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Research article

Topp-Leone Modified Kies-G Family of Distributions: Properties, Actuarial Measures, Inference and Applications

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Abstract: We introduce a new two-parameter generalized family of distributions named the Topp-Leone Modified Kies-G family by combining the Topp-Leone-G and Modified Kies-G families. Several statistical properties of the proposed family were derived, including the moments, moment-generating function, order statistics, entropy, mean deviation, measures of inequality, and actuarial measures. A baseline distribution called the Topp-Leone Modified Kies exponential distribution is defined as a special member of this family. The proposed model supports the left and right-skewed, decreasing and decreasing-increasing-decreasing density forms, as well as the decreasing, increasing and bathtub hazard forms. A simulation study was carried out using different estimation techniques, including maximum likelihood, maximum product spacing, least squares, Cramer-Von-Mises and Anderson-Darling techniques, and the efficiency of the estimators was assessed. The practical applicability of the model is illustrated with three real-life datasets using various model adequacy and goodness-of-fit measures along with Vuong's test and leave-one-out log-likelihood. The findings indicate that the proposed model provides a better fit than the four well-known three-parameter models, demonstrating its applicability in reliability and survival analyses.

Keywords: Topp-Leone-G family, Modified Kies-G family, skewness, actuarial measures.
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Key Notations

Notation	Definition		
cdf	Cumulative distribution function	Mk	Modified Kies distribution
pdf	Probability density function	Mk-G	Modified Kies-G distribution
TL	Topp-Leone distribution	*TLMk-G*	Topp-Leone-Modified Kies G family
TL-G	Topp-Leone generalized family	*TLMkE*	Topp-Leone-Modified Kies Exponential
		mgf	Moment generating function

1. Introduction

The evolving complexity of modern data involves continuous developments in the statistical distributions, leading to the construction of generalized families that improve the behaviour of classical models through various approaches. Several methods have been introduced to generalize probability distributions by adding parameters. One of the earliest approaches was by Mudholkar and Srivatsava [43], who introduced the exponentiated-G (Exp-G) family by raising the baseline *cdf* to a power. Subsequently, Marshall and Olkin [42] developed a simpler method for introducing a parameter to the baseline distribution.

Eugene et al.[27] introduced the beta-G family using the baseline *cdf* as the upper limit in the beta distribution. Although this family lacks a closed form of *cdf*, it can be considered the most useful family in the literature, as few other families have been introduced based on this. Cordeiro and de Castro [24] extended this idea by replacing the random variable in the Kumaraswamy distribution[38] with the baseline *cdf*, leading to the Kumaraswamy-G family. Another approach was proposed by Alzaatreh et al.[14], who introduced a new family using the baseline *pdf* as the integrand and any function of the baseline distribution as the upper limit. Several authors have studied generalized family distributions by Hassan and Nassr including Inverse Weibull Generator family of distributions [31], Power Lindley-G family [32], and the exponentiated generalized power function distribution [33], Type II general inverse exponential family by Farrukh Jamal et al. [35], Log-logistic Tan-G family by Zaidi et al.[61], Gull Alpha power family by Kilai et al. [36], New Sine-G family by Benchicha et al. [21], ratio exponentiated general family by Bantan et al. [20], extended odd inverse Weibull generator family by Abdellal et al. [1], unit inverse exponentiated Weibull distribution by Hasan and Alharbi.[30]. Tahir and Nadarajah [57] and Ahmad et al. [6] provided an overview of the developments in the generalized family of distributions.

Each classical probability distribution or family has unique characteristics, particularly when modelling different data patterns. Combining two families can integrate their features, resulting in a more flexible model. Following this idea, we construct a new two-parameter family of distributions by combining the TL-G [9] and Mk-G families [7]. This approach is not new, and several researchers like [11, 55] have used this method.

The Topp-Leone distribution is a continuous probability distribution developed by Topp and Leone [58] to model J-shaped patterns. It was originally used for real-world datasets, such as equipment failures. Later, the distribution gained wide recognition after Nadarajah and Kotz [45] further explored its mathematical properties. However, since the distribution is defined for a bounded interval, it fails to model the modern data. Several studies on Topp-Leone distribution include the Transmuted Topp-Leone power function distribution introduced by Hassan et al. [50], Type-II Topp-Leone power Lomax distribution by Al-Marzouki et al. [8], New power Topp-Leone distribution by Atchade et al. [18] and El-Saeed et al.[26] introduced the power inverted Topp-Leone distribution. Furthermore, Abushal et al. [2] and Nassr et al. [46] applied the acceptance sampling to the inverted Topp-Leone and power-inverted Topp-Leone distributions, respectively.

The Topp Leone-G family, developed by Al-Shomrani et al.[9], overcomes the limitation of boundedness. They derived statistical properties and proposed an exponential extension named Topp-Leone exponential distribution, and demonstrated its application to the failure rate of component data. The

TL-G family's distribution function and *pdf* are given by:

$$F(x;b) = \left\{1 - [1 - G(x)]^2\right\}^b; \quad x > 0 \tag{1.1}$$

$$f(x;b) = 2bg(x)[1 - G(x)]\left\{1 - [1 - G(x)]^2\right\}^{b-1} \tag{1.2}$$

where, $b > 0$ is the parameter.

Some notable generalized families extending from the TL-G family are Reyad et al. [49] introduced the Topp-Leone odd Lindley-G family and introduced its extensions to uniform, Lomax and Pareto distributions. Sule et al.[53] introduced the Topp-Leone exponentiated-G family. Sule et al.[52] introduced the Topp-Leone Kumaraswamy-G family. Oluyede et al. [47] introduced the Topp-Leone-Gompertz-G family. Chipepea et al. [23] introduced the Topp-Leone Marshall-Olkin-G family. Gabanakgosi and Oluyede [29] introduced the Topp-Leone-Gompertz-G Power series family, and proposed some special cases, derived their properties and estimated the parameters using different estimation methods. Atchade et al.[17] developed a four-parameter family called the Topp-Leone Kumaraswamy Marshall-olkin-G family.

The Mk-G family was introduced by Al-Babtain et al. [7], and developed using the reduced Kies distribution by Kumar and Dharmaja [37] as the baseline model. This represents a specific instance of the Weibull-H family [56] when the scale parameter of the family equals one. The distribution function and *pdf* of the family are as follows:

$$F(x;a) = 1 - e^{-\left(\frac{G(x)}{1-G(x)}\right)^a} \tag{1.3}$$

$$f(x;a) = ag(x)\frac{G(x)^{a-1}}{[1 - G(x)]^{a+1}}ag(x)\frac{G(x)^{a-1}}{[1 - G(x)]^{a+1}}e^{-\left(\frac{G(x)}{1-G(x)}\right)^{a-1}} \tag{1.4}$$

where, $a > 0$ is parameter and $G(x)$ is any baseline distribution.

Several extensions of the Mk-G family have been proposed. Bandar et al. [19] proposed an exponentiated reduced Kies-G family in which the authors considered the base distributions named reduced Kies distribution to introduce the family. Afify et al. [4] contributed to this line of research by considering the special case of the Weibull-H family, which reduces to the Mk-G family and introduces the Marshall-Olkin Weibull-G family. The KMk-G family was proposed by Swetha and Nagarjuna [55], where the family supports heavy-tailed patterns, as shown in the simulation study. An extension of the family named Kumaraswamy power Lomax distribution was recently proposed by Swetha et al [54]. An inverted modified Kies-G (IMk-G) family was introduced by Diab et al. [25] using inverse transformation to the Modified Kies distribution. Recently, Vandana and Nagarjuna [44] proposed a new trigonometric extension of the Mk-G family named the Sine modified Kies-G family. Alghamdi et al. [10] introduced the Half-logistic modified Kies exponential distribution, Aljohani et al. [12] applied ranked set sampling to the modified Kies exponential distribution, and the extended reduced Kies distribution was introduced by Almuqrin et al. [13].

In this study, a novel two-parameter family is proposed by integrating the properties of both the TL-G and Mk-G families. It is the Topp-Leone Modified Kies-G family (*TLMk-G*). The TL-G family is known for handling the tails of the model well, whereas the Mk-G family has flexibility in handling the tails and models various hazard rate forms. Unlike most TL models, including the TL-exponential distribution [9], which primarily produces right-skewed or unimodal patterns, the proposed model can capture complex patterns, such as decreasing-increasing-decreasing patterns. Although such a pattern was not observed in the considered datasets, this capability demonstrates the model's potential to handle a wider range of data patterns. Most existing TL-G and Mk-G extensions fail to adequately capture certain complex data patterns observed in real data, such as a density function with a decreasing-increasing-decreasing pattern. This limitation creates a gap in the modelling of survival, actuarial, and reliability data. To address this gap, the proposed TLMk-G model combines the strengths of both families, resulting in a model that is significantly more flexible than the existing TL and Mk-based models. It's two shape parameters work together to control the skewness, kurtosis, and tail behaviour of the model.

The key contributions of the paper include:

- Various statistical properties of the family such as ordinary moments, incomplete moments, probability-weighted moments, generating functions, order-statistics, and entropy, are derived.
- Income and inequality measures, including Lorenz, Bonferroni and Zenga curves, and actuarial measures such as Value at Risk, expected shortfall, Tail Value at Risk, tail variance and tail variance premium are presented.
- A scale parameter 'c' is added to the new family to enhance its flexibility, based on the *TLMkE* distribution.
- The *TLMkE* model can fit various data patterns, including left-skewed, right-skewed, decreasing, decreasing-increasing-decreasing density patterns, and decreasing, bathtub, and increasing hazard patterns.
- The *TLMkE* model demonstrates greater adaptability than the Topp-Leone exponential and Modified-Kies exponential distributions, with 'a' and 'b' having reasonable control over the tails, and 'c' influencing the scale.
- The variance, skewness, and kurtosis of the model are computed using ordinary moments. The inequality measures of the model show typical shapes, and the actuarial measures demonstrate the model's ability to support heavy-tailed patterns, making it useful for income distribution, finance, and risk assessment.
- The simulation was performed using Maximum Likelihood estimation, maximum product spacing, Least square estimation, Anderson-Darling and Cramer-Von-Mises estimation methods, and it shows the most efficient method for parameter estimation of *TLMkE* parameters.
- The applications of the *TLMkE* model show better performance in reliability and survival analysis fields supported by adequacy and goodness of fit measures, Vuong's test and leave-one-out log-likelihood.

The remainder of this paper is organized as follows: Section 2 introduces the proposed *TLMk-G* family, its distribution function and *pdf* and provides the statistical properties of the *TLMk-G* family; in section 3, the parameter estimates of the *TLMk-G* family are derived using different estimation methods; further, section 4 focuses on the *TLMkE* distribution covering its key properties and parameter estimates; sections 5 and 6 deal with the simulation study and application of the proposed model.

2. Topp-Leone Modified Kies-G family

This section describes the basic properties of the *TLMk-G* family. By replacing $G(x)$ in the TL-G distribution function and *pdf* given by Equation (4.1) and Equation (1.2) with the distribution function and *pdf* of the Mk-G family in Equation (1.3) and Equation (1.4), the distribution and density functions of the *TLMk-G* family are obtained as follows:

$$F(x;\vartheta) = \left(1 - e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a}\right)^b \quad ; x > 0 \tag{2.1}$$

$$f(x;\vartheta) = \frac{2abg(x)\,[G(x)]^{a-1}}{[1-G(x)]^{a+1}} e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a} \left(1 - e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a}\right)^{b-1} \tag{2.2}$$

where, $\vartheta = (a, b) > 0$ is the parameter set and $G(x)$ is the baseline distribution.
The quantile function of the *TLMk-G* family is

$$x = G^{-1}\left\{\frac{\left[\left(-\frac{1}{2}\right)\log\left(1-u^{1/b}\right)\right]^{1/a}}{1+\left[\left(-\frac{1}{2}\right)\log\left(1-u^{1/b}\right)\right]^{1/a}}\right\} \tag{2.3}$$

here, $u \sim U(0,1)$.
The survival and hazard functions of *TLMk-G* family represented by $\bar{F}(x;\vartheta)$ and $h(x;\vartheta)$ respectively are given by the following equations.

$$\bar{F}(x;\vartheta) = 1 - \left(1 - e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a}\right)^b$$

$$h(x;\vartheta) = \frac{2abg(x)\,[G(x)]^{a-1}\, e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a} \left(1 - e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a}\right)^{b-1}}{[1-G(x)]^{a+1}\left[1 - \left(1 - e^{-2\left(\frac{G(x)}{1-G(x)}\right)^a}\right)^b\right]}$$

The distribution function of the *TLMk-G* family in Equation (2.1) can be written as

$$F(x;\vartheta) = \sum_s \omega s, t\,[G(x)]^{aj+k} \tag{2.4}$$

where, $\omega_{s,t} = \binom{b}{i}\binom{-aj}{k}(-1)^{i+j+k}\frac{(2i)^j}{j!}$.
Similarly, the *pdf* of the *TLMk-G* family (Equation (2.2))can be written as

$$f(x;\vartheta) = g(x)\sum_s \delta_{s,t}\,[G(x)]^{p-1} \tag{2.5}$$

where, $\delta_{s,t} = 2ab\binom{b-1}{i}\binom{(-a+aj+1)}{k}(-1)^{i+j+k}2^j\frac{(i+1)^j}{k!(p-1)}$ and $s = (i, j, k)$, $t = (a, b)$.
Hence, the linear representation of the distribution function and *pdf* of the proposed *TLMk-G* family can be represented as the weighted sum of exponentiated-G family with the power parameter p where $p = a(j+1) + k$.

