\rho(\varepsilon_{it} \ )$ is called the objective function, the solution does not scale equivariant, and thus the residuals must be standardized by a robust estimate of their scale $\mathcal{S}$, which is estimated simultaneously. As in the case of M-estimation of location, the median absolute deviation is often used. Which is used to find $\hat{\bar{\beta}}_M$, a robust estimator of $\bar{\beta} = \left(\bar{\beta}_1, \ldots, \bar{\beta}_K\right)'$, defined as:

$$\mathcal{S} = median_i \left| r_i - median_j(r_j) \right| / 0.6745,$$

where r_i are the residuals from an initial fit (e.g., RCRCP). This ensures robustness against outliers and provides a consistent scale estimate under normality. The standardized residuals $\varepsilon_{it}/\mathcal{S}$ are then used in the estimating equations. The solution of the minimization problem in Equation (5.2) is equivalent to simultaneously solving the following equations:

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\psi\left(\frac{\varepsilon_{it}}{\mathcal{S}}\right)\sum_{k=1}^{K}x_{kit}=0; \qquad K=1,\ldots,k \tag{5.3}$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\eta\left(\frac{\varepsilon_{it}}{\mathcal{S}}\right)=\theta, \tag{5.4}$$

where $\psi(\varepsilon_{it})=\rho'(\varepsilon_{it})$, η is turns out to be $\eta(\varepsilon_{it})=[\psi(\varepsilon_{it})]^2/2$, and $\eta(\varepsilon_{it})=\varepsilon_{it}\,\psi(\varepsilon_{it})-\rho(\varepsilon_{it})$. If we want $\mathcal{S}$ to be asymptotically unbiased for normal errors, we take $\theta=[(n-k)/n]E_{\Phi}(\eta)$ with Φ being the normal distribution.

To reduce the detrimental impact of less frequent spurious values on estimates of variables, parameters, or both together, a variety of robust M-estimator objective functions have been proposed in the robust statistics literature, see De Menezes et al. [40]. A few of these objective functions are covered below.

5.1. Huber's M-estimator

In contrast to the typical squared error loss, Huber is a useful loss function in robust statistics and machine learning to lessen the impact of outliers. it was introduced by Huber in 1964, and residuals with a magnitude larger than delta are not squared. Typically, ε_{it} represents residuals, the difference between a model prediction and a dataset, see Huber and Ronchetti [41]. The robust Huber M-estimator has the following objective, score, and weight functions. The Huber objective function (ρ) is first defined as follows:

$$\rho_{HU}(\varepsilon_{it})=\begin{cases}\frac{1}{2}\varepsilon_{it}^2, & \text{for } |\varepsilon_{it}|\le 1.345,\\ 1.345|\varepsilon_{it}|-\frac{1}{2}(1.345)^2, & \text{for } |\varepsilon_{it}|>1.345.\end{cases} \quad ; i=1,\ldots,N \quad t=1,\ldots,T. \tag{5.5}$$

The derivative of the ρ-function concerning the coefficients $\hat{\hat{\beta}}$ is calculated to yield the Huber score "influence" function (ψ). Second, the associated ψ-function may be expressed as follows:

$$\psi_{HU}(\varepsilon_{it})=\begin{cases}\varepsilon_{it}, & \text{for } |\varepsilon_{it}|\le 1.345,\\ 1.345\cdot\text{sign}(\varepsilon_{it}), & \text{for } |\varepsilon_{it}|>1.345.\end{cases} \tag{5.6}$$

Third, by computing the ψ-function by the associated residuals, the weight function (w) is subsequently generated, that is $w(\varepsilon_{it})=\psi(\varepsilon_{it})/\varepsilon_{it}$. Then the w-function can be expressed as follows;

$$w_{HU}(\varepsilon_{it})=\begin{cases}1, & \text{for } |\varepsilon_{it}|\le 1.345,\\ \dfrac{1.345}{|\varepsilon_{it}|}, & \text{for } |\varepsilon_{it}|>1.345.\end{cases} \tag{5.7}$$

When $|\varepsilon_{it}| > 1.345$, the tail of the modified standard normal distribution is replaced by an exponential distribution, and the Huber ρ-function can be thought of as the negative log-likelihood of that distribution. Thus, it can be said that Huber's robust M-estimator uses the least informative distribution $f(\varepsilon_{it}) = \alpha^{-1} exp\{-\rho(\varepsilon_{it})\}$ as a "working likelihood" in the context of parameter estimations, see Wang and Jiang [42].

5.2. *Hampel's M-estimator*

Another possibility for the robust functions is Hampel's three-part M-estimator. It's a type of robust statistical filter used to identify and handle outliers in a dataset it was introduced by Hampel in 1970. The basic idea behind the Hampel function is to replace data points that are considered outliers with more representative values, see Hampel [43]. The Hampel's three-part ρ-function is defined by;

$$\rho_{HM}(\varepsilon_{it}) = \begin{cases} \frac{1}{2}\varepsilon_{it}^2, & \text{for } |\varepsilon_{it}| \leq 1.35, \\ (1.35)|\varepsilon_{it}| - \frac{1}{2}(1.35)^2, & \text{for } 1.35 < |\varepsilon_{it}| \leq 2.70, \\ (1.35)(2.70) - \frac{(1.35)^2}{2} + \frac{1.35(5.40-2.70)}{2}\left[1 - \left(\frac{5.40-|\varepsilon_{it}|}{5.40-2.70}\right)\right], & \text{for } 1.35 < |\varepsilon_{it}| \leq 5.40, \\ (1.35)(2.70) - \frac{(1.35)^2}{2} + \frac{1.35(5.40-2.70)}{2}, & \text{otherwise.} \end{cases} \tag{5.8}$$

The corresponding ψ-function can be expressed as follows;

$$\psi_{HM}(\varepsilon_{it}) = \begin{cases} \varepsilon_{it}, & \text{for } |\varepsilon_{it}| \leq 1.35, \\ 1.35 \cdot \text{sign}(\varepsilon_{it}), & \text{for } 1.35 < |\varepsilon_{it}| \leq 2.70, \\ (1.35)\dfrac{5.40 - |\varepsilon_{it}|}{5.40 - 2.70} \cdot \text{sign}(\varepsilon_{it}), & \text{for } 2.70 < |\varepsilon_{it}| \leq 5.40, \\ 0, & \text{otherwise.} \end{cases} \tag{5.9}$$

The corresponding w-function is given by:

$$w_{HM}(\varepsilon_{it}) = \begin{cases} 1, & \text{for } |\varepsilon_{it}| \leq 1.35, \\ \dfrac{1.35}{|\varepsilon_{it}|}, & \text{for } 1.35 < |\varepsilon_{it}| \leq 2.70, \\ \dfrac{1.35 \cdot (5.40 - |\varepsilon_{it}|)}{(5.40 - 2.70)|\varepsilon_{it}|}, & \text{for } 2.70 < |\varepsilon_{it}| \leq 5.40, \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

The ranges vary from $0 < 1.35 \leq 2.70 < 5.40 < \infty$, which is the slope of the redescending part ($\varepsilon_{it} \in (2.70, 5.40]$) is set to $-1/2$. Note that $\psi_{HM}(\varepsilon_{it})$ can also be tuned to have a downward slope of $-1/3$, see Koller and Stahel [44].

5.3. *Tukey's M-estimator*

The Tukey bisquare function, also known as Tukey's biweight function, is a loss function that is used in robust statistics and was introduced by Beaton and Tukey in 1974. Tukey's function is similar to Huber's function in that it demonstrates quadratic behavior near the origin. However, it is even more insensitive to outliers because the loss incurred by large residuals is constant, rather than scaling linearly as it would for the Huber function. The Tukey's bisquare ρ-function is defined as;

$$\rho_{BI}(\varepsilon_{it}) = \begin{cases} \frac{(4.685)^2}{6}\left\{1-\left[1-\left(\frac{\varepsilon_{it}}{4.685}\right)^2\right]^3\right\}, & \text{for } |\varepsilon_{it}| \le 4.685, \\ \frac{(4.685)^2}{6}, & \text{for } |\varepsilon_{it}| > 4.685. \end{cases} \tag{5.11}$$

The corresponding ψ-function can be expressed as follows:

$$\psi_{BI}(\varepsilon_{it}) = \begin{cases} \varepsilon_{it}\left(1-\left(\frac{\varepsilon_{it}}{4.685}\right)^2\right)^2, & \text{for } |\varepsilon_{it}| \le 4.685, \\ 0, & \text{for } |\varepsilon_{it}| > 4.685. \end{cases} \tag{5.12}$$

The corresponding w-function is given by:

$$w_{BI}(\varepsilon_{it}) = \begin{cases} \left[1-\left(\frac{\varepsilon_{it}}{4.685}\right)^2\right]^2, & \text{for } |\varepsilon_{it}| \le 4.685, \\ 0, & \text{for } |\varepsilon_{it}| > 4.685. \end{cases} \tag{5.13}$$

According to Beaton and Tukey [45], Tukey's bisquare function is known for its smoothness and has been widely and effectively applied across various fields.

5.4. Algorithm of the Proposed Robust M-Estimator

The algorithms required to obtain the proposed robust M-estimator of the RCR model, as stated in Section 2, are introduced below.

Step 1. Estimate the RCR model coefficients using classical estimation methods ($\hat{\bar{\beta}}_{RCRCP}$, $\hat{\bar{\beta}}_{RCRMG}$, and $\hat{\bar{\beta}}_{RCRSW}$), and test all assumptions.