Moments and Moment generating function

Statistical moments were used to characterise the distribution. First, we obtain ordinary moments, which describe the central tendency, variability, and shape of the distribution. Next, we compute incomplete moments, which are further used to derive income inequalities and actuarial measures, providing information on risk and tail behaviour. Finally, we derived probability-weighted moments, which offer a good alternative for parameter estimation, particularly in the presence of extreme values. Using the Linear representation of *pdf* in Equation (2.5), the r^{th} ordinary moment of *TLMk-G* family is given by:

$$\mu_r' = \int_0^\infty x^r g(x) \sum_s \delta_{s,t} \left[G(x)\right]^{p-1} dx = \sum_s \delta_{s,t} \int_0^\infty x^r g(x) \left[G(x)\right]^{p-1} dx$$

$$\implies \mu_r' = \sum_s \delta_{s,t} E[Y^r] \tag{2.6}$$

where 'Y' is a random variable of the Exp-G family with the parameter p.
These moments characterise descriptive measures of the distribution, such as mean μ_1', variance $(\mu_2' - \mu_1'^2)$, skewness and kurtosis.
The incomplete moment (I_k) and probability-weighted moment (P_w) of the *TLMk-G* family are given by the following equations.

$$I_k = \int_0^q x^k f(x;\vartheta) dx = \sum_s \delta_{s,t} \int_0^q x^k g(x)[G(x)]^{p-1} dx \tag{2.7}$$

and

$$P_w = \int x^q [F(x;\vartheta)]^r f(x;\vartheta) dx = \sum_s \phi_{s,t} \int x^q g(x)[G(x)]^{p-1} dx$$

where, $\phi_{s,t} = \binom{b(r+1)-1}{i}(-1)^{i+j+k}\frac{(2i+2)^j}{j!}$ and $s = (i, j, k)$, $t = (a, b, r)$. The mgf provides a powerful tool for characterising the distribution by generating moments. The mgf of the *TLMk-G* family is

$$M_X(t) = \int_0^\infty e^{tx} g(x) \sum_s \delta_{s,t} \left[G(x)\right]^{p-1} dx = \sum_s \delta_{s,t} \int_0^\infty e^{tx} \left[G(x)\right]^{p-1} dx$$

$$\implies M_X(t) = \sum_s \delta_{s,t} M_p$$

where, M_p is mgf of Exp-G family with the power parameter 'p'.

Order statistics

Order statistics are essential for analysing the behaviour of ordered data. The *cdf* of r^{th} order statistic is

$$F_{(r)}(x;\vartheta) = \sum_{i=r}^{n} \sum_j \binom{n}{i}\binom{n-i}{j}(-1)^j \left[F(x;\vartheta)\right]^{i+j}$$

similarly, *pdf* of the r^{th} order statistic is

$$f_{(r)}(x;\vartheta) = \frac{1}{B(r,n-r)}\sum_{i}\binom{n-r}{i}\frac{(-1)^i}{i+r}f(x;\vartheta_0)$$

where, $B(\cdot,\cdot)$ is the incomplete beta function and $f(x;\vartheta_0)$, $\vartheta_0 = a, b(i+r)$ is the *pdf* of the *TLMk-G* family.

Entropy

Entropy measures the uncertainty, which is the randomness associated with a probability distribution. The Renyi entropy [48] I_R and Tsallis entropy [59] I_T of the *TLMk-G* family are

$$I_R = \frac{1}{1-\alpha}\log\left(\sum_{s}\theta_{s,t}\int (g(x))^{\alpha}\,[G(x)]^{aj+\alpha(a-1)+k}\right)dx$$

The Renyi entropy reduces to Shannon entropy when $\alpha \to 1$.

$$I_T = \frac{1}{\alpha-1}\left(1-\sum_{s}\theta_{s,t}\int (g(x))^{\alpha}\,[G(x)]^{aj+\alpha(a-1)+k}\right)dx$$

where, $\theta_{s,t} = (2ab)^{\alpha}\binom{\alpha(b-1)}{i}\binom{-aj-\alpha(a+1)}{k}(-1)^{i+j+k}\frac{(2i+2\alpha)^j}{j!}$ and $s = (i, j, k)$, $t = (a, b, \alpha)$.

Mean deviation

The mean deviation from the mean 'μ' can be obtained using the Equation (2.4). It is as follows

$$E[|X-\mu|] = \int_{d}^{\infty}|x-\mu|f(x;\vartheta)dx = \sum_{s}\zeta_{s,t}\left[(G(\mu))^{aj+k+1} - (G(d))^{aj+k+1}\right]$$

the mean deviation from the median 'M_d' is

$$E[|X-M_d|] = \int_{d}^{\infty}|x-M_d|f(x;\vartheta)dx = (\mu - M_d)\sum_{s}\zeta_{s,t}\left[(G(M_d))^{aj+k+1} - (G(d))^{aj+k+1}\right]$$

where, $\zeta_{s,t} = \frac{\omega_{s,t}}{aj+k+1}$.

Inequality measures

The Lorenz (L_0)[40], Bonferroni (B_0) [22] and Zenga (Z_0) [62] curves are graphical and analytical tools used to understand income inequality or wealth distribution within a population. The L_0 shows the amount of income earned by different segments of the population, with a straight line representing perfect equality. The higher the deviation of the curve from the line, the more significant the inequality will be. Using the incomplete moments ($k = 1$) of the *TLMk-G* family defined in Equation (2.7), they can be defined as follows:

$$L_0 = \frac{I_k}{E(x)} = \frac{\sum_s \delta_{s,t} \int_0^q x g(x)[G(x)]^{p-1} dx}{\sum_s \delta_{s,t} \int_0^\infty x^k g(x)[G(x)]^{p-1} dx} \tag{2.8}$$

The B_0 is similar but assigns more importance to the lower portion of the income distribution, making it more sensitive to poverty, the Bonferroni curve B_0 is given by

$$B_0 = \frac{L_0}{F(q)} = \frac{\sum_s \delta_{s,t} \int_0^q x g(x)[G(x)]^{p-1} dx}{\left\{\left(1 - e^{-2\left(\frac{G(q)}{1-G(q)}\right)^a}\right)^b\right\} \sum_s \delta_{s,t} \int_0^\infty q^k g(q)[G(q)]^{p-1} dq} \tag{2.9}$$

On the other hand, the Zenga curve compares the average income of people below a certain point to those above it, making it useful for detecting extreme differences between the rich and poor. Using L_0, the Zenga curve can be expressed as

$$Z_0 = 1 - \frac{L_0(1-u)}{(1-L_0)u} = \frac{u\left(\sum_s \delta_{s,t} \int_0^\infty x^k g(x)[G(x)]^{p-1} dx\right) - \left(\sum_s \delta_{s,t} \int_0^q x g(x)[G(x)]^{p-1} dx\right)}{u\left(\sum_s \delta_{s,t} \int_0^q x g(x)[G(x)]^{p-1} dx\right)} \tag{2.10}$$

where, $u \in (0, 1)$.

Actuarial measures

This section presents the actuarial measures used to evaluate risk and tail behaviour in financial and insurance applications.

Value at Risk

The Value at Risk (VaR) by Artzner [15] is the quantile function of a model that functions as a standard risk indicator, enabling researchers to understand risk exposure. The VaR of the *TLMk-G* family is obtained by replacing 'u' with 'q' in Equation (2.3).

$$VaR_{TLMkG} = G^{-1}\left\{\frac{\left[\frac{-1}{2}\log\left(1 - q^{1/b}\right)\right]^{1/a}}{1 + \left[\frac{-1}{2}\log\left(1 - q^{1/b}\right)\right]^{1/a}}\right\} \tag{2.11}$$

Expected shortfall

The expected shortfall (ES) introduced by Artzner et al. [16] is the average loss when the losses exceed the VaR threshold. It can be considered a more reliable measure than VaR, which ignores extreme losses beyond the threshold. The ES of *TLMk-G* family is

$$ES_{TLMkG} = \frac{1}{u}\int_0^q VaR(x)dx = \frac{1}{q}\int_0^q G^{-1}\left\{\frac{\left[\left(-\frac{1}{2}\right)\log\left(1-q^{1/b}\right)\right]^{1/a}}{1+\left[\left(-\frac{1}{2}\right)\log\left(1-q^{1/b}\right)\right]^{1/a}}\right\}dx$$

Tail Value at risk

The tail value at risk (TVaR) measures the expected loss when they exceed a designated probability threshold, frequently exceeding the VaR. It concentrates on the tail of the distribution to better understand the extreme losses. In modern usage, it is similar to ES. The TVaR of *TLMk-G* family is

$$TVaR_{TLMkG} = \frac{1}{1-q}\int_{VaR}^q xf(x;\vartheta)dx = \frac{1}{1-q}\sum_s \phi_{s,t}\int_{VaR}^q xg(x)[G(x)]^{p-1}dx$$

Tail Variance

Tail Variance (TV), initially introduced by Landsman [39] is the variability of losses beyond the VaR. A high TV indicates unpredictable extreme losses. The TV of the *TLMk-G* family is

$$TV_{TLMkG} = E(X^2|X>x) - (TVaR)^2 = \frac{1}{1-q}\int_{VaR}^q x^2 f(x;\vartheta)dx - (TVaR)^2$$

$$\implies TV_{TLMkG} = \frac{1}{1-q}\sum_s \phi_{s,t}\int_{VaR}^q x^2 g(x)[G(x)]^{p-1}dx - (TVaR)^2$$

Tail varaince premium

The Tail variance premium (TVP) Edward Furman et al. [28] is the cost that insures extreme risks, which includes both their level of severity and unpredictability. This is used for insurance pricing. The TVP *TLMk-G* family is

$$TVP_{TLMkG} = TVaR_{TLMkG} + \lambda TV_{TLMkG}$$

where, $0 < \lambda < 1$.

3. Estimation Methods

To assess the parameter estimation of the proposed model, we consider five commonly used estimation methods: Maximum Likelihood estimation (MLE), Maximum Product Spacing (MPS), Least Squares estimation (LS), Cramer-von Mises (CvM) and Anderson-Darling (AD). These methods represent different approaches to parameter estimation, including the likelihood-based approach, spacing-based approach, least squares and goodness-of-fit based approaches which helps in evaluating various properties of the distribution.

Consider $x_1, x_2, \ldots, x_m$ representing a sample of observations from the *TLMk-G* family. Let $x_{(1)} < x_{(2)} <, \ldots, < x_{(m)}$ denote the corresponding ordered sample drawn from the family. The estimates of parameters 'a' and 'b' can be estimated using the following methods.

Maximum Likelihood estimation

The parameter estimates $\hat{a}$ and $\hat{b}$ are obtained by maximising the likelihood function of *TLMk-G* family which is given by

$$L = \prod_{k=1}^{m} \left\{ \frac{2abG(x_k)\,[G(x_k)]^{a-1}}{[1-G(x_k)]^{a+1}} e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a} \left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)^{b-1} \right\}$$

Now, the log-likelihood function

$$\log L = m\log 2 + m\log a + m\log b + \sum_{k=1}^{m} \log G(x_k) + (a-1)\sum_{k=1}^{m} \log G(x_k) - (a+1)\sum_{k=1}^{m} \log 1 - G(x_k) - 2\sum_{k=1}^{m} \left(\frac{G(x_k)}{1-G(x_k)}\right)^a + (b-1)\sum_{k=1}^{m} \log\left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)$$

The estimates $\hat{a}$ and $\hat{b}$ of the parameters a and b can be obtained by solving the following equations

$$\frac{\partial \log L}{\partial a} = \frac{m}{a} + \sum_{k=1}^{m} \log G(x_k) - \sum_{k=1}^{m} \log(1 - G(x_k)) - 2\sum_{k=1}^{m} \left(\frac{G(x_k)}{1-G(x_k)}\right)^a \log\left(\frac{G(x_k)}{1-G(x_k)}\right) + 2(b-1)\sum_{k=1}^{m} \frac{\left(\frac{G(x_k)}{1-G(x_k)}\right)^a \log\left(\frac{G(x_k)}{1-G(x_k)}\right)}{e^{2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a} - 1} = 0$$

$$\frac{\partial \log L}{\partial b} = \frac{m}{b} + \sum_{k=1}^{m} \log\left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right) = 0$$

Maximum Product Spacing

The Maximum Product spacing is based on the concept of information contained in the spacing of the distribution function and considers parameter estimates that maximise this measure. The MPS function is given by:

$$M = \left(\prod_{k=1}^{m+1} D_k\right)^{1/(m+1)} \tag{3.1}$$

where, $D_k = F(x_{(k)};\vartheta) - F(x_{(k-1)};\vartheta)$. Upon applying the logarithm on both sides, we obtain the following

$$\log M = \sum_{k=1}^{m} \log\{F(x_{(k)};\vartheta) - F(x_{(k-1)};\vartheta)\}$$

Using the distribution function of *TLMkG* family in Equation (2.1), the following equation is obtained

$$\log M = \log\left\{\left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)^b - \left(1 - e^{-2\left(\frac{G(x_{k-1})}{1-G(x_{k-1})}\right)^a}\right)^b\right\}$$

The estimates of the parameters 'a' and 'b' are obtained as

$$\frac{\partial \log M}{\partial a} = \frac{1}{m+1}\sum_{i=1}^{m+1}\left(\frac{P_{a_k} - P_{a_{k-1}}}{D_k}\right) = 0 \quad ; \frac{\partial \log M}{\partial b} = \frac{1}{m+1}\sum_{i=1}^{m+1}\left(\frac{P_{b_k} - P_{b_{k-1}}}{D_k}\right) = 0$$

where,

$$P_{a_k} = \frac{\partial F(x_{(k)};\vartheta)}{\partial a} = 2b\left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)^{b-1} e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\left(\frac{G(x_k)}{1-G(x_k)}\right)^a \log\left(\frac{G(x_k)}{1-G(x_k)}\right)$$

$$P_{b_k} = \frac{\partial F(x_{(k)};\vartheta)}{\partial b} = \left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)^b \log\left(1 - e^{-2\left(\frac{G(x_k)}{1-G(x_k)}\right)^a}\right)$$

Least Squares estimation

The least-squares estimation follows the principle of minimizing the overall squared errors between the theoretical distribution function and empirical plotting positions. The LSE function for the *TLMk-G* family is

$$LS = \sum_{k=1}^{m}\left[F(x_{(k)};\vartheta) - \frac{k}{m+1}\right]^2$$

The LS estimates can be obtained by solving the following equations

$$\frac{\partial \log LS}{\partial a} = 2\sum_{k=1}^{m} P_{a_k}\left[F(x_{(k)};\vartheta) - \frac{k}{m+1}\right] = 0 \quad ; \frac{\partial \log LS}{\partial b} = 2\sum_{k=1}^{m} P_{b_k}\left[F(x_{(k)};\vartheta) - \frac{k}{m+1}\right] = 0$$

Cramer Von Mises estimation

The Cramer-Von Mises method depends on minimising the squared distance between the empirical and theoretical distribution function's, assigning equal weights to all parts of the distribution uniformly. The CvM measure of *TLMk-G* family is

$$CE = \frac{1}{12m} + \sum_{k=1}^{m}\left[F(x_{(k)};\vartheta) - \frac{2k-1}{m}\right]^2$$

The estimates are obtained by

$$\frac{\partial CE}{\partial a} = 2\sum_{k=1}^{m} P_{a_k}\left[F(x_{(k)};\vartheta) - \frac{2k-1}{m}\right] = 0 \quad ; \frac{\partial CE}{\partial b} = 2\sum_{k=1}^{m} P_{b_k}\left[F(x_{(k)};\vartheta) - \frac{2k-1}{m}\right] = 0$$

Anderson Darling estimation

The Anderson-Darling estimation modifies the CvM method by applying a weighted function that increases sensitivity to fluctuations in the distribution tails. The AD function of *TLMk-G* family is given by

$$AE = -m - \frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\log(F(x_{(k)};\vartheta)) + \log(1 - F(x_{(m-k+1)};\vartheta))\right] = 0$$

The estimates of the proposed family can be obtained using the following equations

$$\frac{\partial AE}{\partial a} = -\frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\frac{P_{a_k}}{F(x_{(k)};\vartheta)} - \frac{P_{a_{m-k+1}}}{F(x_{(m-k+1)};\vartheta)}\right] = 0,$$

$$\frac{\partial AE}{\partial b} = -\frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\frac{P_{b_k}}{F(x_{(k)};\vartheta)} - \frac{P_{b_{m-k+1}}}{F(x_{(m-k+1)};\vartheta)}\right] = 0.$$

4. Topp-Leone Modified Kies Exponential distribution

The distribution function and *pdf* of *TLMkE* distribution are derived by substituting the distribution function and *pdf* of the exponential distribution into Equations (2.1) and (2.2). The resulting equations are presented in Equations (4.1) and (4.2), respectively.

$$F(x;\vartheta_1) = \left(1 - e^{-2(e^{cx}-1)^a}\right)^b \quad ; x > 0, \tag{4.1}$$

$$f(x;\vartheta_1) = 2abce^{acx-2(e^{cx}-1)^a}\left(1 - e^{-cx}\right)^{a-1}\left(1 - e^{-2(e^{cx}-1)^a}\right)^{b-1}, \tag{4.2}$$

where, $\vartheta_1 = (a, b, c) > 0$ are parameters.
The quantile function of the *TLMkE* distribution with $u \sim U(0, 1)$ is

$$x = \frac{1}{c}\log\left\{\left[\left(-\frac{1}{2}\right)\log(1 - u^{1/b})\right]^{1/a} + 1\right\}. \tag{4.3}$$

Figure 1 shows the various density patterns supported by *TLMkE* distribution. It accommodated left-skewed, right-skewed, decreasing and unknown shapes. Figure 1(a) demonstrates the influence of parameter 'c' on the scale of the distribution. It clearly shows both platykurtic with long-tail and leptokurtic with short-tail behaviour of the distribution for smaller and higher values of 'c' respectively. Increasing the value of 'c' is also reducing the spread. This indicates that the parameter 'c' influences only the scale of the distribution.

The Figure 1(b) shows the effect of the parameter 'b' on the left-tail of *TLMkE* distribution. Increasing the value of 'b' is resulting in a sharper peak and a reduction in the left tail. Figure 1(c), shows a decreasing pattern for $0 < b < 1$ and $0 < c < 1$ and increasing values of 'a'. The Figure 1(d) shows a decreasing-increasing-decreasing shape for fixed values of 'a' and 'b' and changing values of 'c'. It can be observed that higher values of parameter 'a' result in a left-skewed plot. Smaller values of 'a' and 'c' and higher values of 'b' resulted in a right-skewed plot.
The survival and hazard functions of the *TLMkE* distribution are given in the Equation (4.4) and Equation (4.5), respectively.

$$\bar{F}(x;\vartheta_1) = 1 - \left(1 - e^{-2(e^{cx}-1)^a}\right)^b, \tag{4.4}$$

and

$$h(x;\vartheta_1) = \frac{2abce^{acx-2(e^{cx}-1)^a}\left(1 - e^{-cx}\right)^{a-1}\left(1 - e^{-2(e^{cx}-1)^a}\right)^{b-1}}{1 - \left(1 - e^{-2(e^{cx}-1)^a}\right)^b}. \tag{4.5}$$

The Figure 2 illustrates the various hazard rate patterns supported by the *TLMkE* distribution. In Figure 2(a), a decreasing hazard pattern is observed when the parameters 'b' and 'c' take very small

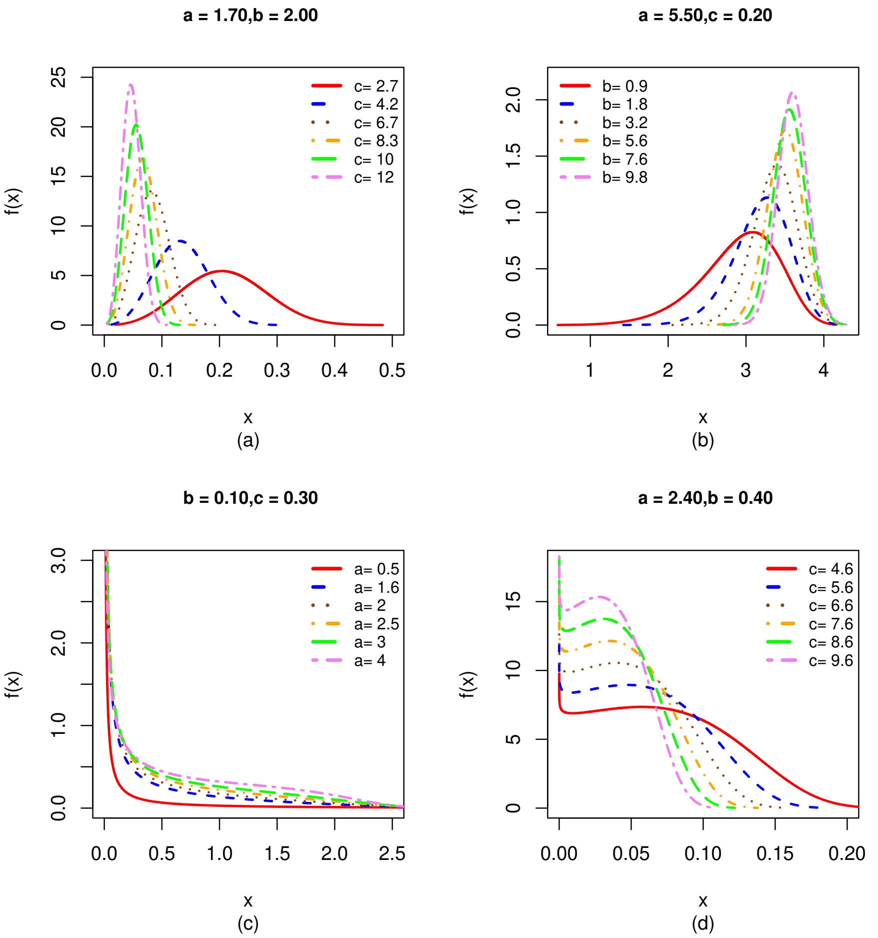

Figure 1. Density plots of TLMkE distribution

values and increasing values of 'a'. A bathtub pattern appears in Figure 2(b) for higher values of 'a' and $b < 0.1$. The Figure 2(c) displays an increasing hazard pattern for fixed values of 'a' and 'c' with varying 'b' values. From these observations, it is clear that for $b < 0.1$ and higher values of 'a' a bathtub pattern is observed and when the values of the parameter 'b' are gradually increasing, the hazard pattern shifts to an increasing shape, as seen in Figures 2(b) and 2(c).

Stochastic ordering of TLMkE distribution

Stochastic ordering was used to compare the distributions of two random variables based on the key characteristics of the model. We examine the stochastic ordering of the *TLMkE* distribution by analysing the monotonicity of its distribution function presented in Equation (4.1), with respect to each parameter a, b and c.

Now let us consider the partial derivative of the *cdf* $F(x; \vartheta_1)$:

$$\varphi_1 = \frac{\partial F(x;\vartheta_1)}{\partial a} = 2be^{-2(e^{cx}-1)^a}(e^{cx}-1)^a \log\left((e^{cx}-1)^a\right)\left(1-e^{-2(e^{cx}-1)^a}\right)^{b-1}$$

$$\varphi_2 = \frac{\partial F(x;\vartheta_1)}{\partial b} = \left(1-e^{-2(e^{cx}-1)^a}\right)^b \log\left(1-e^{-2(e^{cx}-1)^a}\right)$$

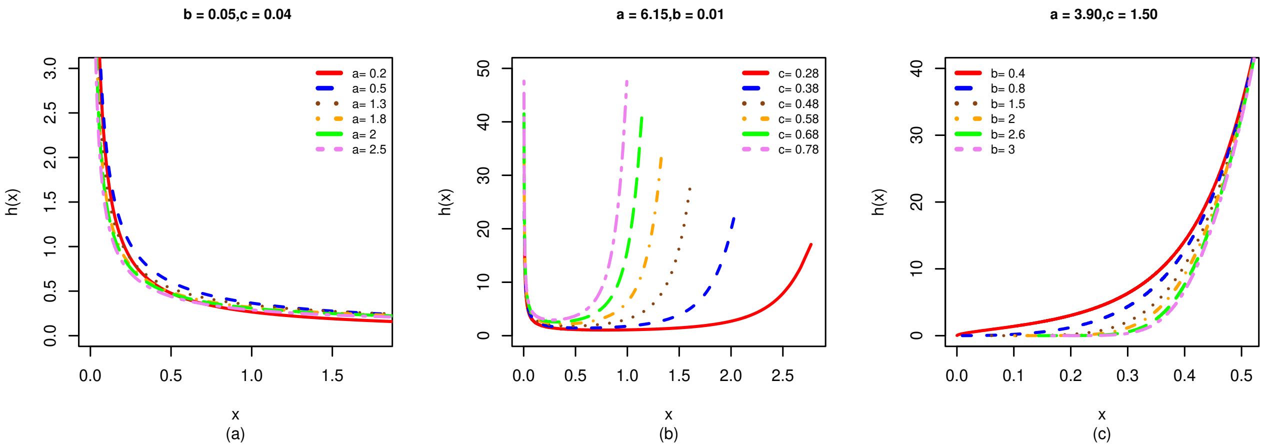

Figure 2. Hazard plots of TLMkE distribution

$$\varphi_3 = \frac{\partial F(x;\vartheta_1)}{\partial c} = 2abxe^{cx-2(e^{cx}-1)^a}(e^{cx}-1)^{a-1}\left(1-e^{-2(e^{cx}-1)^a}\right)^{b-1}$$

The sign of each derivative determines the effect of the corresponding parameter on the *cdf*.

Parameter a: For $x > 0$, the sign of φ_1 depends on $\log(e^{cx}-1)$. There are three possible cases:

(i) if $e^{cx}-1 > 1$ then $\log(e^{cx}-1) > 0$ and φ_1, so F increase in 'a'.
(ii) if $0 < e^{cx}-1 < 1$ then $0 < \log(e^{cx}-1) < 0$ and $\varphi_1 < 0$ so F is decreasing in 'a'.
(iii) if $(e^{cx}-1) = 1$ then $\varphi_1 = 0$

Parameter b: For $x > 0$, term $\left(1-e^{-2(e^{cx}-1)^a}\right)^b$ lies in $(0,1)$. Therefore, its logarithm is negative and thus, $\varphi_2 < 0$ for all $x > 0$. Therefore, F decrease in 'b'.
Parameter c: All the terms in φ_3 are positive for $x > 0$ and $\vartheta_1 > 0$. Hence, $\varphi_3 > 0$ and F increase in 'c'.

These monotonicity properties of the *TLMkE* model lead to the following first-order stochastic dominance results.
Now, let us consider parameter sets: (a,b,c_1) and (a,b,c_2), (a,b_1,c) and (a,b_1,c) where $a_1 \leq a_2$, $b_1 \leq b_2$ and $c_1 \leq c_2$. Thus, we have the following dominance properties:

$$F(x;a,b,c_1) \leq F(x;a,b,c_2)$$
$$F(x;a,b_1,c) \leq F(x;a,b_2,c)$$

This means that the parameter sets (a,b,c_1) and(a,b_1,c) stochastically dominate sets (a,b,c_2) and (a,b_2,c) respectively. For parameter 'a', the stochastic ordering depends on x: $F(x;\vartheta_1)$ increases with 'a' for $x > \log(2)/c$ and decreases for $0 < x < \log(2)/c$.