Step 2. Calculate initial regression coefficients by the RCR model ($\hat{\bar{\beta}}^0$).

Step 3. Detect the presence of outliers in the dataset.

Step 4. Calculate the variance-covariance matrix of ($\hat{\bar{\beta}}$).

Step 5. Similarly, as in Huber and Dutter [46], we can obtain a proposed robust M-estimator by minimizing;

$$\frac{1}{T}\sum_{t=1}^{T}\rho\left(\frac{y_{it}-\sum_{k=1}^{K}\beta_{ki}x_{kit}}{\mathcal{S}_i}\right)\mathcal{S}_i + \theta_i\mathcal{S}_i, \tag{5.14}$$

to obtain $\hat{\bar{\beta}}_i^M$ a robust M-estimator of $\bar{\beta}_i = \left(\bar{\beta}_{1i}, \ldots, \bar{\beta}_{ki}\right)'$ and $\mathcal{S}_\text{i}$, for i = 1, ..., N. In practice, we must compute $\hat{\bar{\beta}}_i^M$ and scale estimator $\mathcal{S}_\text{i}$ using simultaneous iterations. The minimization problem in Equation (5.14) is solved by simultaneously calculating the following equations:

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\psi\left[\frac{y_{it}-\sum_{k=1}^{K}\beta_{ki}x_{kit}}{\mathcal{S}_i}\right]\right\}\sum_{t=1}^{T}x_{kit} = 0, \tag{5.15}$$

$$\left(\frac{1}{T}\right)\sum_{t=1}^{T}\left\{\eta\left[\frac{y_{it}-\sum_{k=1}^{K}\beta_{ki}x_{kit}}{\mathcal{S}_i}\right]\right\} = \theta_i, \tag{5.16}$$

where $\rho(\varepsilon_{it}), \psi(\varepsilon_{it}), \eta(\varepsilon_{it}), \mathcal{S}_i$, and θ_i (for unit $i = 1, \ldots, N$) are as defined in Sections 4 and 5. A sophisticated technique for simultaneously solving Equations (5.15) and (5.16) can be found in Huber and Dutter [46]. Letting $\bar{\beta}_i^v$ and $\mathcal{S}_i^v$ be trial values for $\bar{\beta}_i$ and $\mathcal{S}_i$, then we obtain;

$$(\mathcal{S}_i^{(v+1)})^2 = \frac{1}{T-k} \sum_{t=1}^{T} \left(y_{it} - \sum_{k=1}^{K} \beta_{ki} x_{kit} \right)^2 \tag{5.17}$$

Step 6. Evaluate the robust estimate of σ_{ii}^M as follows;

$$\hat{\sigma}_{ii}^M = \frac{1}{(T-k)} \cdot \frac{\sum_{t=1}^{T} \psi\left(\frac{\varepsilon_{it}}{\mathcal{S}_i}\right)^2 \mathcal{S}_i^2}{\left[\frac{1}{T} \sum_{t=1}^{T} \psi'\left(\frac{\varepsilon_{it}}{\mathcal{S}_i}\right)\right]^2}, \tag{5.18}$$

where $\varepsilon_{it} = y_{it} - \sum_{k=1}^{K} \beta_{ki} x_{kit}$ and $\psi'(\varepsilon_{it}) = \frac{\partial}{\partial \varepsilon_{it}} \psi(\varepsilon_{it})$.

Step 7. Calculate the weight values $w_\varepsilon(\varepsilon_{it})$ using weight functions.

Step 8. Estimate the robust RCR coefficients of $\hat{\bar{\beta}}_{RCRHU}$, $\hat{\bar{\beta}}_{RCRHM}$, and $\hat{\bar{\beta}}_{RCRBI}$ estimator based on $w_\varepsilon(\varepsilon_{it})$.

Step 9. Repeat the steps from 4 to 7 until the algorithm converges to obtain a convergent value of $\hat{\bar{\beta}}_{RCRHU}$, $\hat{\bar{\beta}}_{RCRHM}$, and $\hat{\bar{\beta}}_{RCRBI}$.

Step 10. Test to determine whether explanatory variables have a significant effect on the response variable, and evaluate the performance and results of the estimates using some criteria.

When comparing various robust estimators, two commonly utilized key metrics are efficiency and BP. When the error distribution is precisely normal and there are no outliers, the efficiency is utilized to calculate the relative efficiency of the robust estimates in comparison to the classical estimation methods. The goal of BP is to quantify the percentage of outliers that an estimate can handle before reaching infinity. Therefore, an estimator is more resilient the larger it's BP. A BP cannot, intuitively, be more than 0.5. As Yu and Yao [47] shows, the BP of the M-estimator is really BP $=1/n \to 0$.

6. Monte Carlo Simulation Study

In this section, a comprehensive Monte Carlo simulation study is conducted to evaluate the overall performance of the proposed robust M-estimators. We performed a simulation study to compare the performance of the classical estimation methods and the robust M-estimators. A Monte Carlo study takes into account several factors that may impact the fitting of the RCR model and the results of the various approaches.

6.1. Monte Carlo Simulation Algorithms

The datasets have been simulated using varying amounts of contamination. Therefore, the following procedure for creating data served as the foundation for the Monte Carlo experiments:

$$y_{it} = \sum_{k=1}^{5} \beta_{ki} x_{kit} + \varepsilon_{it}, \qquad i = 1, \ldots, N; \qquad t = 1, \ldots, T. \tag{6.1}$$

Using the general RCR assumptions as in Section 2, the Monte Carlo simulation study was conducted by generating the model in Equation (6.1). The following serves as the foundation for the algorithms used in the Monte Carlo simulation study for the RCR model:

1. The explanatory variables,(x_{kit}), were generated from a standard normal distribution with a mean (μ_x) equal to zero and a fixed standard deviation (σ_x) equal to one, where the number of the explanatory variables was $k = 1, 2, 3, 4$, and 5.
2. The coefficients, β_{ki}, were generated as in Assumption **(A4)**: $\beta_i = \bar{\beta} + \gamma_i$, where the vector of coefficients $\bar{\beta} = (1, 1, 1, 1, 1)'$, and γ_i were generated from a multivariate normal distribution with a mean equal to zero and a variance-covariance matrix $\Psi = \text{diag}\left\{\phi_k^2\right\}$; for $k = 1, 2, 3, 4$, and 5. The values of ϕ_k^2 were chosen to be fixed for all k and equal to 0, 10, and 20. Take notice that when $\phi_k^2 = 0$, the coefficients are fixed, as in Abonazel [10].
3. The errors, ε_{it}, were generated from a normally distributed, independent of the x_{kit} values, with a mean equal to zero and a standard deviation ($\sigma_{\varepsilon_{ii}}$) equal to 5 and 15, and maintained fixed across all cross-sectional units.
4. For N and T values were selected to be 5, 10, 15, 20, and 25 to symbolize moderate and small samples in terms of both the total number of individuals and the time dimension. Three distinct sampling schemes, each containing four pairings of N and T, have been created in our simulation to examine the sample performance for the various robust and non-robust estimators. The following designs were created:
   - Case I: $N < T$, the pairs are $(N, T) = (5, 10), (10, 15), (15, 20)$, or $(20, 25)$.
   - Case II: $N > T$, the pairs are $(N, T) = (10, 5), (15, 10), (20, 15)$, or $(25, 20)$.
   - Case III: $N = T$, the pairs are $(N, T) = (5, 5), (15, 15), (20, 20)$, or $(25, 25)$.
5. The outlier's percentage (τ%), were generated from a normal distribution with a mean equal to ($10 \times$ IQR), where IQR is the interquartile range of (y_{it}) values, with different percentages of outliers ($\tau = 0\%, 5\%, 15\%$, and 25%) in the RCR model, as in [20, 48].
6. We performed 1000 replications for all experiments of the Monte Carlo simulation, and the findings of every separate experiment are derived utilizing the same set of arbitrary integers. Also, to increase the effectiveness of evaluating the classical estimation methods and robust M-estimators' performances, we compute the total mean squared error (TMSE) for $\hat{\bar{\beta}}$;

$$\text{TMSE} = \frac{1}{1000} \sum_{l=1}^{1000} (\hat{\bar{\beta}}_l - \bar{\beta})'(\hat{\bar{\beta}}_l - \bar{\beta}) \tag{6.2}$$

where $\hat{\bar{\beta}}_l$ is the estimated coefficient vector of $\bar{\beta}$ in Equation (6.1) at *l-th* experiment of 1000 Monte Carlo experiments, while $\bar{\beta}$ is the vector of true coefficients.

6.2. *Monte Carlo Simulation Results*

The Monte Carlo simulation results are summarized in Tables 1- 6. The TMSE values of various estimators are specifically shown in Tables 1-3 for $\sigma_{\varepsilon ii}$= 5, and in cases of $N < T$, $N > T$, and $N = T$, respectively. While Tables 3-6 depict the situation of $\sigma_{\varepsilon ii}$= 15 for the same cases of N and T. According to the results of the Monte Carlo simulation, we concluded the following:

1. The non-robust (RCRCP, RCRMG, and RCRSW) estimators' performance was the worst of the given estimators in the presence of an outlier problem, as expected.

2. An increase in the value of the percentage of outliers ($\tau\%$) leads to higher TMSE values in most scenarios.
3. In the absence of outliers ($\tau = 0\%$), the non-robust estimators (RCRCP, RCRMG, and RCRSW) perform better than the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) across all N, T, $\sigma_{\varepsilon ii}$, and ϕ_k^2 in the RCR model. Moreover, it can be noted that the RCRMG estimator performs well even in small samples, while the RCRSW estimator performs well even in large samples.
4. Increasing the value of ($\sigma_{\varepsilon ii}$), leads to an increase in TMSE values in all cases of the Monte Carlo simulation.
5. In most cases, the values of TMSE increase as the value of (ϕ_k^2) increases for the various scenarios adopted in the simulation.
6. When outliers' problem existed in the RCR model (i.e.,$\tau > 0\%$), the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) were better than the non-robust (RCRCP, RCRMG, and RCRSW) estimators, for all N, T, $\sigma_{\varepsilon ii}$, and ϕ_k^2 values.
7. The values of TMSE for the various scenarios adopted in the simulation decrease when N and T are raised.
8. Finally, the proposed robust M-estimator (RCRBI) achieved the best performance among all given estimators when the outlier problem existed in the RCR model in most situations of the Monte Carlo simulation.

Graphically, we depend on another comparative performance level called relative efficiency (RE). The RE values are given by dividing the TMSE of RCRCP by the TMSE of the non-robust (RCRMG, and RCRSW) and the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI). The RE values of the estimates for each N and T are independent.

We report the results of RE values by 2D graphs are shown in Figures 1 and 2, respectively. Figure 1, indicates that when the absence of outliers ($\tau = 0\%$), the RCRMG estimator performs well even in small samples, while the RCRSW estimator performs well even in large samples for each cross-section (N) value. Moreover, the proposed robust M-estimator (RCRBI) RE values are greater than those of different robust M-estimators (RCRHU and RCRHM) for each cross-section (N) value. Since the proposed robust M-estimator (RCRBI) has the largest RE values, and as a result, we can conclude that the proposed robust M-estimator (RCRBI) is more efficient and reliable than the other robust M-estimators (RCRHU and RCRHM) for different N, T, $\tau\%$ and ϕ_k^2 values.

However, when N and $\tau\%$ increase, the efficiency of the proposed robust M-estimator increases. In Figure 2, the efficiency of the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) is close, but the RCRBI estimator is still more efficient than the different robust M-estimators.

7. Application of Energy Management Systems: Results and Discussion

Using a real energy dataset, we investigate the effectiveness of the non-robust estimators and proposed robust M-estimators of the RCR model in this application. By examining the dependence of carbon dioxide (CO_2) emissions on nuclear, renewable, and non-renewable energy sources in the top 5 G20 countries between 2000 and 2024, this study investigates attitudes and opinions regarding nuclear and renewable energy production technologies. China, the United States, India, Russia, and Japan are the top 5 G20 countries in terms of carbon dioxide emissions, as shown in Figure 3. Large industrial

Table 1. TMSE values for various estimators when $N < T$ and $\sigma_{\varepsilon ii} = 5$.

	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N, T)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)
								$\phi_k^2 = 0$								
RCRCP	1.996	0.166	0.129	0.073	14.157	3.300	1.319	0.848	45.934	9.130	3.244	1.980	68.437	13.259	5.381	2.803
RCRMG	0.674	0.205	0.084	0.053	5.150	1.918	0.905	0.606	16.752	5.119	2.402	1.479	23.985	7.153	3.695	2.231
RCRSW	0.679	0.276	0.082	0.051	5.160	1.942	0.910	0.606	16.946	5.091	2.416	1.475	23.927	7.247	3.688	2.233
RCRHU	0.696	0.165	0.089	0.055	0.764	0.216	0.114	0.065	1.443	0.401	0.187	0.108	13.117	1.737	0.634	0.345
RCRHM	0.676	0.175	0.085	0.053	0.666	0.178	0.092	0.056	2.027	0.213	0.097	0.059	25.318	7.428	3.897	2.361
RCRBI	0.708	0.305	0.089	0.055	0.729	0.184	0.095	0.058	0.760	0.215	0.099	0.061	9.790	0.219	0.134	0.061
								$\phi_k^2 = 10$								
RCRCP	4.605	0.851	0.380	0.204	37.358	8.044	3.284	2.107	117.928	21.369	7.881	4.787	161.872	30.947	12.512	6.805
RCRMG	1.671	0.532	0.282	0.154	13.058	4.803	2.235	1.492	40.402	12.217	5.994	3.481	57.288	16.563	8.669	5.476
RCRSW	1.679	0.532	0.281	0.148	13.109	4.855	2.249	1.495	40.950	12.203	6.021	3.473	56.889	16.772	8.663	5.469
RCRHU	1.715	0.574	0.295	0.166	1.876	0.684	0.355	0.192	4.432	1.160	0.622	0.309	28.676	4.735	1.916	1.066
RCRHM	1.649	0.551	0.282	0.152	1.625	0.582	0.292	0.157	7.260	0.622	0.350	0.179	60.678	17.052	9.011	5.657
RCRBI	1.799	0.612	0.304	0.170	1.783	0.615	0.319	0.172	2.282	0.627	0.351	0.183	22.320	1.702	0.403	0.220
								$\phi_k^2 = 20$								
RCRCP	7.296	1.394	0.627	0.334	58.902	12.768	5.170	3.320	181.197	33.801	12.424	7.491	254.016	48.179	19.631	10.556
RCRMG	2.765	0.901	0.475	0.257	20.554	7.631	3.532	2.353	62.768	19.476	9.434	5.411	90.118	26.020	13.576	8.501
RCRSW	2.770	0.902	0.477	0.241	20.627	7.711	3.555	2.359	63.651	19.467	9.472	5.397	89.447	26.322	13.568	8.486
RCRHU	2.824	0.958	0.497	0.278	3.052	1.148	0.607	0.330	7.529	1.906	1.047	0.510	47.459	7.653	3.121	1.774
RCRHM	2.733	0.917	0.476	0.252	2.646	0.981	0.493	0.268	11.933	1.044	0.592	0.300	95.344	26.720	14.074	8.721
RCRBI	3.020	1.036	0.522	0.291	2.973	1.057	0.560	0.304	3.669	1.056	0.600	0.306	35.375	3.520	0.674	0.384

Table 2. TMSE values for various estimators when $N > T$ and $\sigma_{\varepsilon ii} = 5$.

$\tau\%$	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N, T)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)
								$\phi_k^2 = 0$								
RCRCP	2.038	0.519	0.139	0.072	28.382	5.962	1.727	0.849	67.597	15.468	4.572	2.443	92.492	22.808	7.021	3.146
RCRMG	0.332	0.171	0.084	0.046	4.004	2.083	0.941	0.574	9.057	4.542	2.748	1.527	13.651	7.647	3.430	1.990
RCRSW	0.335	0.182	0.080	0.037	4.066	2.087	0.941	0.572	9.204	4.570	2.740	1.543	13.749	7.661	3.418	2.002
RCRHU	0.354	0.195	0.089	0.048	0.473	0.212	0.102	0.075	0.804	0.349	0.208	0.112	6.731	2.165	0.609	0.326
RCRHM	0.338	0.184	0.085	0.046	0.393	0.182	0.086	0.063	0.413	0.198	0.107	0.062	14.347	7.953	3.611	2.105
RCRBI	0.359	0.195	0.090	0.048	0.412	0.188	0.090	0.064	0.427	0.204	0.108	0.063	4.081	1.077	0.101	0.069
								$\phi_k^2 = 10$								
RCRCP	4.293	1.299	0.403	0.253	68.770	14.750	4.350	2.013	155.045	37.360	10.959	5.751	217.119	56.690	17.572	7.571
RCRMG	0.871	0.461	0.293	0.174	9.543	5.005	2.465	1.386	20.756	11.340	6.533	3.698	31.988	18.648	8.667	4.724
RCRSW	0.888	0.477	0.282	0.161	9.739	5.019	2.466	1.384	21.047	11.388	6.513	3.729	32.205	18.683	8.649	4.746
RCRHU	0.957	0.503	0.303	0.176	1.384	0.668	0.351	0.216	2.230	1.171	0.613	0.348	16.498	6.253	1.935	0.949
RCRHM	0.895	0.474	0.289	0.170	1.150	0.564	0.288	0.190	1.272	0.649	0.347	0.189	33.571	19.140	8.993	4.885
RCRBI	1.041	0.532	0.310	0.183	1.192	0.619	0.306	0.197	1.206	0.609	0.309	0.175	10.528	1.878	0.404	0.222
								$\phi_k^2 = 20$								
RCRCP	6.748	2.184	0.672	0.425	109.218	22.836	6.810	3.139	245.408	58.998	16.936	9.078	344.459	88.860	27.355	11.626
RCRMG	1.458	0.803	0.496	0.294	15.100	7.688	3.886	2.172	33.035	17.979	10.102	5.835	49.153	28.781	13.480	7.297
RCRSW	1.487	0.812	0.491	0.283	15.422	7.712	3.890	2.169	33.489	18.041	10.075	5.881	49.563	28.839	13.447	7.327
RCRHU	1.557	0.835	0.512	0.301	2.246	1.109	0.600	0.355	3.633	1.938	1.006	0.591	25.908	10.369	3.227	1.560
RCRHM	1.491	0.796	0.485	0.287	1.878	0.943	0.492	0.315	2.117	1.083	0.584	0.322	51.564	29.473	13.926	7.496
RCRBI	1.744	0.904	0.538	0.318	1.944	1.041	0.534	0.330	2.214	1.102	0.588	0.337	18.124	4.192	0.697	0.373

Table 3. TMSE values for various estimators when $N = T$ and $\sigma_{\varepsilon ii} = 5$.