Moments of TLMkE distribution

The moments of the *TLMkE* distribution are derived by substituting the distribution function and *pdf* of the exponential distribution into Equation (2.6). The Table 1 provides the first, second, third and fourth moments along with the variance, skewness and kurtosis values computed from them. Based on

Table 1. Moments of TLMkE distribution

Parameter values		μ_1'	μ_2'	μ_3'	μ_4'	Var	sk	kur
	a							
b=1.40,c=1.20	0.2	0.37372	0.7684	2.26697	8.07447	0.62873	9.17238	13.33412
	2.5	0.45800	0.22663	0.11903	0.06561	0.01687	0.00992	2.74837
	4.8	0.50825	0.26422	0.14007	0.07555	0.00590	0.22629	3.28537
	7.7	0.53247	0.28608	0.15497	0.08458	0.00256	0.41324	3.69912
	11	0.54534	0.29872	0.16431	0.09074	0.00132	0.52955	3.95306
	14	0.55199	0.30554	0.16956	0.09434	0.00084	0.59344	4.09204
	b							
a=2.50, c=1.20	0.7	0.36807	0.16025	0.07736	0.04019	0.02477	0.00129	2.43403
	4.3	0.56966	0.33291	0.19923	0.12189	0.00840	0.00004	2.96234
	7.5	0.61201	0.38072	0.24060	0.15438	0.00616	0.01059	3.00659
	9.2	0.62592	0.39733	0.25570	0.16677	0.00555	0.01910	3.02467
	12	0.64291	0.41820	0.27518	0.18315	0.00487	0.03334	3.05091
	18	0.66669	0.44852	0.30448	0.20857	0.00404	0.06059	3.09682
	c							
a=0.60, b=1.90	0.5	0.93444	1.56000	3.49557	9.43536	0.68683	1.75583	4.78110
	3.8	0.12295	0.02701	0.00796	0.00283	0.01189	1.75582	4.78109
	4.2	0.11124	0.02211	0.00590	0.00190	0.00973	1.75578	4.78106
	7.4	0.06314	0.00712	0.00108	0.00020	0.00314	1.75588	4.78108
	9.8	0.04768	0.00406	0.00046	0.00006	0.00179	1.75576	4.78016
	12	0.03893	0.00271	0.00025	0.00003	0.00119	1.75580	4.78959
	a							
b=5.9, c=4.2	2	0.17147	0.0303	0.00551	0.00103	0.0009	0.01743	2.98112
	4	0.16778	0.02837	0.00484	0.00083	0.00022	0.00253	3.02497
	6	0.16676	0.02791	0.00469	0.00079	0.0001	0.01297	3.05793
	8	0.16629	0.02771	0.00463	0.00077	0.00006	0.0211	3.05300
	10	0.16602	0.0276	0.00459	0.00077	0.00004	0.02745	3.10035
	12	0.16585	0.02753	0.00457	0.00076	0.00002	0.03148	3.0683
	b							
a=5.60, c=2.20	1.2	0.27651	0.07802	0.02240	0.00653	0.00156	0.33207	3.44806
	3.5	0.3079	0.09541	0.02974	0.00933	0.0006	0.06545	3.15306
	5.2	0.31623	0.10045	0.03205	0.01027	0.00045	0.01944	3.07296
	8.4	0.3248	0.10582	0.03458	0.01134	0.00033	0.00022	3.05542
	10	0.32758	0.1076	0.03544	0.01171	0.0003	0.00069	3.02638
	13	0.33149	0.11014	0.03668	0.01224	0.00026	0.00697	3.03016
	c							
a=0.30, b=0.40	1	0.12391	0.17526	0.38596	1.05703	0.15991	25.77052	34.45981
	3	0.04130	0.01947	0.01429	0.01305	0.01777	25.77059	34.45988
	5	0.02478	0.00701	0.00309	0.00169	0.00640	25.77076	34.46007
	7	0.01771	0.00358	0.00113	0.00044	0.00326	25.77171	34.4611
	9	0.01377	0.00216	0.00053	0.00016	0.00197	25.77198	34.46139
	11	0.01127	0.00145	0.00029	0.00007	0.00132	25.77218	34.46168

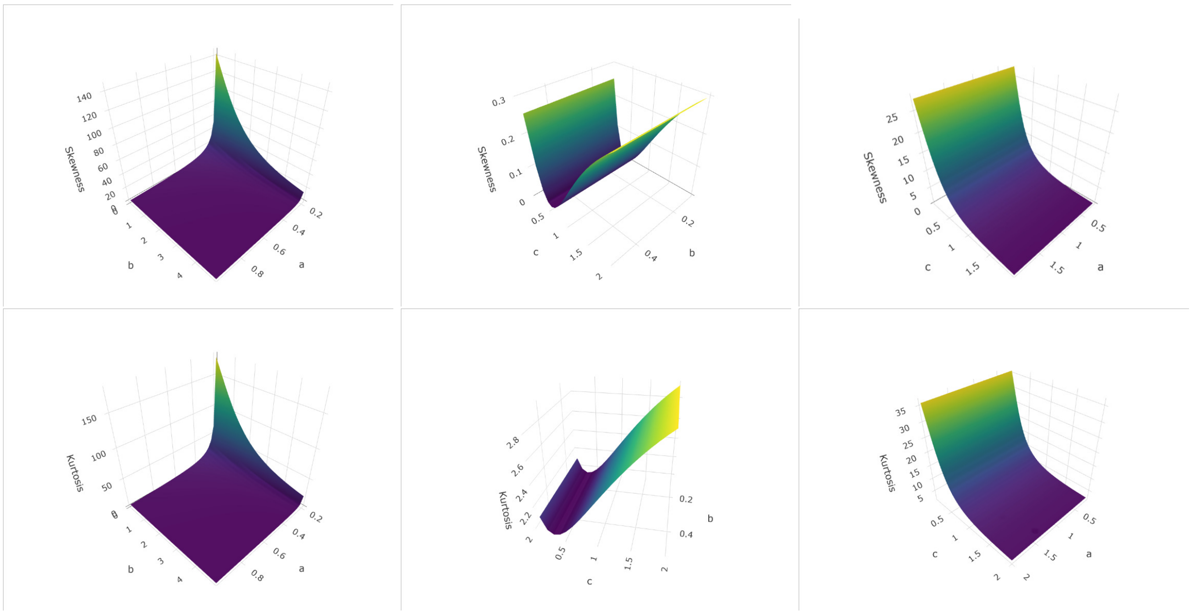

Figure 3. 3-D plots of skewness and kurtosis for different parameter pairs

the results presented in the table, several patterns are observed in the behaviour of *TLMkE* distribution with respect to parameters 'a','b' and 'c'. They are:

- As parameter 'a' increases, the mean steadily increases, while variance, skewness and kurtosis decrease, indicating a shift towards symmetric and light-tailed distributions. Although kurtosis shows slight fluctuations it tends to decrease, indicating a reduced tail heaviness.
- With increasing 'b', the mean and higher moments increase slightly, whereas the variance and kurtosis decrease. The skewness remains low, suggesting near-symmetric distributions at higher 'b' values.
- In contrast, increasing 'c' leads to a decline in moments and variance, with skewness and kurtosis showing minimal variation.
- Smaller values of the parameters yield right-skewed, heavy-tailed shapes, while higher values, especially of 'b' and 'c', show a more symmetric and light-tailed distribution.
- The findings indicate that the modified Kies parameter 'a' and Topp-Leone parameter'b' have a more significant influence on the skewness and kurtosis of *TLMkE* distribution, whereas 'c' has a comparatively minimal impact.

The Figure 3 presents 3D plots of skewness and kurtosis for various parameter combinations.

Inequality measures

The L_0, B_0 and Z_0 curves of *TLMkE* distribution are obtained upon substitution of the distribution function and *pdf* of the baseline distribution in Equations (2.8), (2.9) and (2.10). The Figure 4 demonstrates how the Bonferroni, Lorenz and Zenga curves show that the inequality in the *TLMkE* distribution varies with the parameters.

- The Bonferroni curve's exhibited both concave and convex shapes, indicating the model's sensitivity to low-income segments. This makes the *TLMkE* model flexible for modelling populations with widespread poverty, which is a common scenario.

- The Lorenz curve exhibits a typical increasing and convex shape. The changing convex shape of the curve with parameter changes shows the ability of the model to accommodate different levels of inequality, including mild and severe inequality.
- The Zenga curve displays a U-shaped curve, which shows major inequality among the poor themselves and even more among the rich. This shows that the *TLMkE* is a better choice for modern economic data.

This shows the flexibility of *TLMkE* distribution for modelling data from different economies with different levels of income inequality.

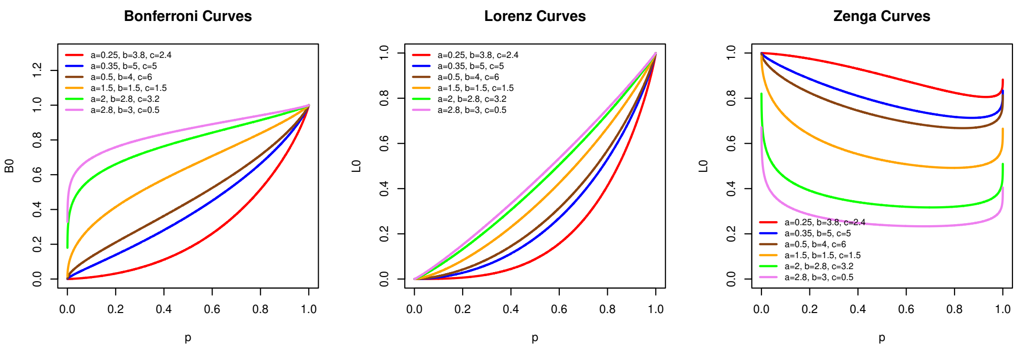

Figure 4. Bonferroni, Lorenz and Zenga curves of TLMkE distribution

Actuarial measures of TLMkE distribution

The VaR of the *TLMkE* distribution is obtained by substituting the exponential distribution in Equation (2.11), and using this equation, we can further obtain other measures. The Table 2 and Figure 5 collectively show these actuarial measures for *TLMkE* distribution across different parameter sets, highlighting how the parameters affect tail risk behaviour. The parameter sets are ordered by the ability of the model to represent the risks ranging from mild to extreme. The impact of the parameters of *TLMkE* distribution on the measures are that small values of the Mk-G parameter 'a', TL-G parameter 'b' and exponential parameter 'c' that are affecting the risk measures in different ways, which is as follows

- It is observed that parameter values $0 < a < 1$, $b > 1$ and $0 < c < 1$ show extreme risk and $a > 1, 0 < b < 1$, $c > 1$ exhibit lighter tails reducing risk measures. This flexibility allows the model to handle both extreme and low-risk scenarios.
- This parameter sensitivity demonstrates that the *TLMkE* distribution can model both light and heavy-tailed scenarios by adjusting the parameter combinations. This makes the model valuable for modelling different risk scenarios.

The trends in risk measures are as follows:

- VaR increases as the 'q' increases, indicating that higher quantiles correspond to greater risk.
- ES is always lower than VaR, because it represents the average loss beyond VaR. It also increases with 'q' but at a slower rate than VaR.

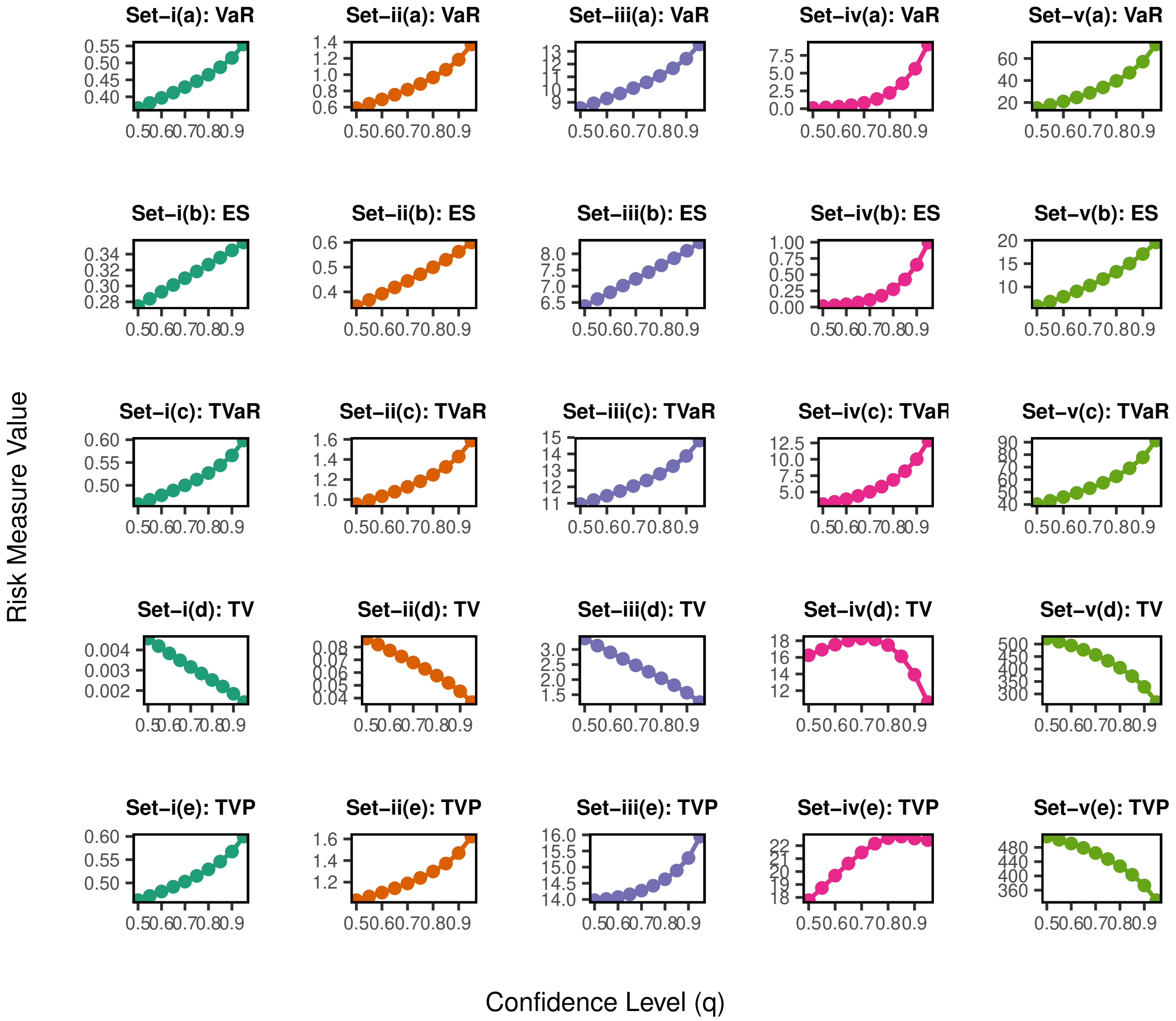

Figure 5. The plot of risk measures vs confidence level q

- TVaR exceeds both VaR and ES because it accounts for tail loss severity. Similar to VaR and ES, it increases with'q', indicating higher extreme losses. Figure 5 Set-iv(a) shows a steeper TVaR curve with heavier tails.
- TV is highly sensitive to parameter choices, as seen in both figure and the table.
- TVP increases with higher 'λ', which means that greater risk resistance comes at a higher financial cost.
- This consistent behaviour shows the model's ability to handle practical data such as pricing insurance premiums.