$\tau\%$	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N,T)	(5,5)	(15,15)	(20,20)	(25,25)	(8,8)	(15,15)	(20,20)	(25,25)	(5,5)	(15,15)	(20,20)	(25,25)	(5,5)	(15,15)	(20,20)	(25,25)
								$\phi_k^2 = 0$								
RCRCP	2.606	0.206	0.095	0.056	31.400	2.602	1.183	0.622	85.356	5.641	2.869	1.411	111.300	8.901	3.830	2.502
RCRMG	0.474	0.117	0.057	0.040	4.590	1.329	0.795	0.468	13.501	3.302	1.997	1.003	17.897	4.447	2.537	1.762
RCRSW	0.475	0.118	0.057	0.040	4.626	1.321	0.794	0.469	13.566	3.307	2.006	1.002	18.111	4.472	2.534	1.764
RCRHU	0.512	0.119	0.058	0.041	0.537	0.157	0.080	0.051	1.410	0.250	0.153	0.078	10.441	0.804	0.439	0.266
RCRHM	0.489	0.117	0.057	0.040	0.426	0.128	0.066	0.043	1.274	0.136	0.079	0.045	18.640	4.699	2.695	1.855
RCRBI	0.535	0.120	0.059	0.041	0.459	0.131	0.068	0.045	0.558	0.142	0.080	0.047	8.409	0.141	0.076	0.053
								$\phi_k^2 = 10$								
RCRCP	4.093	0.397	0.208	0.116	55.572	4.438	2.026	1.090	144.456	9.296	4.796	2.362	194.797	15.476	6.631	4.208
RCRMG	0.908	0.242	0.138	0.088	8.097	2.324	1.364	0.823	23.412	5.393	3.412	1.679	29.866	7.485	4.298	3.028
RCRSW	0.914	0.253	0.133	0.082	8.174	2.308	1.361	0.826	23.539	5.402	3.427	1.678	30.105	7.533	4.291	3.033
RCRHU	0.941	0.263	0.144	0.094	1.106	0.298	0.161	0.112	2.746	0.498	0.325	0.160	16.611	1.526	0.888	0.563
RCRHM	0.914	0.253	0.137	0.089	0.896	0.245	0.129	0.095	2.413	0.286	0.173	0.097	31.097	7.866	4.491	3.147
RCRBI	0.989	0.270	0.147	0.095	0.989	0.250	0.135	0.098	1.153	0.296	0.175	0.099	12.139	0.330	0.189	0.121
								$\phi_k^2 = 20$								
RCRCP	5.828	0.590	0.310	0.177	78.565	6.124	2.853	1.544	193.886	12.958	6.607	3.288	287.534	21.636	9.154	5.814
RCRMG	1.341	0.384	0.213	0.136	11.546	3.260	1.928	1.169	31.177	7.536	4.734	2.337	43.543	10.466	5.930	4.199
RCRSW	1.349	0.388	0.206	0.131	11.665	3.237	1.924	1.173	31.384	7.547	4.755	2.335	43.918	10.534	5.922	4.206
RCRHU	1.382	0.400	0.224	0.146	1.641	0.449	0.244	0.170	3.959	0.742	0.480	0.245	24.473	2.230	1.305	0.848
RCRHM	1.353	0.384	0.211	0.137	1.351	0.369	0.196	0.144	3.355	0.438	0.260	0.150	45.352	10.970	6.161	4.331
RCRBI	1.487	0.416	0.232	0.150	1.474	0.380	0.209	0.151	1.732	0.455	0.263	0.155	17.699	0.512	0.303	0.185

Table 4. TMSE values for various estimators when $N < T$ and $\sigma_{\varepsilon ii} = 15$.

$\tau\%$	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N,T)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)	(5,10)	(10,15)	(15,20)	(20,25)
								$\phi_k^2 = 0$								
RCRCP	17.966	2.742	1.158	0.656	116.144	25.988	10.270	6.463	350.290	70.449	25.303	15.073	504.585	101.476	41.223	21.532
RCRMG	6.069	1.477	0.755	0.478	40.265	14.875	7.049	4.672	124.580	39.100	18.798	11.220	181.313	55.403	27.991	16.963
RCRSW	6.109	1.490	0.756	0.457	40.367	15.064	7.089	4.668	126.037	38.883	18.913	11.186	180.586	56.077	27.931	16.983
RCRHU	6.260	1.584	0.805	0.495	6.877	1.941	1.023	0.584	12.988	3.605	1.686	0.971	104.732	15.589	5.703	3.104
RCRHM	6.080	1.486	0.764	0.480	5.992	1.602	0.829	0.512	17.509	1.919	0.895	0.535	190.962	57.478	29.590	17.940
RCRBI	6.374	1.577	0.805	0.496	6.559	1.560	0.854	0.509	6.841	1.934	0.885	0.543	80.718	2.895	1.205	0.548
								$\phi_k^2 = 10$								
RCRCP	20.290	3.300	1.426	0.787	140.543	31.318	12.561	7.970	456.617	83.870	29.955	18.131	618.624	120.593	49.831	25.778
RCRMG	6.773	1.857	0.966	0.570	49.289	18.184	8.549	5.719	154.999	46.997	22.705	13.347	222.409	64.624	33.726	20.536
RCRSW	6.786	1.843	0.967	0.568	49.399	18.413	8.607	5.722	157.083	46.765	22.832	13.306	221.434	65.555	33.690	20.532
RCRHU	7.053	2.033	1.049	0.606	7.831	2.448	1.240	0.685	16.831	4.350	2.163	1.159	117.729	17.851	7.238	3.892
RCRHM	6.750	1.896	0.987	0.572	6.720	2.032	1.021	0.576	16.284	2.260	1.158	0.661	235.233	66.915	35.560	21.563
RCRBI	7.324	2.052	1.053	0.605	7.163	2.124	1.055	0.601	8.635	2.300	1.169	0.672	81.880	2.429	1.468	0.690
								$\phi_k^2 = 20$								
RCRCP	22.864	3.848	1.680	0.917	167.312	36.373	14.902	9.210	533.504	96.345	34.888	21.306	717.488	139.850	57.152	30.031
RCRMG	7.731	2.123	1.165	0.669	58.258	21.366	10.140	6.582	178.777	54.669	26.501	15.595	258.211	74.813	38.894	23.987
RCRSW	7.745	2.207	1.161	0.662	58.389	21.621	10.213	6.589	181.365	54.398	26.644	15.553	256.875	75.832	38.855	23.968
RCRHU	8.017	2.448	1.259	0.718	8.869	2.918	1.476	0.809	19.763	5.130	2.612	1.355	134.506	20.953	8.451	4.556
RCRHM	7.723	2.308	1.190	0.671	7.672	2.435	1.220	0.675	26.079	2.663	1.417	0.778	273.701	77.431	40.930	25.086
RCRBI	8.255	2.517	1.269	0.717	8.141	2.542	1.267	0.708	10.239	2.705	1.423	0.794	103.299	2.970	1.740	0.848

Table 5. TMSE values for various estimators when $N > T$ and $\sigma_{\varepsilon ii} = 15$.

$\tau\%$	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N,T)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)	(10,5)	(15,10)	(20,15)	(25,20)
$\phi_k^2 = 0$																
RCRCP	18.344	4.669	1.252	0.652	232.193	46.510	13.435	6.657	496.529	119.073	35.075	18.727	712.808	173.866	53.450	24.008
RCRMG	2.986	1.630	0.752	0.414	31.568	16.061	7.306	4.493	67.943	34.420	20.692	11.645	103.149	57.945	25.970	15.010
RCRSW	3.014	1.637	0.743	0.402	32.044	16.098	7.303	4.486	68.996	34.628	20.627	11.761	104.111	58.112	25.869	15.105
RCRHU	3.189	1.753	0.803	0.435	4.257	1.911	0.922	0.674	7.236	3.143	1.872	1.006	54.824	19.343	5.477	2.932
RCRHM	3.043	1.652	0.769	0.415	3.537	1.638	0.773	0.563	3.984	1.779	0.960	0.556	107.862	60.249	27.290	15.889
RCRBI	3.234	1.757	0.808	0.435	3.710	1.688	0.808	0.579	3.839	1.836	0.968	0.565	36.441	9.056	0.910	0.620
$\phi_k^2 = 10$																
RCRCP	19.919	5.093	1.500	0.861	281.482	56.926	16.217	7.966	600.254	147.303	42.516	22.735	869.769	208.688	65.923	28.851
RCRMG	3.361	1.777	0.981	0.567	38.035	19.404	8.960	5.392	80.064	42.982	25.133	14.239	128.260	69.785	32.146	17.941
RCRSW	3.408	1.798	0.962	0.557	38.573	19.478	8.961	5.384	81.296	43.190	25.041	14.379	129.551	69.940	32.061	18.045
RCRHU	3.695	1.920	1.044	0.587	5.312	2.397	1.184	0.821	8.661	4.130	2.300	1.227	64.941	22.301	6.913	3.585
RCRHM	3.461	1.811	1.003	0.565	4.264	2.031	0.990	0.691	4.620	2.273	1.199	0.672	134.232	72.119	33.668	18.828
RCRBI	3.765	1.952	1.050	0.588	4.571	2.169	1.021	0.720	4.784	2.307	1.227	0.692	41.422	11.364	1.236	0.779
$\phi_k^2 = 20$																
RCRCP	22.092	5.830	1.762	1.045	326.764	67.178	18.933	9.338	695.060	171.753	48.712	26.107	984.759	244.148	77.223	34.031
RCRMG	3.880	2.050	1.193	0.698	44.246	22.910	10.585	6.352	92.545	50.944	28.831	16.424	146.611	81.209	37.814	21.121
RCRSW	3.942	2.076	1.193	0.687	44.967	22.988	10.586	6.343	93.952	51.191	28.723	16.577	148.199	81.393	37.724	21.242
RCRHU	4.282	2.225	1.262	0.717	6.289	2.853	1.432	0.972	10.092	4.992	2.723	1.466	69.727	26.065	8.192	4.255
RCRHM	4.012	2.090	1.211	0.692	5.052	2.420	1.191	0.820	5.500	2.732	1.442	0.796	153.993	83.805	39.499	22.113
RCRBI	4.463	2.271	1.269	0.720	5.461	2.607	1.230	0.856	5.691	2.773	1.473	0.819	37.843	10.369	1.545	0.934

Table 6. TMSE values for various estimators when $N = T$ and $\sigma_{\varepsilon ii} = 15$.

$\tau\%$	$\tau = 0\%$				$\tau = 5\%$				$\tau = 15\%$				$\tau = 25\%$			
(N,T)	(5,5)	(15,15)	(20,20)	(25,25)	(5,5)	(15,15)	(20,20)	(25,25)	(5,5)	(15,15)	(20,20)	(25,25)	(5,5)	(15,15)	(20,20)	(25,25)
$\phi_k^2 = 0$																
RCRCP	23.453	1.858	0.854	0.504	245.171	20.407	9.146	4.880	632.772	43.183	22.223	10.615	864.609	67.671	28.842	18.971
RCRMG	4.264	1.057	0.511	0.362	36.438	10.421	6.185	3.636	103.511	25.292	15.454	7.439	136.110	33.738	18.823	13.156
RCRSW	4.277	1.063	0.504	0.351	36.715	10.370	6.172	3.645	104.015	25.343	15.512	7.432	137.734	33.932	18.795	13.176
RCRHU	4.606	1.068	0.522	0.371	4.832	1.410	0.718	0.459	12.694	2.253	1.375	0.702	85.778	7.237	3.953	2.396
RCRHM	4.405	1.051	0.516	0.362	3.821	1.148	0.593	0.389	9.732	1.224	0.712	0.404	141.752	35.629	19.998	13.914
RCRBI	4.814	1.077	0.530	0.372	4.129	1.182	0.608	0.403	5.020	1.275	0.717	0.420	69.250	1.270	0.688	0.480
$\phi_k^2 = 10$																
RCRCP	24.093	2.042	1.007	0.565	258.329	22.200	9.949	5.331	714.302	48.370	24.967	11.713	925.760	73.492	32.089	20.807
RCRMG	4.704	1.197	0.615	0.429	40.182	11.426	6.710	3.975	117.176	28.004	17.488	8.199	147.152	36.335	20.803	14.577
RCRSW	4.742	1.205	0.608	0.409	40.489	11.359	6.696	3.986	117.599	28.060	17.562	8.192	148.719	36.558	20.778	14.602
RCRHU	4.946	1.236	0.631	0.429	5.374	1.527	0.788	0.527	14.522	2.483	1.608	0.769	94.172	7.890	4.350	2.718
RCRHM	4.798	1.206	0.618	0.409	4.313	1.226	0.637	0.446	15.042	1.362	0.819	0.452	153.139	38.340	22.018	15.375
RCRBI	5.136	1.253	0.635	0.427	4.705	1.263	0.655	0.463	5.723	1.414	0.833	0.465	70.581	1.476	0.801	0.557
$\phi_k^2 = 20$																
RCRCP	28.555	2.614	1.347	0.746	331.105	28.230	12.745	6.787	880.369	59.609	30.716	14.787	1154.869	93.885	40.987	26.430
RCRMG	6.008	1.602	0.860	0.552	50.667	14.638	8.543	5.099	143.373	34.561	21.767	10.411	183.401	46.232	26.603	18.702
RCRSW	6.062	1.610	0.852	0.541	51.163	14.536	8.529	5.114	144.039	34.635	21.857	10.402	184.908	46.496	26.563	18.735
RCRHU	6.260	1.669	0.886	0.588	7.090	1.953	1.033	0.709	18.178	3.223	2.125	1.018	111.844	10.035	5.742	3.624
RCRHM	6.066	1.611	0.857	0.556	5.747	1.577	0.828	0.604	17.580	1.806	1.105	0.606	190.640	48.671	27.979	19.585
RCRBI	6.441	1.708	0.896	0.589	6.354	1.614	0.853	0.623	7.483	1.876	1.122	0.623	80.351	2.041	1.136	0.762

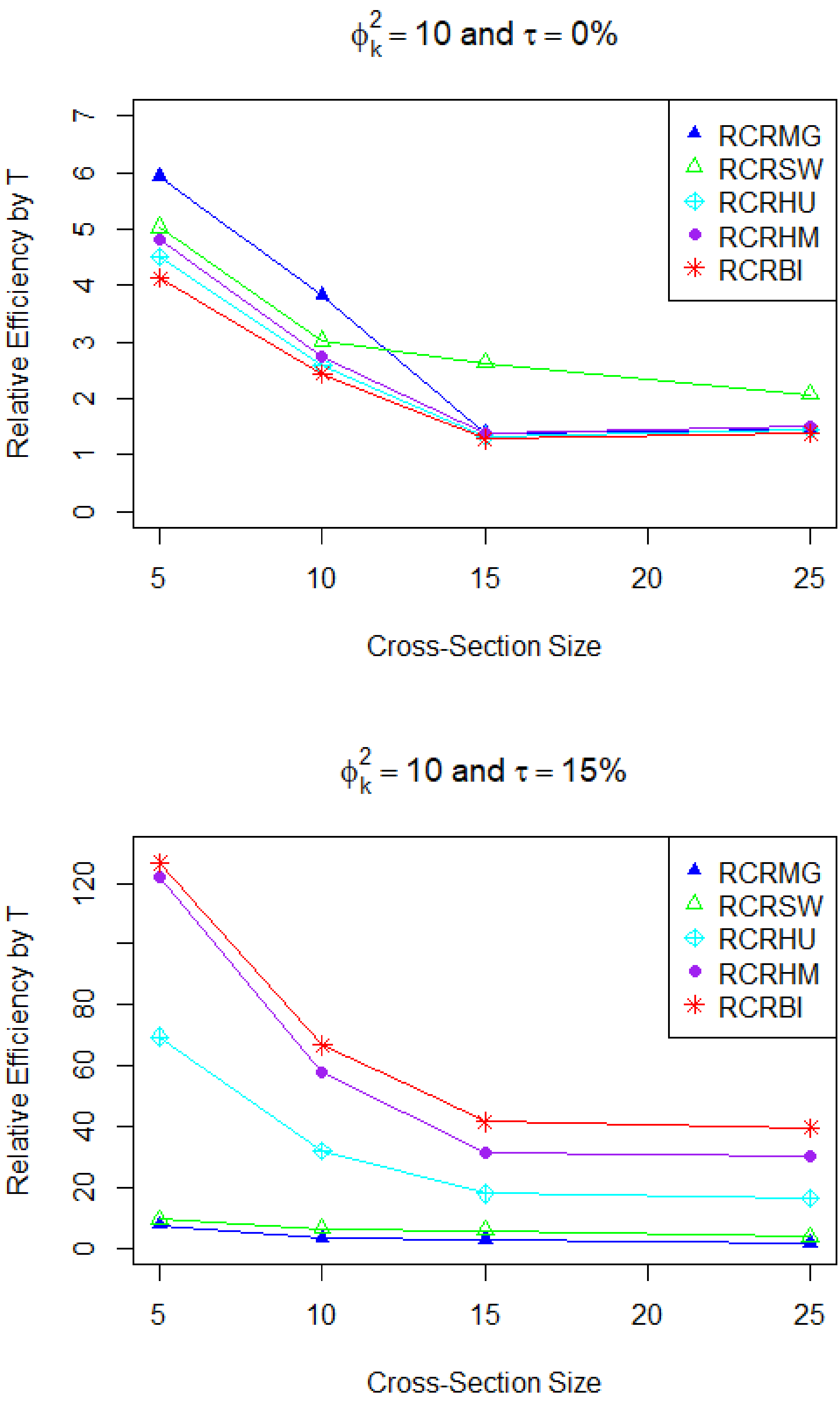

Figure 1. Relative efficiency for various estimators when $\sigma_{\varepsilon_{ii}} = 5$.