Table 2. Actuarial measures of TLMkE distribution

Parameter values	q	VaR	ES	TVaR	TV	TVP			
						$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 0.9$
a=2.1,b=1.6 c=1.5	0.50	0.36703	0.27478	0.45889	0.00456	0.45981	0.46118	0.46209	0.46300
	0.55	0.38187	0.28385	0.46828	0.00419	0.46911	0.47037	0.47121	0.47204
	0.60	0.39690	0.29264	0.47814	0.00383	0.47891	0.48006	0.48082	0.48159
	0.65	0.41237	0.30125	0.48865	0.00349	0.48935	0.49040	0.49109	0.49179
	0.70	0.42859	0.30976	0.50002	0.00316	0.50066	0.50161	0.50224	0.50287
	0.75	0.44597	0.31825	0.51260	0.00284	0.51317	0.51402	0.51459	0.51516
	0.80	0.46516	0.32682	0.52690	0.00252	0.52741	0.52816	0.52867	0.52917
	0.85	0.48730	0.33559	0.54390	0.00220	0.54434	0.54500	0.54544	0.54587
	0.90	0.51475	0.34475	0.56564	0.00184	0.56601	0.56656	0.56693	0.56730
	0.95	0.55456	0.35466	0.59828	0.00143	0.59856	0.59899	0.59928	0.59956
a=0.9,b=2.1 c=0.8	0.50	0.59000	0.34281	0.95548	0.08666	0.97281	0.99881	1.01614	1.03347
	0.55	0.64126	0.36759	0.99326	0.08199	1.00966	1.03426	1.05065	1.06705
	0.60	0.69529	0.39263	1.03392	0.07733	1.04939	1.07258	1.08805	1.10352
	0.65	0.75304	0.41810	1.07822	0.07263	1.09275	1.11454	1.12907	1.14359
	0.70	0.81583	0.44423	1.12727	0.06785	1.14084	1.16119	1.17476	1.18833
	0.75	0.88560	0.47129	1.18272	0.06289	1.19529	1.21416	1.22674	1.23931
	0.80	0.96548	0.49961	1.24726	0.05764	1.25879	1.27609	1.28762	1.29914
	0.85	1.06105	0.52973	1.32582	0.05192	1.33620	1.35178	1.36216	1.37255
	0.90	1.18434	0.56250	1.42896	0.04534	1.43802	1.45163	1.46070	1.46977
	0.95	1.37133	0.59970	1.58862	0.03684	1.59599	1.60704	1.61441	1.62178
a=1.5,b=2.7 c=0.07	0.50	8.55567	6.39126	10.95364	3.36065	11.62577	12.63396	13.30609	13.97823
	0.55	8.92392	6.60471	11.19968	3.12743	11.82517	12.76340	13.38888	14.01437
	0.60	9.30138	6.81362	11.46069	2.90376	12.04144	12.91257	13.49332	14.07407
	0.65	9.69463	7.01999	11.74130	2.68678	12.27866	13.08469	13.62205	14.15940
	0.70	10.11210	7.22577	12.04803	2.47357	12.54275	13.28482	13.77953	14.27425
	0.75	10.56568	7.43307	12.39059	2.26079	12.84275	13.52099	13.97314	14.42530
	0.80	11.07388	7.64439	12.78470	2.04402	13.19350	13.80671	14.21551	14.62431
	0.85	11.66956	7.86309	13.25879	1.81646	13.62209	14.16702	14.53031	14.89361
	0.90	12.42250	8.09448	13.87418	1.56510	14.18720	14.65673	14.96975	15.28277
	0.95	13.54113	8.34912	14.81569	1.25492	15.06668	15.44315	15.69413	15.94512
a=0.2,b=1.5 c=0.3	0.50	0.09965	0.01605	3.16349	16.25100	6.41369	11.28899	14.53919	17.78939
	0.55	0.17300	0.02670	3.50019	16.92295	6.88478	11.96167	15.34626	18.73085
	0.60	0.29518	0.04356	3.90910	17.53334	7.41576	12.67577	16.18243	19.68910
	0.65	0.49720	0.07004	4.41214	18.01322	8.01478	13.41875	17.02139	20.62403
	0.70	0.82869	0.11143	5.03925	18.26098	8.69145	14.16974	17.82194	21.47414
	0.75	1.36635	0.17575	5.83185	18.13908	9.45967	14.90140	18.52921	22.15703
	0.80	2.22313	0.27491	6.84924	17.48326	10.34589	15.59087	19.08752	22.58417
	0.85	3.55942	0.42597	8.18467	16.12806	11.41028	16.24870	19.47431	22.69992
	0.90	5.62609	0.65322	10.01872	13.92370	12.80346	16.98057	19.76531	22.55005
	0.95	9.03726	0.99602	12.87101	10.62160	14.99533	18.18181	20.30613	22.43045
a=0.5,b=1.9 c=0.02	0.50	15.04948	5.91009	40.34902	522.13470	144.77600	301.41640	405.84330	510.27020
	0.55	17.83804	6.86510	43.00834	509.35810	144.87990	297.68740	399.55900	501.43060
	0.60	21.00144	7.90847	45.96118	494.45020	144.85120	293.18630	392.07630	490.96640
	0.65	24.61846	9.05152	49.27447	477.10690	144.69590	287.82790	383.24930	478.67070
	0.70	28.80155	10.30905	53.04407	456.91290	144.42670	281.50050	372.88310	464.26570
	0.75	33.72001	11.70113	57.41484	433.27110	144.06910	274.05040	360.70460	447.35890
	0.80	39.64816	13.25632	62.62251	405.25880	143.67430	265.25190	346.30370	427.35540
	0.85	47.08034	15.01819	69.09395	371.29750	143.35340	254.74270	329.00220	403.26170
	0.90	57.07300	17.06102	77.74638	328.21900	143.39020	241.85590	307.49970	373.14350
	0.95	72.76878	19.54019	91.32756	267.39260	144.80610	225.02380	278.50240	331.98090

Estimation methods of TLMkE distribution

Consider a sample of observations $x_1, x_2, \ldots, x_m$ drawn from the *TLMkE* distribution with *pdf* $f(x;\vartheta_1)$ and distribution function $F(x;\vartheta_1)$. The parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{c}$ of *TLMkE* distribution can obtained using different estimation techniques in this section.

Maximum Likelihood estimation of TLMkE distribution

The likelihood function of *TLMkE* distribution is

$$\tau_1 = \prod_{k=1}^{m} f(x;\vartheta_1) = \prod_{k=1}^{m} \left\{ 2abce^{acx-2(e^{cx}-1)^a} \left(1-e^{-cx}\right)^{a-1} \left(1-e^{-2(e^{cx}-1)^a}\right)^{b-1} \right\}$$

Now, the log-likelihood function of *TLMkE* distribution is

$$\log\tau_1 = m\log 2 + m\log a + m\log b + m\log c + ac\sum_{k=1}^{m} x_i - 2\sum_{k=1}^{m}(e^{cx_i}-1)^a + (a-1)\sum_{k=1}^{m}\log\left(1-e^{-cx_i}\right) + (b-1)\sum_{k=1}^{m}\log\left(1-e^{-2(e^{cx}-1)^a}\right)$$

By solving the following equations, we can obtain the estimates $\hat{a}$, $\hat{b}$ and $\hat{c}$ of parameters a, b and c respectively:

$$\frac{\partial\log\tau_1}{\partial a} = \frac{m}{a} + c\sum_{k=1}^{m} x_i - 2\sum_{k=1}^{m}(e^{cx_i}-1)^a\log(e^{cx_i}-1) + \sum_{k=1}^{m}\log(1-e^{-cx_i}) + 2(b-1)\sum_{k=1}^{m}\frac{(e^{cx_i}-1)^a\log(e^{cx_i}-1)}{e^{2(e^{cx_i}-1)^a}-1} = 0$$

$$\frac{\partial\log\tau_1}{\partial b} = \frac{m}{b} + \sum_{k=1}^{m}\log\left(1-e^{-2(e^{cx}-1)^a}\right) = 0$$

$$\frac{\partial\log\tau_1}{\partial c} = \frac{m}{c} + a\sum_{k=1}^{m} x_i - 2a\sum_{k=1}^{m} x_i e^{cx_i}(e^{cx_i}-1)^{a-1} - (a-1)\sum_{k=1}^{m}\frac{x_i}{e^{cx_i}-1} + 2a(b-1)\sum_{k=1}^{m}\frac{x_i e^{cx_i}(e^{cx_i}-1)^{a-1}}{e^{2(e^{cx_i}-1)^a-1}} = 0$$

Maximum Product Spacing

Using the distribution function of *TLMkG* family in Equation (2.1), the following equation is obtained

$$\log\tau_2 = \log\left\{\left(1-e^{-2(e^{cx_i}-1)^a}\right)^b - \left(1-e^{-2(e^{cx_{k-1}}-1)^a}\right)^b\right\}$$

the estimates $\hat{a}$, $\hat{b}$ and $\hat{c}$ are obtained using following equations:

$$\frac{\partial\log\tau_2}{\partial a} = \frac{1}{m+1}\sum_{i=1}^{m+1}\left(\frac{R_{a_k}-R_{a_{k-1}}}{D_k}\right) = 0 \quad ; \frac{\partial\log\tau_2}{\partial b} = \frac{1}{m+1}\sum_{i=1}^{m+1}\left(\frac{R_{b_k}-R_{b_{k-1}}}{D_k}\right) = 0$$

$$\frac{\partial \log \tau_2}{\partial c} = \frac{1}{m+1}\sum_{i=1}^{m+1}\left(\frac{R_{c_k} - R_{c_{k-1}}}{D_k}\right) = 0$$

where,

$$R_{a_k} = \frac{\partial F(x_{(k)};\vartheta_1)}{\partial a} = 2b\left(1 - e^{-2(e^{cx}-1)^a}\right)^{b-1} e^{-2(e^{cx}-1)^a} (e^{cx} - 1)^a \log(e^{cx} - 1)$$

$$R_{b_k} = \frac{\partial F(x_{(k)};\vartheta_1)}{\partial b} = \left(1 - e^{-2(e^{cx}-1)^a}\right)^{b} \log\left(1 - e^{-2(e^{cx}-1)^a}\right)$$

$$R_{c_k} = 2abxe^{cx}\left(1 - e^{-2(e^{cx}-1)^a}\right)^{b-1} e^{-2(e^{cx}-1)^a} (e^{cx} - 1)^{a-1}$$

Least Squares estimation

The Least square estimate function of *TLMkE* distribution is

$$\tau_3 = \sum_{k=1}^{m}\left[F(x_{(k)};\vartheta_1) - \frac{k}{m+1}\right]^2$$

The LS estimates can be obtained by solving following equations

$$\frac{\partial \log \tau_3}{\partial a} = 2\sum_{k=1}^{m} R_{a_k}\left[F(x_{(k)};\vartheta_1) - \frac{k}{m+1}\right] = 0 \quad ; \frac{\partial \log \tau_3}{\partial b} = 2\sum_{k=1}^{m} R_{b_k}\left[F(x_{(k)};\vartheta_1) - \frac{k}{m+1}\right] = 0$$

$$\frac{\partial \log \tau_3}{\partial c} = 2\sum_{k=1}^{m} R_{c_k}\left[F(x_{(k)};\vartheta_1) - \frac{k}{m+1}\right] = 0$$

Cramer-Von-Mises estimation

The CvM estimation function of *TLMkE* distribution is

$$\tau_4 = \frac{1}{12m} + \sum_{k=1}^{m}\left[F(x_{(k)};\vartheta_1) - \frac{2k-1}{m}\right]^2$$

The estimates are obtained by

$$\frac{\partial \tau_4}{\partial a} = 2\sum_{k=1}^{m} R_{a_k}\left[F(x_{(k)};\vartheta_1) - \frac{2k-1}{m}\right] = 0 \quad ; \frac{\partial \tau_4}{\partial b} = 2\sum_{k=1}^{m} R_{b_k}\left[F(x_{(k)};\vartheta_1) - \frac{2k-1}{m}\right] = 0$$

$$\frac{\partial \tau_4}{\partial c} = 2\sum_{k=1}^{m} R_{c_k}\left[F(x_{(k)};\vartheta_1) - \frac{2k-1}{m}\right] = 0$$

Anderson Darling estimation

The AD estimation function of *TLMkE* distribution is

$$\tau_5 = -m - \frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\log(F(x_{(k)};\vartheta_1)) + \log(1 - F(x_{(m-k+1)};\vartheta_1))\right]$$

The estimates of the proposed family can be obtained using the following equations

$$\frac{\partial \tau_5}{\partial a} = -\frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\frac{R_{a_k}}{F(x_{(k)};\vartheta_1)} - \frac{R_{a_{m-k+1}}}{F(x_{(m-k+1)};\vartheta_1)}\right] = 0$$

$$\frac{\partial \tau_5}{\partial b} = -\frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\frac{R_{b_k}}{F(x_{(k)};\vartheta_1)} - \frac{R_{b_{m-k+1}}}{F(x_{(m-k+1)};\vartheta_1)}\right] = 0$$

$$\frac{\partial \tau_5}{\partial c} = -\frac{1}{m}\sum_{k=1}^{m}(2k-1)\left[\frac{R_{c_k}}{F(x_{(k)};\vartheta_1)} - \frac{R_{c_{m-k+1}}}{F(x_{(m-k+1)};\vartheta_1)}\right] = 0$$

5. Simulation study of TLMkE dsitribution

In this simulation study section, we present the results of a simulation study conducted for the *TLMkE* distribution to evaluate the performance of the estimation methods. Based on $N = 1000$ simulations, we assessed five estimation methods: MLE (τ_1), MPS (τ_2), LS (τ_3), CvM (τ_4) and AD (τ_5) across different sample sizes. Samples were generated using the quantile function in Equation (4.3).

The Tables 3,4, 5, 6 and 7 display the mean, bias, and mean square error (MSE) for each method across different sample sizes $n = \{20, 50, 75, 100, 200, 350, 500\}$. Parameter sets were selected to represent a diverse range of underlying density shapes of the *TLMkE* model, including symmetric, skewed and decreasing forms. The estimators were ranked based on their MSE values, with the lowest MSE value given the best rank. These ranks are demonstrated in Table 8. The total ranks are presented within the parentheses in the Table 8. The key observations from the simulation study of *TLMkE* distribution are as follows:

- The MPS and AD methods demonstrated consistent performance by achieving top ranks across all sample sizes and parameter combinations. This indicates their efficiency in estimating parameters of the *TLMkE* model regardless of the underlying shape.
- Their consistent performance is likely due to the ability of MPS to handle complex models and small sample sizes, whereas the AD is known for its sensitivity to tail behaviour.
- The performance of MLE was highly dependent on the sample size, and often performed poorly for small sample sizes. However, its performance improves as the sample size increases, aligned with the asymptotic theory that MLE estimators become efficient with large sample sizes.
- The improved performance is specifically observed for the underlying skewed density form.
- The LS method showed limited effectiveness. It is occasionally competitive with MLE for small sample sizes, but is consistently outperformed by MPS, AD and MLE with increasing sample size.
- Overall, the CvM method was the least effective. It consistently produced the highest MSE values across the different sample sizes and parameter sets.