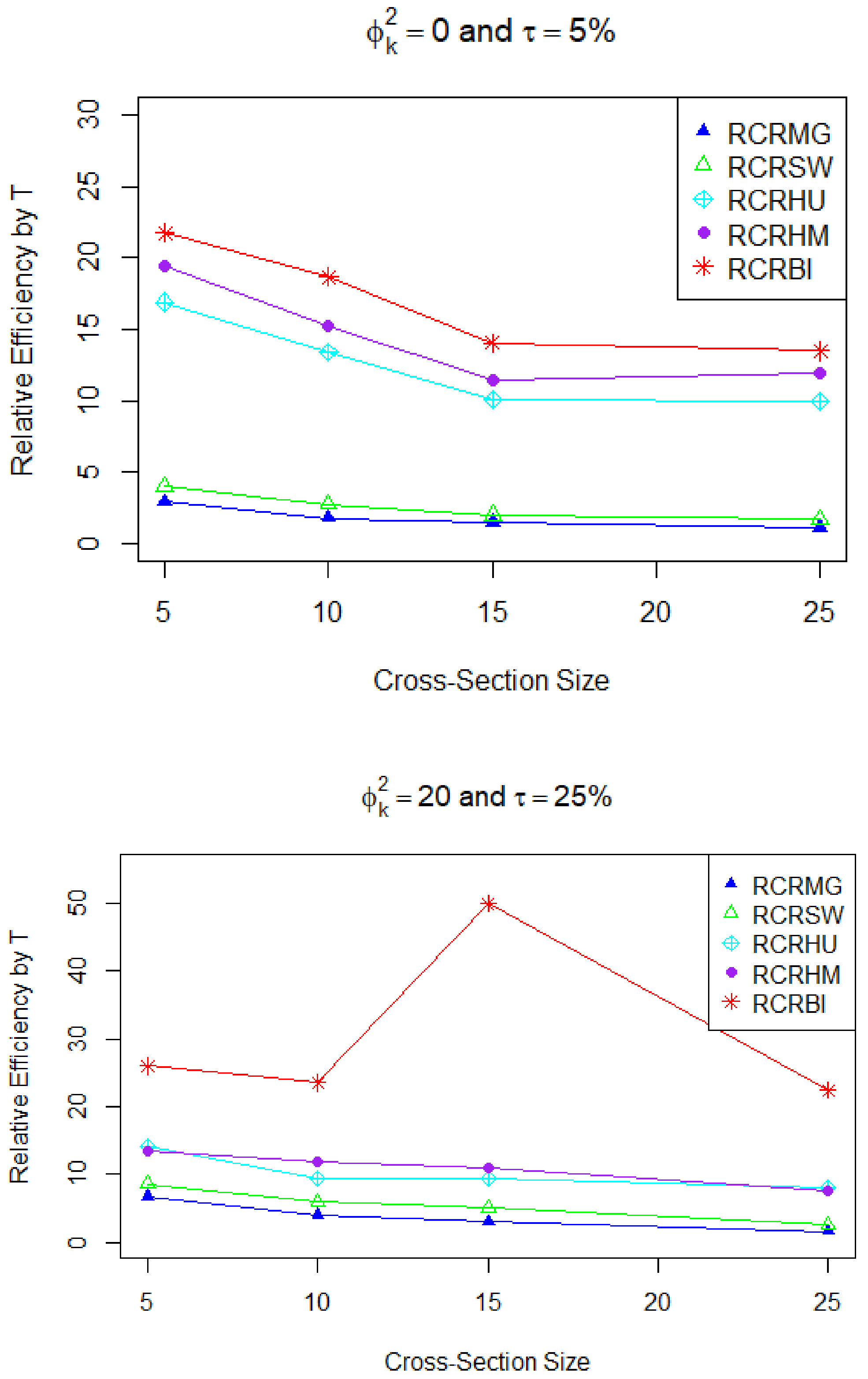

Figure 2. Relative efficiency for various estimators when $\sigma_{\varepsilon_{ii}} = 15$.

sectors, high energy consumption, and reliance on fossil fuels are some of the reasons why these nations routinely rank at the top. The International Energy Agency (IEA) [49], BP Statistical Review of World Energy [50], and World Bank Development Indicators [51], an online database for G20 countries, were the sources of annual data from 2000 to 2024. With 125 observations for each of the top 5 G20 countries, the balanced panel data are matched by year and country. For more details about the dataset, see Alghamdi et al. [52].

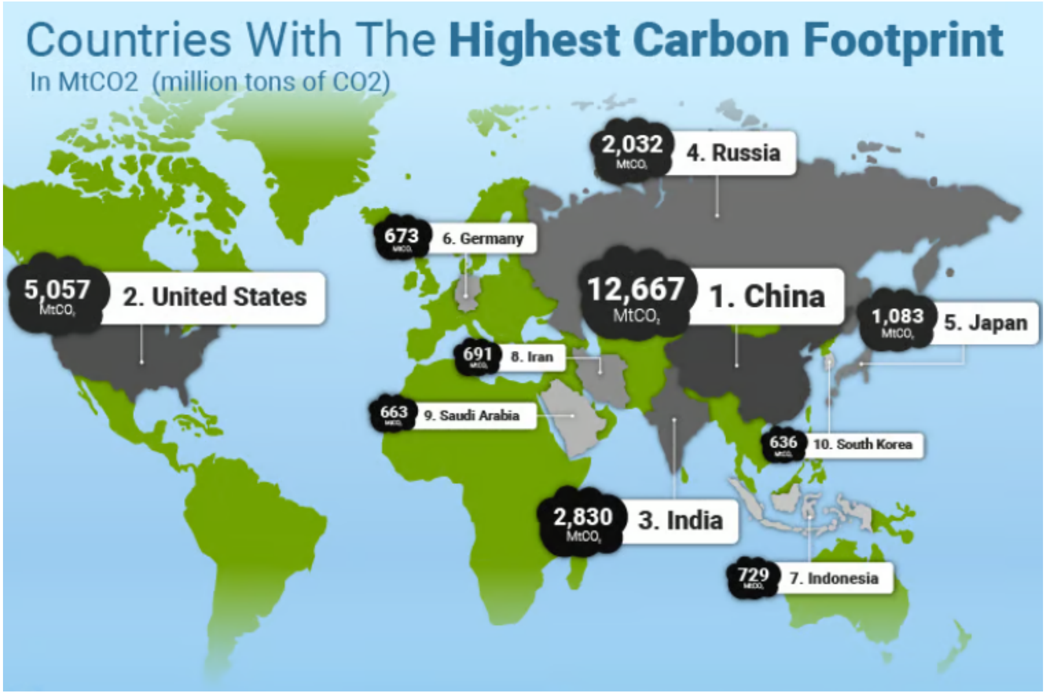

Figure 3. The top 5 G20 countries with the highest CO_2 emissions, source [53].

According to historical data, the energy sector, specifically the generation of heat and electricity, is the largest contributor to world emissions, accounting for 76% of total greenhouse gas emissions. Figure 4, provides important information about the increase and decrease in carbon emissions over the past 20 years in the top 5 G20 countries. The fact that nations like China and India have been on an upward trajectory since the year 2000 is noteworthy. However, the USA has experienced a slight increase or continuous drop since 2010, even if it is still at number two on the list. Additionally, there is evidence that almost all nations saw a reduction in emissions during the first year of the COVID-19 epidemic, which was 2019–2020. This was probably brought on by the worldwide lockdown, which led fewer people to travel, businesses to close, and less fuel to be consumed worldwide.

In our energy management systems application, the response variable (y_{it}) is the total CO_2 emissions, while the explanatory variables include: the non-renewable energy ($X_{1,it}$), the renewable energy ($X_{2,it}$), and the nuclear energy ($X_{3,it}$). In this application, we will use the algorithm described in Section 5 to obtain the results of the proposed robust M-estimators. The descriptive statistics for the variables taken into consideration in this investigation are shown in Table 7. From the results, it can be noted that for all variables, the Positive skewness (skewness > 0) and high kurtosis (kurtosis > 3) suggest heavy-tailed distributions with outliers. Also, from Jarque-Bera test results, all variables have p-values of 0.0000, confirming that the null hypothesis of normality is rejected at any significance level. Fur-

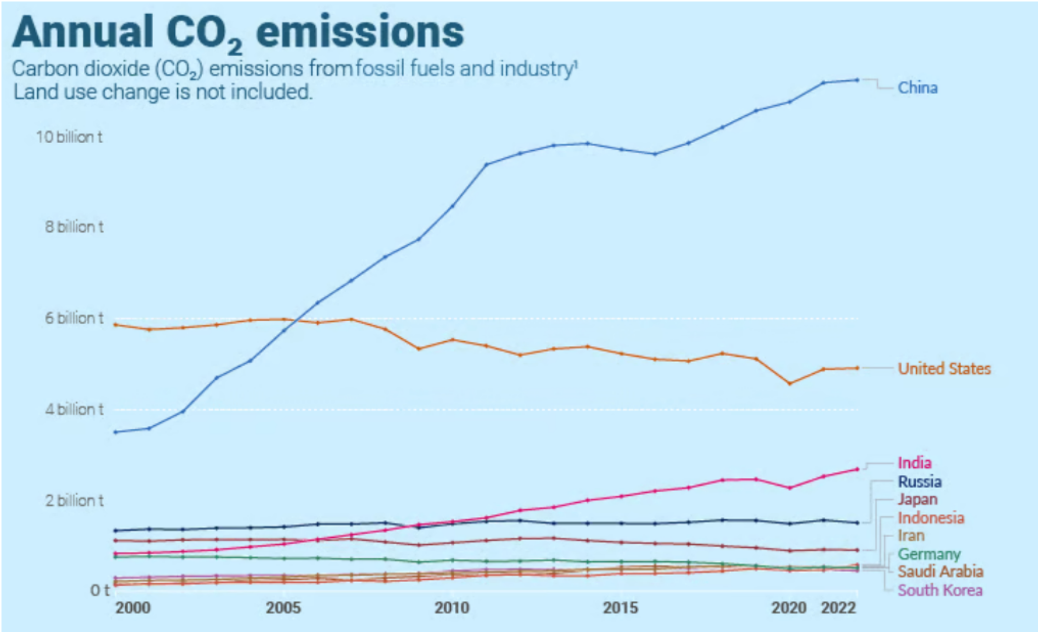

Figure 4. Annual CO_2 emissions for the top 5 G20 countries, source [54].

thermore, as indicated by Table 8, the insurance dataset does not exhibit a multicollinearity problem, since all the Variance Inflation Factor (VIF) values for all variables are less than 5 [55]. Figure 5, displays the description of the study variables. It is clear that some bubbles stand out due to their size or position; these could represent outliers. Based on the correlation coefficient results shown in Figure 6, it can be observed that the highest correlation exists between $X_{1,it}$ and $X_{3,it}$, whereas the lowest correlation is found between y_{it} and $X_{3,it}$, while y_{it} and $X_{3,it}$ have a very weak and non-significant negative correlation.