Furthermore, our simulation results confirm the consistency of the parameter estimators. For all methods, the mean of the estimates approaches the true parameter value whereas the metrics bias and MSE exhibit a consistent decreasing trend with increasing sample size. Figure 6 shows the relationship between the MSE and sample size, providing a clear presentation of these performance trends across all five parameter sets and estimation methods.

Table 3. Simulation results of different estimation techniques for Set-I

		Set-I: a=1.2, b=0.5, c=0.5														
		τ_1			τ_2			τ_3			τ_4			τ_5		
		a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
20	Mean	2.10750	1.36024	0.856	1.943	0.89610	0.62184	2.14716	0.98524	0.68782	2.19622	1.08574	0.77853	1.77432	1.05082	0.72143
	Bias	0.90750	0.86024	0.356	0.743	0.39610	0.12184	0.94716	0.48524	0.18782	0.99622	0.58574	0.27853	0.57432	0.55082	0.22143
	MSE	4.89107	3.55980	0.486	3.772	1.36976	0.14854	4.91916	1.45845	0.19050	5.42928	1.86943	0.29589	2.98366	1.66371	0.22187
50	Mean	1.53451	0.96340	0.684	1.359	0.80712	0.59254	1.58835	0.93937	0.66309	1.59977	0.94912	0.67861	1.38573	0.87208	0.63516
	Bias	0.33451	0.46340	0.184	0.159	0.30712	0.09254	0.38835	0.43937	0.16309	0.39977	0.44912	0.17861	0.18573	0.37208	0.13516
	MSE	1.27765	1.82013	0.237	0.774	0.94317	0.11051	1.84897	1.15931	0.14552	1.86881	1.17861	0.14421	0.83118	0.98910	0.11524
75	Mean	1.45499	0.81121	0.630	1.326	0.70708	0.56365	1.51779	0.79398	0.60645	1.51813	0.82731	0.62711	1.30849	0.76183	0.60079
	Bias	0.25499	0.31121	0.130	0.126	0.20708	0.06365	0.31779	0.29398	0.10645	0.31813	0.32731	0.12711	0.10849	0.26183	0.10079
	MSE	0.77343	1.24541	0.174	0.517	0.66635	0.08248	1.27066	0.77235	0.08393	1.29770	0.85922	0.09308	0.46633	0.66399	0.08884
100	Mean	1.32070	0.76504	0.604	1.256	0.65955	0.54950	1.43048	0.72727	0.58355	1.42259	0.78080	0.60466	1.28478	0.70080	0.57748
	Bias	0.12070	0.26504	0.104	0.056	0.15955	0.04950	0.23048	0.22727	0.08355	0.22259	0.28080	0.10466	0.08478	0.20080	0.07748
	MSE	0.41790	0.96085	0.126	0.276	0.45650	0.05868	0.82347	0.59381	0.06381	0.82003	0.74564	0.08311	0.38330	0.44277	0.05603
200	Mean	1.25918	0.59048	0.537	1.211	0.57205	0.51642	1.27822	0.65588	0.55182	1.32969	0.63173	0.55010	1.24752	0.57675	0.52845
	Bias	0.05918	0.09048	0.037	0.011	0.07205	0.01642	0.07822	0.15588	0.05182	0.12969	0.13173	0.05010	0.04752	0.07675	0.02845
	MSE	0.14166	0.26449	0.035	0.103	0.16305	0.01904	0.31204	0.35921	0.03849	0.34253	0.31108	0.03096	0.14563	0.14423	0.01539
350	Mean	1.23967	0.52042	0.509	1.191	0.53821	0.50906	1.26412	0.54830	0.51704	1.24775	0.57233	0.52714	1.22191	0.54027	0.51356
	Bias	0.03967	0.02042	0.009	-0.009	0.03821	0.00906	0.06412	0.04830	0.01704	0.04775	0.07233	0.02714	0.02191	0.04027	0.01356
	MSE	0.06570	0.03764	0.005	0.056	0.04820	0.00598	0.15782	0.07574	0.00648	0.15514	0.11726	0.01128	0.07359	0.05150	0.00571
500	Mean	1.22363	0.51064	0.504	1.191	0.52424	0.50429	1.22609	0.54048	0.51206	1.25344	0.53298	0.51288	1.23038	0.51926	0.50795
	Bias	0.02363	0.01064	0.004	-0.009	0.02424	0.00429	0.02609	0.04048	0.01206	0.05344	0.03298	0.01288	0.03038	0.01926	0.00795
	MSE	0.03904	0.01269	0.001	0.038	0.01454	0.00157	0.08985	0.05298	0.00500	0.10225	0.05033	0.00441	0.06164	0.02226	0.00216

Table 4. Simulation results of different estimation techniques for Set-II

		Set-II: a=0.6, b=1.8, c=2														
		τ_1			τ_2			τ_3			τ_4			τ_5		
		a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
20	Mean	1.28437	2.85661	2.92557	1.22247	1.95402	2.16553	1.39291	2.00719	2.44250	1.46085	2.19408	2.65304	1.20047	1.97436	2.27807
	Bias	0.68437	1.05661	0.92557	0.62247	0.15402	0.16553	0.79291	0.20719	0.44250	0.86085	0.39408	0.65304	0.60047	0.17436	0.27807
	MSE	4.30615	5.99376	3.75406	2.96933	2.88894	1.93927	3.95598	3.03610	2.98991	4.59667	3.57754	3.62407	2.64702	2.68796	1.82458
50	Mean	0.79346	2.53948	2.59261	0.82664	1.80765	2.01561	1.06779	1.81883	2.17028	0.97775	2.00933	2.32653	0.79540	2.02949	2.23740
	Bias	0.19346	0.73948	0.59261	0.22664	0.00765	0.01561	0.46779	0.01883	0.17028	0.37775	0.20933	0.32653	0.19540	0.22949	0.23740
	MSE	0.56915	3.93793	2.06471	0.40502	1.80656	0.92561	1.48750	1.84618	1.04825	1.15393	2.14722	1.30660	0.41764	1.69733	0.91376
75	Mean	0.70486	2.50856	2.56064	0.77376	1.75327	1.96647	0.84119	1.83779	2.12365	0.87076	1.92938	2.20891	0.74610	1.98375	2.20053
	Bias	0.10486	0.70856	0.56064	0.17376	-0.04673	-0.03353	0.24119	0.03779	0.12365	0.27076	0.12938	0.20891	0.14610	0.18375	0.20053
	MSE	0.30722	3.29606	1.78210	0.24866	1.42077	0.68408	0.39166	1.53551	0.78155	0.53859	1.72597	0.91566	0.20579	1.52677	0.77792
100	Mean	0.64249	2.45874	2.48989	0.69176	1.87091	2.06278	0.77596	1.88517	2.13711	0.77775	1.96953	2.21236	0.70500	1.91606	2.13822
	Bias	0.04249	0.65874	0.48989	0.09176	0.07091	0.06278	0.17596	0.08517	0.13711	0.17775	0.16953	0.21236	0.10500	0.11606	0.13822
	MSE	0.10961	2.78656	1.41676	0.08509	1.36524	0.69006	0.23316	1.28479	0.64526	0.25248	1.48521	0.74357	0.11043	1.18689	0.58638
200	Mean	0.61837	2.27684	2.34985	0.67148	1.81427	2.01074	0.69018	1.94592	2.15519	0.68214	1.97738	2.18806	0.67249	1.92514	2.12721
	Bias	0.01837	0.47684	0.34985	0.07148	0.01427	0.01074	0.09018	0.14592	0.15519	0.08214	0.17738	0.18806	0.07249	0.12514	0.12721
	MSE	0.05453	1.85343	0.88925	0.05121	0.92839	0.43199	0.10150	1.14610	0.56864	0.08995	1.15551	0.56911	0.06326	1.00103	0.49405
350	Mean	0.59049	2.29552	2.35996	0.63819	1.87617	2.05873	0.65382	1.97958	2.15725	0.65629	1.96283	2.15364	0.61551	2.07676	2.22100
	Bias	-0.00951	0.49552	0.35996	0.03819	0.07617	0.05873	0.05382	0.17958	0.15725	0.05629	0.16283	0.15364	0.01551	0.27676	0.22100
	MSE	0.03640	1.67864	0.83592	0.02979	0.78369	0.37970	0.05699	0.99123	0.45590	0.05580	0.93571	0.44508	0.03472	1.02508	0.50650
500	Mean	0.59074	2.19837	2.29105	0.62562	1.87997	2.05624	0.63732	1.97460	2.14814	0.63304	1.99418	2.16407	0.61235	2.03456	2.18114
	Bias	-0.00926	0.39837	0.29105	0.02562	0.07997	0.05624	0.03732	0.17460	0.14814	0.03304	0.19418	0.16407	0.01235	0.23456	0.18114
	MSE	0.02763	1.31273	0.64463	0.02180	0.65387	0.31271	0.04098	0.86929	0.39745	0.03920	0.87243	0.39857	0.02809	0.81818	0.39317

Table 5. Simulation results of different estimation techniques for Set-III

		Set-III: a=2.1, b=1.8, c=0.7														
		τ_1			τ_2			τ_3			τ_4			τ_5		
		a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
20	Mean	4.53023	9.03638	0.95321	3.91057	7.16968	0.88975	4.70925	9.07503	0.94293	4.689831852	12.03215	1.00241	3.57693	9.22847	0.94605
	Bias	2.43023	7.23638	0.25321	1.81057	5.36968	0.18975	2.60925	7.27503	0.24293	2.58983	10.23215	0.30241	1.47693	7.42847	0.24605
	MSE	49.40706	303.09852	0.31231	37.15603	176.43943	0.20246	53.68255	301.00402	0.26854	52.91449	874.85612	0.33926	28.14896	374.34157	0.26497
50	Mean	2.47451	5.02612	0.81995	2.27090	4.18139	0.79505	2.79176	5.94679	0.86555	2.78542	6.41562	0.87474	2.33665	4.76193	0.82184
	Bias	0.37451	3.22612	0.11995	0.17090	2.38139	0.09505	0.69176	4.14679	0.16555	0.68542	4.61562	0.17474	0.23665	2.96193	0.12184
	MSE	3.37256	105.10124	0.13225	2.43544	51.79073	0.09176	8.20471	104.88820	0.15867	7.61796	128.04449	0.17233	2.07377	72.29961	0.11289
75	Mean	2.27054	3.69586	0.77808	2.20238	3.02025	0.74804	2.48681	4.75320	0.81563	2.42124	4.88027	0.82556	2.26631	3.54193	0.77708
	Bias	0.17054	1.89586	0.07808	0.10238	1.22025	0.04804	0.38681	2.95320	0.11563	0.32124	3.08027	0.12556	0.16631	1.74193	0.07708
	MSE	1.06124	46.61188	0.07323	0.86465	24.64248	0.04578	3.28717	68.54060	0.10497	2.76940	71.96474	0.11103	1.11374	33.55419	0.06108
100	Mean	2.22406	2.93701	0.75065	2.19841	2.49315	0.72763	2.32382	4.26276	0.79990	2.33838	4.46316	0.81012	2.22074	3.24896	0.76217
	Bias	0.12406	1.13701	0.05065	0.09841	0.69315	0.02763	0.22382	2.46276	0.09990	0.23838	2.66316	0.11012	0.12074	1.44896	0.06217
	MSE	0.74275	22.54298	0.03823	0.74413	10.52414	0.02306	1.88937	52.08373	0.08637	1.99459	58.80157	0.09532	0.76838	28.73751	0.05271
200	Mean	2.14944	2.10737	0.71536	2.11837	2.04256	0.71016	2.13920	2.75489	0.74493	2.19034	2.77245	0.74394	2.16455	2.36750	0.72611
	Bias	0.04944	0.30737	0.01536	0.01837	0.24256	0.01016	0.03920	0.95489	0.04493	0.09034	0.97245	0.04394	0.06455	0.56750	0.02611
	MSE	0.24434	3.10727	0.00697	0.25554	1.12061	0.00384	0.57067	11.49362	0.02412	0.65765	13.69624	0.02655	0.35486	9.39176	0.01863
350	Mean	2.16191	1.89480	0.70461	2.09276	1.97330	0.70836	2.15452	2.28842	0.72370	2.13000	2.30595	0.72473	2.12642	1.99286	0.71006
	Bias	0.06191	0.09480	0.00461	-0.00724	0.17330	0.00836	0.05452	0.48842	0.02370	0.03000	0.50595	0.02473	0.02642	0.19286	0.01006
	MSE	0.15197	0.56417	0.00208	0.14422	0.63739	0.00240	0.37678	4.60711	0.01060	0.32735	4.47945	0.01036	0.18188	0.78274	0.00292
500	Mean	2.12129	1.88365	0.70428	2.09967	1.89135	0.70381	2.13030	2.04692	0.71195	2.14554	2.04052	0.71262	2.13809	1.90946	0.70507
	Bias	0.02129	0.08365	0.00428	-0.00033	0.09135	0.00381	0.03030	0.24692	0.01195	0.04554	0.24052	0.01262	0.03809	0.10946	0.00507
	MSE	0.08572	0.28812	0.00123	0.09374	0.30176	0.00129	0.21713	1.22935	0.00390	0.23813	1.18192	0.00376	0.13713	0.48843	0.00186

Table 6. Simulation results of different estimation techniques for Set-IV

		Set-IV: a=2, b=2, c=2														
		τ_1			τ_2			τ_3			τ_4			τ_5		
		a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
20	Mean	3.87978	11.68628	2.83046	3.47556	7.56838	2.53946	4.43249	10.02381	2.71240	4.59850	11.11387	2.79542	3.20502	8.35678	2.61812
	Bias	1.87978	9.68628	0.83046	1.47556	5.56838	0.53946	2.43249	8.02381	0.71240	2.59850	9.11387	0.79542	1.20502	6.35678	0.61812
	MSE	39.40574	466.88153	2.88646	28.38097	214.98613	1.67308	46.62489	417.80409	2.20769	53.02286	498.69549	2.46170	20.81361	232.77344	1.80807
50	Mean	2.33562	5.76526	2.39223	2.18582	5.01000	2.33276	2.62900	6.74822	2.52622	2.72991	6.78730	2.51177	2.26241	5.31147	2.37158
	Bias	0.33562	3.76526	0.39223	0.18582	3.01000	0.33276	0.62900	4.74822	0.52622	0.72991	4.78730	0.51177	0.26241	3.31147	0.37158
	MSE	2.97008	115.72207	1.21123	2.92679	71.63129	0.91928	7.46926	121.17868	1.42718	7.65895	132.95811	1.46138	2.33084	82.79888	0.99850
75	Mean	2.19652	4.02710	2.21776	2.05979	3.64502	2.18222	2.37544	5.43521	2.37389	2.34484	5.80654	2.41047	2.20137	3.83225	2.20479
	Bias	0.19652	2.02710	0.21776	0.05979	1.64502	0.18222	0.37544	3.43521	0.37389	0.34484	3.80654	0.41047	0.20137	1.83225	0.20479
	MSE	1.30660	55.68015	0.61354	0.90223	35.88807	0.48950	3.60789	80.85984	0.97206	3.03387	95.86380	1.09315	1.00086	39.64348	0.51104
100	Mean	2.16545	3.32503	2.15262	2.07959	2.89296	2.09649	2.23648	4.48362	2.27910	2.19644	4.98216	2.34134	2.11511	3.37818	2.16145
	Bias	0.16545	1.32503	0.15262	0.07959	0.89296	0.09649	0.23648	2.48362	0.27910	0.19644	2.98216	0.34134	0.11511	1.37818	0.16145
	MSE	0.75834	32.49804	0.43459	0.56587	15.20822	0.23832	1.83552	49.28906	0.66283	1.58449	64.98159	0.82223	0.69493	27.36028	0.38153
200	Mean	2.08795	2.24385	2.03054	2.00998	2.35941	2.04397	2.09405	3.16739	2.14157	2.11820	2.96531	2.11489	2.05060	2.63168	2.08154
	Bias	0.08795	0.24385	0.03054	0.00998	0.35941	0.04397	0.09405	1.16739	0.14157	0.11820	0.96531	0.11489	0.05060	0.63168	0.08154
	MSE	0.25441	1.57541	0.03928	0.24377	2.95984	0.06330	0.65064	17.25774	0.25933	0.60742	14.26524	0.21313	0.34438	7.54809	0.12699
350	Mean	2.02766	2.16819	2.02407	2.00162	2.13618	2.01615	2.06305	2.39908	2.05033	2.07631	2.64869	2.08113	2.02550	2.26352	2.03677
	Bias	0.02766	0.16819	0.02407	0.00162	0.13618	0.01615	0.06305	0.39908	0.05033	0.07631	0.64869	0.08113	0.02550	0.26352	0.03677
	MSE	0.12956	0.65128	0.01870	0.11421	0.57931	0.01733	0.29537	3.33523	0.06633	0.37784	8.93378	0.13734	0.18119	1.43840	0.03397
500	Mean	2.00777	2.13092	2.01931	2.00627	2.08869	2.00950	2.02299	2.31904	2.04136	2.04887	2.32950	2.04211	2.02409	2.15939	2.02424
	Bias	0.00777	0.13092	0.01931	0.00627	0.08869	0.00950	0.02299	0.31904	0.04136	0.04887	0.32950	0.04211	0.02409	0.15939	0.02424
	MSE	0.08315	0.49509	0.01355	0.08138	0.36390	0.01129	0.21938	1.98681	0.04186	0.23159	2.68676	0.04997	0.12418	0.63629	0.01807

Table 7. Simulation results of different estimation techniques for Set-V

		Set-V: a=2.4, b=0.4, c=3.6														
		τ_1			τ_2			τ_3			τ_4			τ_5		
		a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
20	Mean	4.27376	1.38011	4.91004	3.71129	0.84947	4.12214	3.73886	1.32493	4.62773	4.04213	1.44463	4.92459	3.38767	1.10736	4.54340
	Bias	1.87376	0.98011	1.31004	1.31129	0.44947	0.52214	1.33886	0.92493	1.02773	1.64213	1.04463	1.32459	0.98767	0.70736	0.94340
	MSE	15.77148	7.82039	10.04395	10.57103	2.61069	4.03877	13.41930	5.26434	5.68019	16.48366	6.01155	7.32977	9.48883	3.47039	5.31535
50	Mean	3.10091	0.63725	3.97434	2.81492	0.58717	3.76923	3.19854	0.71174	4.00296	3.32439	0.83679	4.20351	2.83688	0.66335	3.96735
	Bias	0.70091	0.23725	0.37434	0.41492	0.18717	0.16923	0.79854	0.31174	0.40296	0.92439	0.43679	0.60351	0.43688	0.26335	0.36735
	MSE	3.77555	1.47614	2.38590	2.91206	0.83171	1.24521	5.80159	1.15837	1.65253	6.96567	1.81525	2.63047	3.18850	0.85631	1.43317
75	Mean	2.83444	0.47711	3.74467	2.59208	0.49949	3.68845	2.90122	0.67563	3.93959	2.96382	0.68379	3.98870	2.58793	0.60618	3.87312
	Bias	0.43444	0.07711	0.14467	0.19208	0.09949	0.08845	0.50122	0.27563	0.33959	0.56382	0.28379	0.38870	0.18793	0.20618	0.27312
	MSE	2.03007	0.30649	0.48225	1.46469	0.36045	0.72831	3.61265	1.02117	1.36881	4.28700	1.03919	1.49708	1.56277	0.73957	1.07087
100	Mean	2.72154	0.44773	3.69685	2.53152	0.44511	3.63223	2.85563	0.54186	3.78170	2.89238	0.56292	3.83511	2.65346	0.46870	3.69192
	Bias	0.32154	0.04773	0.09685	0.13152	0.04511	0.03223	0.45563	0.14186	0.18170	0.49238	0.16292	0.23511	0.25346	0.06870	0.09192
	MSE	1.24742	0.25705	0.34194	0.96635	0.05235	0.10344	2.75327	0.44121	0.57169	3.17281	0.45931	0.61212	1.31090	0.09369	0.16362
200	Mean	2.54621	0.41464	3.63418	2.43273	0.42251	3.60416	2.63193	0.45998	3.68241	2.58631	0.48087	3.71848	2.48102	0.43484	3.65409
	Bias	0.14621	0.01464	0.03418	0.03273	0.02251	0.00416	0.23193	0.05998	0.08241	0.18631	0.08087	0.11848	0.08102	0.03484	0.05409
	MSE	0.44024	0.02091	0.04921	0.35889	0.01806	0.04236	1.20991	0.11934	0.18327	1.12841	0.13791	0.19720	0.48201	0.03099	0.05534
350	Mean	2.49126	0.40580	3.61767	2.40327	0.41640	3.60004	2.52290	0.42717	3.63852	2.52130	0.42578	3.64032	2.46583	0.41850	3.63355
	Bias	0.09126	0.00580	0.01767	0.00327	0.01640	0.00004	0.12290	0.02717	0.03852	0.12130	0.02578	0.04032	0.06583	0.01850	0.03355
	MSE	0.23534	0.00982	0.02405	0.21049	0.01056	0.02464	0.55740	0.02878	0.03935	0.52058	0.02552	0.03540	0.30183	0.01491	0.02753
500	Mean	2.44059	0.40838	3.61947	2.37135	0.41790	3.61372	2.47460	0.42065	3.62674	2.48755	0.41521	3.62836	2.43353	0.41368	3.61937
	Bias	0.04059	0.00838	0.01947	-0.02865	0.01790	0.01372	0.07460	0.02065	0.02674	0.08755	0.01521	0.02836	0.03353	0.01368	0.01937
	MSE	0.13837	0.00682	0.01685	0.12348	0.00700	0.01685	0.36247	0.01608	0.02242	0.34073	0.01561	0.02250	0.18300	0.00954	0.01884

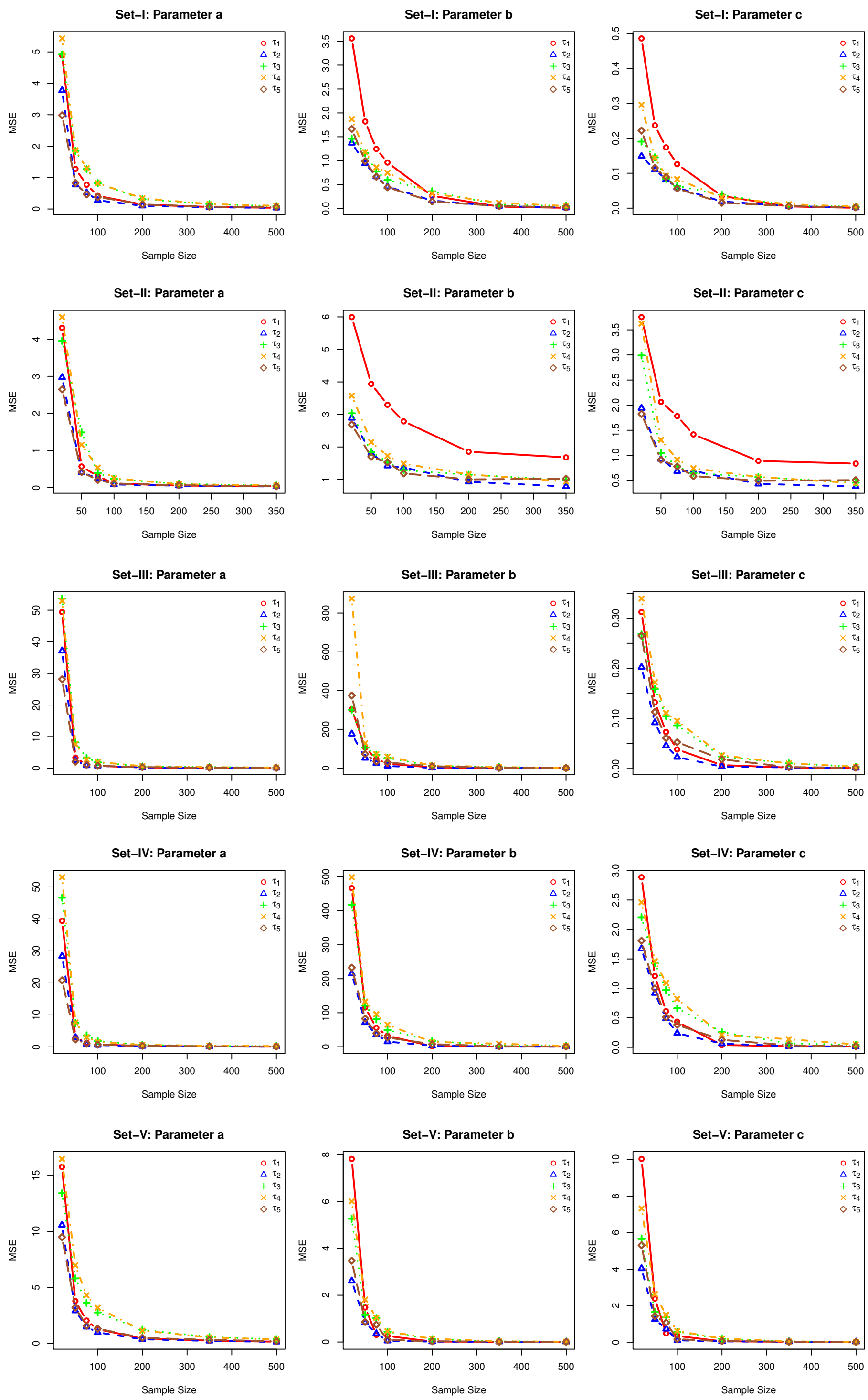

Figure 6. The sample size vs MSE plots for different estimation techniques across parameter sets