Table 7. Descriptive statistics for all variables.

Statistics	y_{it}	$X_{1,it}$	$X_{2,it}$	$X_{3,it}$
Mean	3922.2770	37.4023	19.9649	255.9833
Median	2021.2710	18.8090	3.0678	150.3420
Maximum	14524.4800	205.5945	160.1453	890.7809
Minimum	994.8629	-2.9667	0.0680	-14.5179
Std. Dev.	3423.0310	50.1913	34.7762	297.3365
Skewness	1.3986	1.9368	2.3943	1.2021
Kurtosis	4.0498	5.7124	8.1843	2.8728
Jarque-Bera	46.4942	116.4651	259.4185	30.1915
Probability	0.0000	0.0000	0.0000	0.0000
Number of observations	125	125	125	125
VIF	—	3.9660	3.0474	4.3716

In the R programming language, we used the "*lmtest*" package to get the likelihood ratio test (LRT) statistic for testing the random coefficients of the RCR model, see Baltagi [2]. The results of the LRT showed that the alternative hypothesis (H_1) was accepted, meaning that the coefficients are random.

Since the LRT results chisq = 13.932, df = 6, with p-value < 0.001.

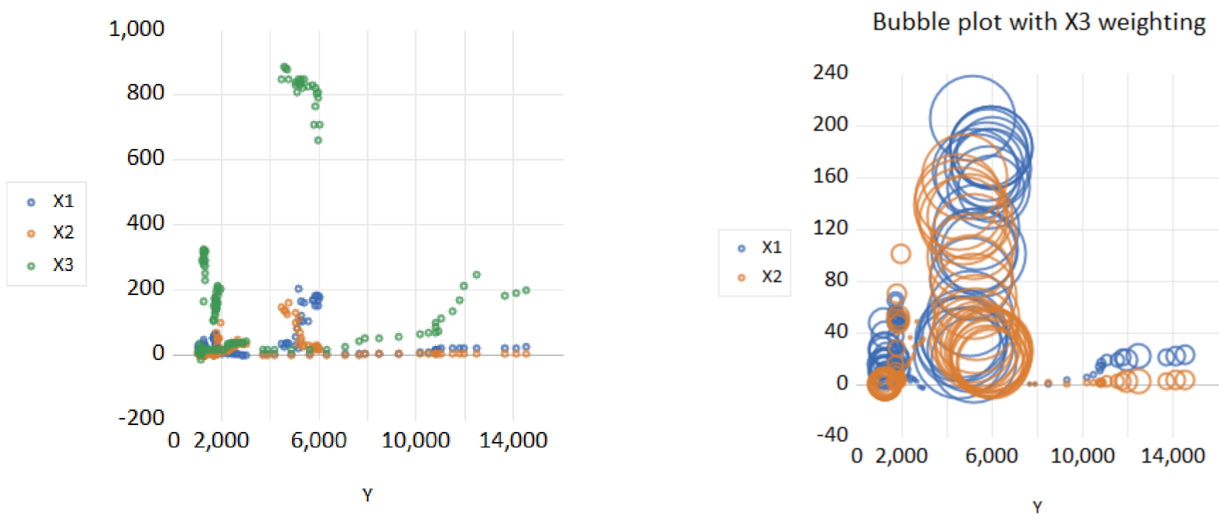

Figure 5. Description of the study variables.

Figure 7, indicates that the residuals of the RCR model are not normally distributed. This is confirmed by Shapiro-Wilk test results of the residuals: W-statistic = 0.81867 with P-value < 0.0001. Since the p-value less than 0.05, then reject H_0, this means that the residuals are not assumed to be normally distributed. Moreover, we will check the outliers by plotting the Cook's distance and boxplots of the CO_2 emissions for each top 5 G20 countries, as in Figure 8, the values are plotted to identify outlier points, the outlier points identified by the Boxplots appeared to be the same as was observed in the leverage values (h_{ii}) as is evident from the Cook's distance. This figure shows that the energy management systems panel dataset contains outlier values.

7.1. Non-robust RCR Estimators Results

The existence of outlier values in this dataset of energy management systems has been confirmed. The RCR model's coefficients will then be estimated using non-robust estimators (RCRCP, RCRMG, and RCRSW); Table 8 displays the estimation outcomes. For each variable in the estimation results using non-robust estimators, Table 8, shows the estimated coefficient values and standard errors. These results imply that the RCRSW estimator has the lowest standard errors. We came to the conclusion that the RCRSW estimate outperforms the RCRCP and RCRMG estimators based on the data in Table 8. When compared to the RCRCP and RCRMG estimators, it is evident that the RCRSW estimator has the lowest values of all goodness-of-fit metrics, as indicated by the mean absolute error (MAE), mean square error (MSE), root of mean square error (RMSE), Akaike's information criterion (AIC), and Bayesian information criterion (BIC).

7.2. Proposed Robust RCR M-Estimator Results

The robust RCR model was employed using the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) in an effort to get around the outlier problem, which violates the RCR Model's assump-

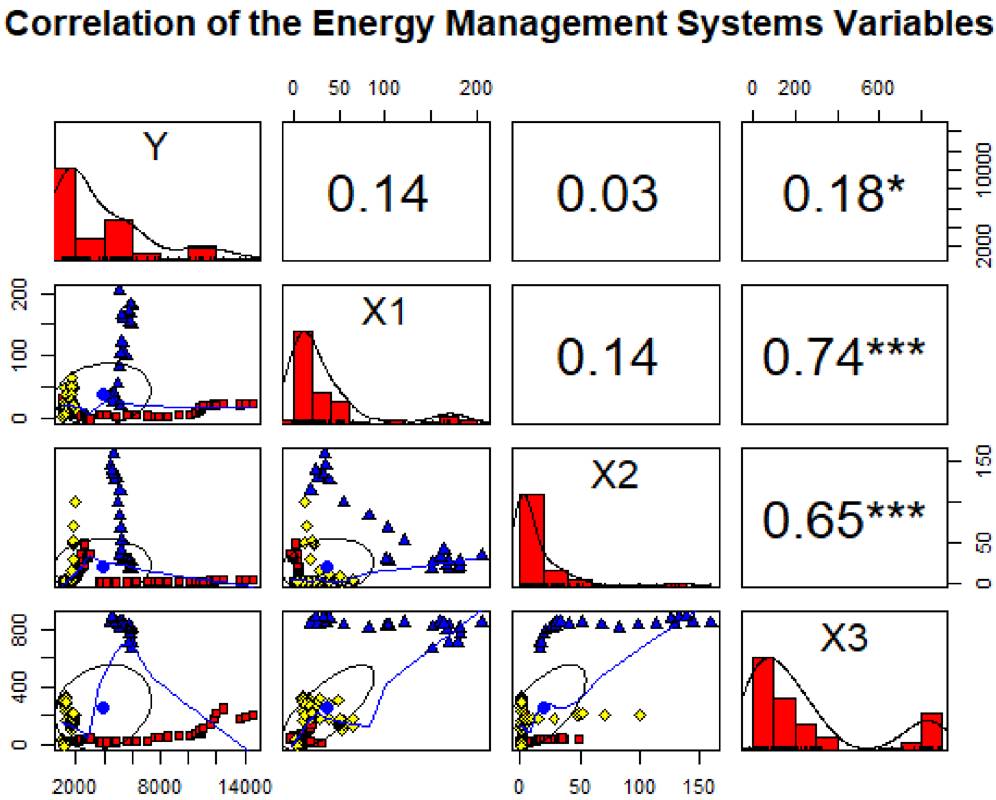

Figure 6. Correlation coefficients of the study variables.

Table 8. Non-robust RCR estimators results.

Variables	RCRCP		RCRMG		RCRSW	
	Coefficient	**Std. Error**	**Coefficient**	**Std. Error**	**Coefficient**	**Std. Error**
Intercept	3456.7244***	404.5349	1300.5662***	352.5499	1465.6321***	321.8854
$X_{1,it}$	-11.8757	12.0087	35.8617***	9.3844	17.3237**	8.2138
$X_{2,it}$	-24.3222**	10.9126	-25.1939***	7.4555	-15.7299**	6.8863
$X_{3,it}$	5.4508**	2.6291	-3.1673	2.1215	-6.3068***	1.7754
Goodness-of-fit Measures						
MAE	2398.4426		1958.5027		1835.6429	
MSE	994841.7526		127496.9248		110882.2819	
RMSE	997.4175		357.0671		332.9899	
AIC	2391.3527		1438.3658		1262.5682	
BIC	2405.4936		1504.2843		1318.9531	

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

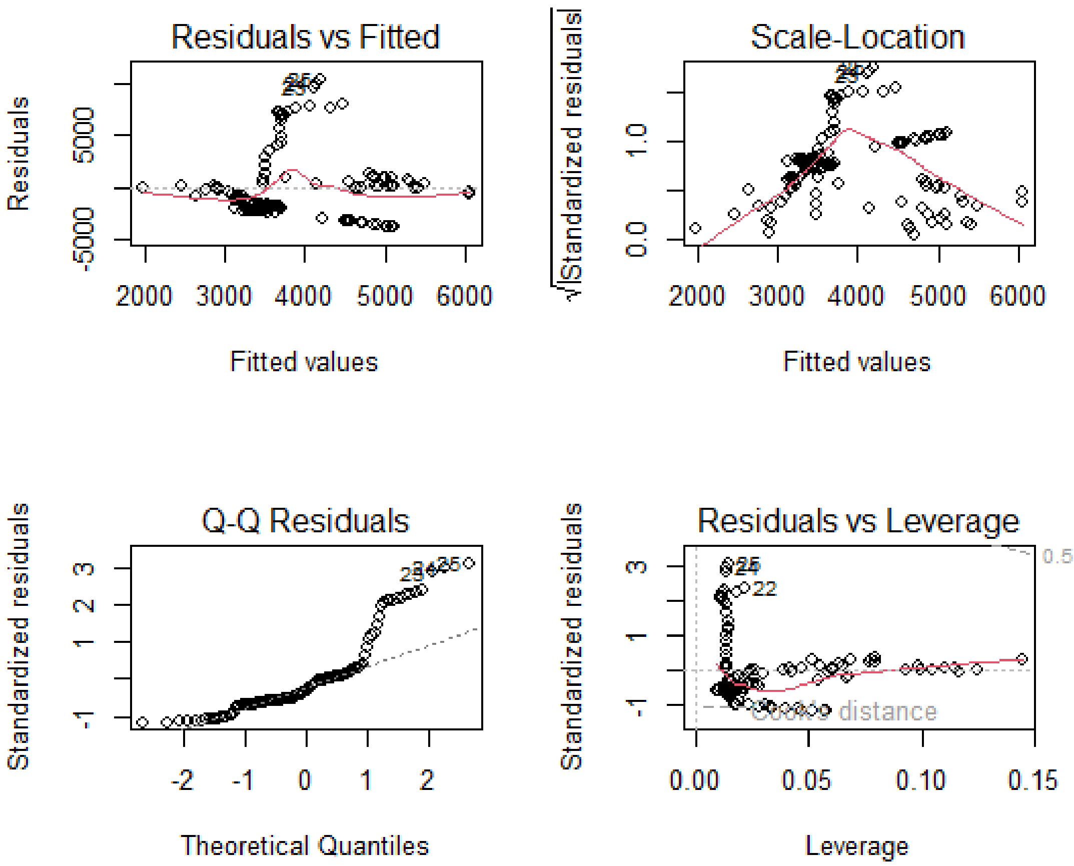

Figure 7. Residual analysis of the energy management systems dataset.

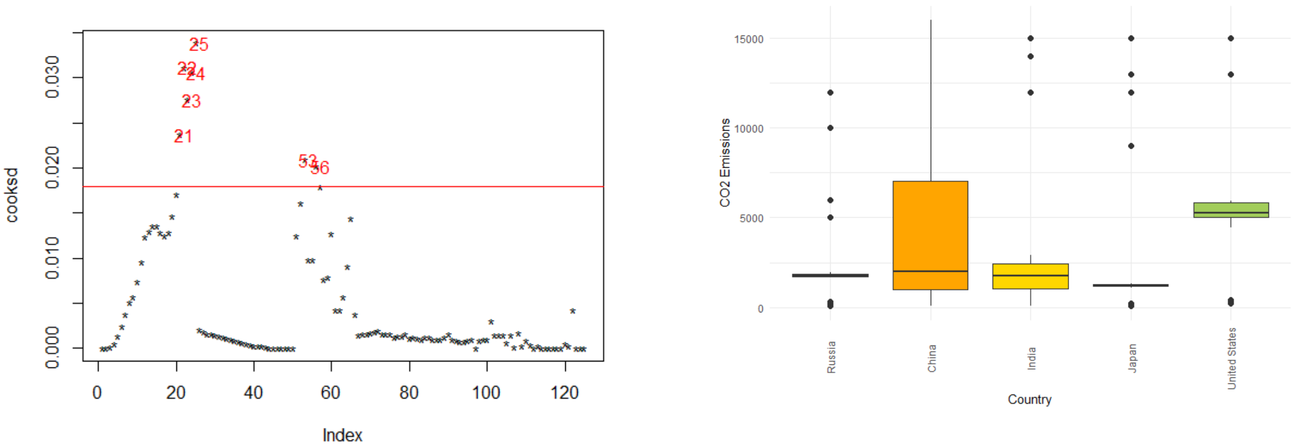

Figure 8. Cook's distance and boxplots of the CO_2 emissions for the top 5 G20 countries of the energy management systems dataset.

tions, and to get good results for examining the three key variables influencing CO_2 emissions in the top 5 G20 countries. The results are displayed in Table 9.

Table 9, demonstrates that most independent variables for the proposed M-estimator are significant because their p-values are less than 5%. These findings show that the proposed M-estimator (RCRHU, RCRHM, and RCRBI) has the smallest standard errors when compared to non-robust (RCRCP, RCRMG, and RCRSW) estimators. Furthermore, as compared to the M-estimator (RCRHU, RCRHM, and RCRBI) estimator, the RCRBI approach has the lowest values of all goodness-of-fit indices, MAE, MSE, RMSE, AIC, and BIC values. This indicates that, as compared to the non-robust estimators (RCRCP, RCRMG, and RCRSW) approaches, the robust M-estimation method increased the RCR model's significance and efficiency. We may conclude that the proposed robust M-estimator is the best estimation technique for the energy management systems dataset.

Furthermore, the RCRBI estimate results in Table 9 indicate that CO_2 emissions will rise by 18.8775% for each percentage point of energy consumption that is classified as non-renewable for all five of the top 5 G20 countries. Additionally, all five of the top 5 G20 countries saw a 17.5483% reduction in CO_2 emissions for every 1% increase in their use of renewable energy. Likewise, all five of the top 5 G20 countries will see a 1.8645% decrease in CO_2 emissions for every 1% rise in nuclear energy usage. Across the top 5 G20 countries, the results show that nuclear and renewable energy consumption are considerably inversely correlated with CO_2 emissions, but non-renewable energy consumption is significantly positively correlated with CO_2 emissions.

Table 9. Proposed robust RCR M-estimator results.

Variables	RCRHU		RCRHM		RCRBI	
	Coefficient	Std. Error	Coefficient	Std. Error	Coefficient	Std. Error
Intercept	1770.9111***	163.7294	1185.0288***	102.743	986.1811***	86.9888
$X_{1,it}$	10.3421**	4.8603	17.3763***	3.0563	18.8775***	2.5823
$X_{2,it}$	-8.4061**	3.9482	-15.8953***	3.8586	-17.5483***	3.2669
$X_{3,it}$	-2.0523*	1.0641	-1.9012**	0.6677	-1.8645***	0.5654
Goodness-of-fit Measures						
MAE	1042.7174		993.6138		941.8425	
MSE	82901.2482		76207.6931		70448.0936	
RMSE	287.9258		276.0574		265.4206	
AIC	972.3394		925.1749		901.8626	
BIC	989.4817		939.3165		913.0435	

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

8. Conclusion and Recommendations

In the RCR model, the problem of outliers causes the classical estimators to be inefficient. In this case, the robust RCR reduces the effects of the outlier values. Therefore, in this paper, we proposed a novel robust M-estimator with different objective functions and compared these with the non-robust (RCRCP, RCRMG, and RCRSW) estimator. To examine the performance of the estimators, we conducted a Monte Carlo simulation study and a practical empirical application to Energy Management Systems. The simulation result shows that, in case there are no outliers (τ = 0%), the classical estimators are better, while the robust M-estimators have lower performance. However, it can be noted that the RCRMG estimator performs well even in small samples, while the RCRSW estimator performs well even in large samples. While in the presence of outliers, the proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) are more effective compared to the classical estimators (RCRCP, RCRMG, and RCRSW) regardless of whether the coefficients of regression are random or fixed. Also, the findings of the energy management systems application show that proposed robust M-estimators (RCRHU, RCRHM, and RCRBI) are better than non-robust estimators (RCRCP, RCRMG, and RCRSW) with outlier values. In addition, the RCRBI estimator is more efficient than RCRHU and RCRHM. We conclude by advising practitioners to estimate the regression parameters of the RCR model using the novel robust M-estimator if an outlier problem arises. For academics and policymakers engaged in scientific research and development, this study is anticipated to yield valuable information. Additionally, it makes the statistical techniques for analyzing the panel data better, especially when handling outliers. Future work will concentrate on expanding the model's scalability and suitability for contemporary large-scale panel datasets by utilizing penalized robust estimates, shrinkage techniques, and distributed computing frameworks to manage high-dimensional settings. To address the issue of outliers in the RCR and generalized RCR models, we also want to create new robust estimators, such as the robust S-estimator, the robust MM-estimator, and others.

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