Table 8. Overall Ranks

		τ_1				τ_2				τ_3				τ_4				τ_5			
	n	a	b	c	$\sum Ranks$	a	b	c	$\sum Ranks$	a	b	c	$\sum Ranks$	a	b	c	$\sum Ranks$	a	b	c	$\sum Ranks$
Set-I	20	3	5	5	$13^{\{3.5\}}$	2	1	1	$4^{\{1\}}$	4	3	2	$9^{\{3\}}$	5	4	4	$13^{\{4.5\}}$	1	2	3	$6^{\{2\}}$
	50	3	5	5	$13^{\{5\}}$	1	1	1	$3^{\{1\}}$	4	3	4	$11^{\{3\}}$	5	4	3	$12^{\{4\}}$	2	2	2	$6^{\{2\}}$
	75	3	5	5	$13^{\{5\}}$	1	2	1	$4^{\{1\}}$	3	3	2	$8^{\{3\}}$	4	4	4	$12^{\{4\}}$	2	1	3	$6^{\{2\}}$
	100	3	5	5	$13^{\{5\}}$	1	2	2	$5^{\{2\}}$	5	3	3	$11^{\{3\}}$	4	4	4	$12^{\{4\}}$	2	1	1	$4^{\{1\}}$
	200	2	3	5	$10^{\{3\}}$	1	2	2	$5^{\{1.5\}}$	4	5	4	$13^{\{5\}}$	5	4	3	$12^{\{4\}}$	3	1	1	$5^{\{1.5\}}$
	350	2	1	1	$4^{\{1\}}$	1	2	3	$6^{\{2\}}$	5	4	4	$13^{\{4\}}$	4	5	5	$14^{\{5\}}$	3	3	2	$8^{\{3\}}$
	500	2	1	1	$4^{\{1\}}$	1	2	2	$5^{\{2\}}$	4	5	5	$14^{\{5\}}$	5	4	4	$13^{\{4\}}$	3	3	3	$9^{\{3\}}$
Set-II	20	4	5	5	$14^{\{5\}}$	2	2	2	$6^{\{2\}}$	3	3	3	$9^{\{3\}}$	5	4	4	$13^{\{4\}}$	1	1	1	$3^{\{1\}}$
	50	3	5	5	$13^{\{5\}}$	1	2	2	$5^{\{2\}}$	5	3	3	$11^{\{3\}}$	4	4	4	$12^{\{4\}}$	2	1	1	$4^{\{1\}}$
	75	3	5	5	$13^{\{4.5\}}$	2	1	1	$4^{\{1\}}$	4	3	3	$10^{\{3\}}$	5	4	4	$13^{\{4.5\}}$	1	2	2	$5^{\{2\}}$
	100	2	5	5	$12^{\{4\}}$	1	3	3	$7^{\{2\}}$	4	2	2	$8^{\{3\}}$	5	4	4	$13^{\{5\}}$	3	1	1	$5^{\{1\}}$
	200	2	5	5	$12^{\{4.5\}}$	1	1	1	$3^{\{1\}}$	5	3	3	$11^{\{3\}}$	4	4	4	$12^{\{4.5\}}$	3	2	2	$7^{\{2\}}$
	350	3	5	5	$13^{\{5\}}$	1	1	1	$3^{\{1\}}$	4	3	3	$10^{\{3.5\}}$	5	2	2	$9^{\{2\}}$	2	4	4	$10^{\{3.5\}}$
	500	2	5	5	$12^{\{4.5\}}$	1	1	1	$3^{\{1\}}$	5	3	3	$11^{\{3\}}$	4	4	4	$12^{\{4.5\}}$	3	2	2	$7^{\{2\}}$
Set-III	20	3	3	4	$10^{\{3.5\}}$	2	1	1	$4^{\{1\}}$	5	2	3	$10^{\{3.5\}}$	4	5	5	$14^{\{5\}}$	1	4	2	$7^{\{2\}}$
	50	3	4	3	$10^{\{3\}}$	2	1	1	$4^{\{1\}}$	5	3	4	$12^{\{4\}}$	4	5	5	$14^{\{5\}}$	1	2	2	$5^{\{2\}}$
	75	2	3	3	$8^{\{3\}}$	1	1	1	$3^{\{1\}}$	5	5	4	$14^{\{5\}}$	4	4	5	$13^{\{4\}}$	3	2	2	$7^{\{2\}}$
	100	1	2	2	$5^{\{2\}}$	2	1	1	$4^{\{1\}}$	4	4	4	$12^{\{4\}}$	5	5	5	$15^{\{5\}}$	3	3	3	$9^{\{3\}}$
	200	1	2	2	$5^{\{2\}}$	2	1	1	$4^{\{1\}}$	4	4	4	$12^{\{4\}}$	5	5	5	$15^{\{5\}}$	3	3	3	$9^{\{3\}}$
	350	2	1	1	$4^{\{1\}}$	1	2	2	$5^{\{2\}}$	5	5	5	$15^{\{5\}}$	4	4	4	$12^{\{4\}}$	3	3	3	$9^{\{3\}}$
	500	1	2	1	$4^{\{1\}}$	2	1	2	$5^{\{2\}}$	4	5	5	$14^{\{5\}}$	5	4	4	$13^{\{4\}}$	3	3	3	$9^{\{3\}}$
Set-IV	20	3	4	5	$12^{\{4\}}$	2	1	1	$4^{\{1\}}$	4	3	3	$10^{\{3\}}$	5	5	4	$14^{\{5\}}$	1	2	2	$5^{\{2\}}$
	50	3	3	3	$9^{\{4\}}$	2	1	1	$4^{\{1\}}$	4	4	4	$12^{\{3\}}$	5	5	5	$15^{\{5\}}$	1	2	2	$5^{\{2\}}$
	75	3	3	3	$9^{\{4\}}$	1	1	1	$3^{\{1\}}$	5	3	4	$12^{\{3\}}$	4	4	5	$13^{\{5\}}$	2	2	2	$6^{\{2\}}$
	100	3	3	3	$9^{\{1\}}$	1	1	1	$3^{\{1\}}$	4	4	4	$12^{\{3\}}$	5	5	5	$15^{\{5\}}$	2	2	2	$6^{\{2\}}$
	200	2	1	1	$4^{\{1\}}$	1	2	2	$5^{\{2\}}$	5	5	5	$15^{\{4\}}$	4	4	4	$12^{\{5\}}$	3	3	3	$9^{\{3\}}$
	350	2	2	2	$6^{\{2\}}$	1	1	1	$3^{\{1\}}$	4	4	4	$12^{\{3\}}$	5	5	5	$15^{\{5\}}$	3	3	3	$9^{\{4\}}$
	500	2	2	2	$6^{\{2\}}$	1	1	1	$3^{\{1\}}$	4	4	4	$12^{\{3\}}$	5	5	5	$15^{\{5\}}$	3	3	3	$9^{\{4\}}$
Set-V	20	4	5	5	$14^{\{5\}}$	2	1	1	$4^{\{1\}}$	3	3	3	$9^{\{3\}}$	5	4	4	$13^{\{4\}}$	1	2	2	$5^{\{2\}}$
	50	3	4	4	$11^{\{4\}}$	1	1	1	$3^{\{1\}}$	4	3	3	$10^{\{3\}}$	5	5	5	$15^{\{5\}}$	2	2	2	$6^{\{2\}}$
	75	3	1	1	$5^{\{1.5\}}$	1	2	2	$5^{\{1.5\}}$	4	4	4	$12^{\{4\}}$	5	5	5	$15^{\{5\}}$	2	3	3	$8^{\{3\}}$
	100	2	3	3	$8^{\{3\}}$	1	1	1	$3^{\{1\}}$	4	4	4	$12^{\{4\}}$	5	5	5	$15^{\{5\}}$	3	2	2	$7^{\{2\}}$
	200	2	2	2	$6^{\{2\}}$	1	1	1	$3^{\{1\}}$	5	4	4	$13^{\{4\}}$	4	5	5	$14^{\{5\}}$	3	3	3	$9^{\{3\}}$
	350	2	1	1	$4^{\{1\}}$	1	2	2	$5^{\{2\}}$	5	5	5	$15^{\{5\}}$	4	4	4	$12^{\{4\}}$	3	3	3	$9^{\{3\}}$
	500	2	1	1.5	$4.5^{\{1.5\}}$	1	2	1.5	$4.5^{\{1.5\}}$	5	5	4	$14^{\{5\}}$	4	4	5	$13^{\{4\}}$	3	3	3	$9^{\{3\}}$

6. Applications

This section deals with the application of the *TLMkE* model to three real-world datasets: blood cancer data, turbocharger suits data and emissions data. The model is compared to four recent exponential-based three-parameter models: Marshall-Olkin logistic exponential (MoLEx) [41], Kumaraswamy Exponential (KuEx) [3], Power Modified Kies exponential (PMkEx) [5], and Half-logistic Modified Kies exponential (HLMkEx) [10]. The performance of the model was evaluated based on various goodness-of-fit measures: Kolmogorov-Smirnov (κ_3), AD (κ_2) and CvM (κ_1)and information measures: Akaike information criterion (κ_4), Consistent Akaike information criterion(κ_5), Bayesian information criterion (κ_6) and Hannan-Quin information criterion (κ_7).

Blood-Cancer data

The first dataset represents the lifetime of patients diagnosed with Leukaemia, a type of cancer that affects blood cells by increasing the number of white blood cells, which reduces the human body's ability to fight infections and weakens the immune system. The dataset was recorded at one of the hospitals of the Ministry of Health facilities in Saudi Arabia and was recently used by Sakthivel et al. [51]. The ordered lifetimes (in years) data is: 0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381 and the Table 9 and Figure 7 give the summary of the data. The blood Cancer data shows a slight left skew, with most values clustered toward higher life expectancies, as seen in the violin plot and histogram in Figure 7. The TTT-plot shows increasing hazard pattern. The distribution has a flatter peak in the histogram matches the low kurtosis in the Table 9, indicating a wider and less peaked distribution.

Table 9. Summary of blood Cancer data

Mean	Q_1	Median	Q_3	Var.	Sk	Kur.	Min.	Max.
3.135	2.187	3.348	4.284	1.893952	-0.400217	-0.786664	0.315	5.381

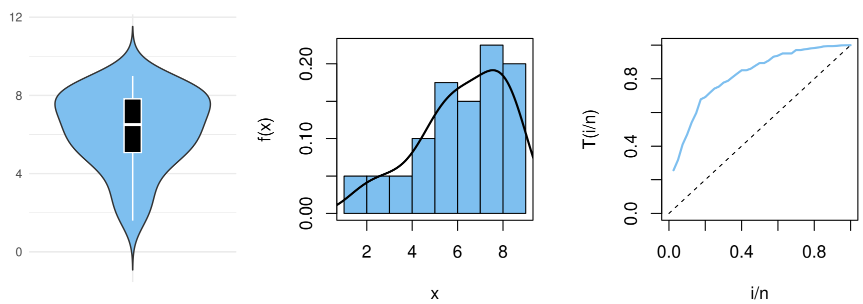

Figure 7. Violin plot, Histogram and TTT-plot of blood Cancer data

Figure 8 shows the estimated distribution function, *pdf* and PP plots of the proposed and competing models for blood cancer data. The estimated hazard plot of the *TLMkE* model in Figure 8 shows an increasing pattern which aligns with the TTT-plot shown in Figure 7. The smooth distribution function curve and well-fitted *pdf* reflect the distribution of the data, while the PP-plot further supports the suitability of the proposed model by showing that the empirical points closely follow the theoretical line, indicating a good fit to the model.

The Table 10 presents the goodness-of-fit measures and various information measures for the considered models. These results support the visual representations shown in Figure 7. The *TLMkE* model consistently exhibited the lowest values for κ_1,κ_2 and κ_3, as well as for the κ_4, κ_6, κ_5 and κ_7 values. This demonstrates the superior performance of the *TLMkE* model over its competing models. The reason for this superiority can be attributed to the built-in flexibility of the *TLMkE* in handling skewness and kurtosis. The *TLMkE* model with TL-G parameter 'a' and Mk-G parameter 'b' provides more control over both skewness and kurtosis than the competing models. In addition, the higher p-value of the model confirms its better fit to the data compared with other models. The Table 11 represents the maximum likelihood estimates and standard errors of *TLMkE* and its competing models for the blood cancer data.

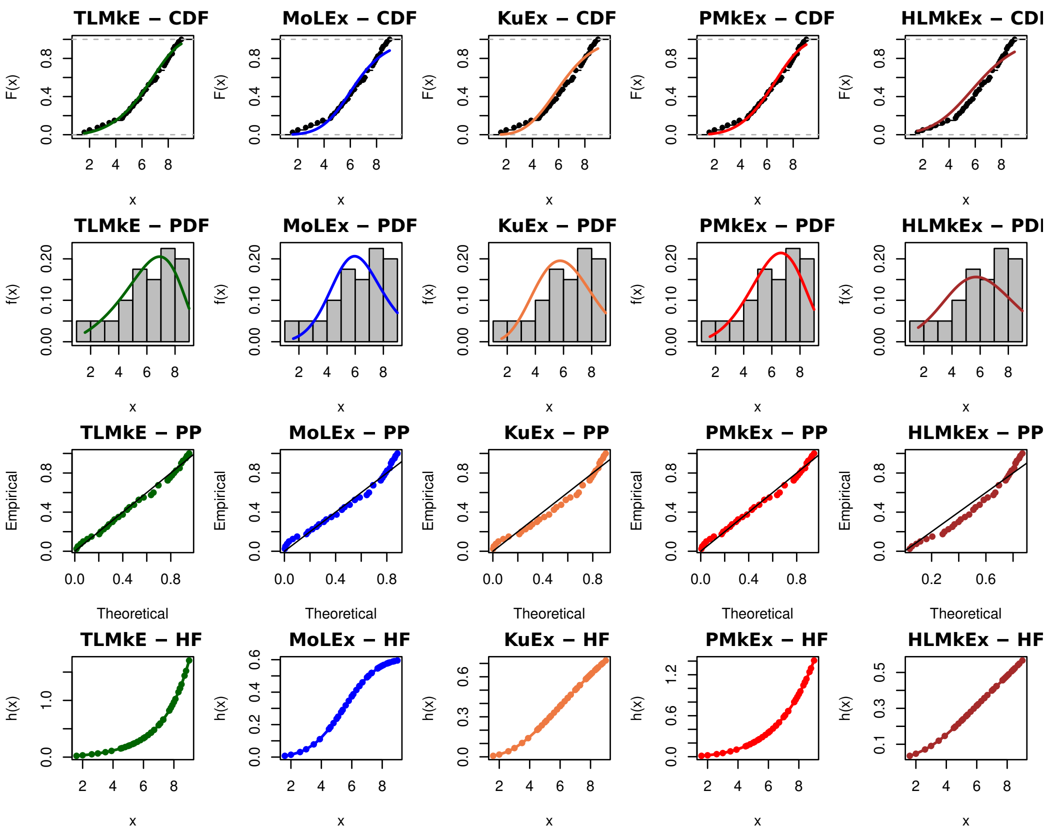

Figure 8. Estimated cdf, pdf and PP-plots of blood Cancer data

Table 10. Goodness-of-fit and adequacy measures of blood Cancer data

Distribution	κ_1	κ_2	κ_3	p-value	κ_4	κ_5	κ_6	κ_7
TLMkE	0.0135	0.1108	0.0575	0.9986	133.5502	134.236	138.5409	135.3409
PMKEx	0.0451	0.3222	0.0818	0.9373	136.6102	137.2959	141.6009	138.4008
HLMkEx	0.0870	0.5863	0.0959	0.8327	140.9743	141.6600	145.9649	142.7649
MoLEx	0.1500	0.9685	0.1062	0.7313	145.6786	146.3643	150.6693	147.4692
KuEx	0.1759	1.1198	0.1337	0.4493	146.6735	147.3592	151.6642	148.4641

Table 11. MLE estimates and standard errors of blood Cancer data

Distribution	$\hat{a}$	$\hat{b}$	$\hat{c}$	E_1	E_2	E_3
TLMkE	5.9067	0.2428	0.1347	4.0080	0.1990	0.0068
PMKEx	1.0036	0.0586	1.9260	1.0371	0.1451	1.919
HLMkEx	1.9903	0.1381	10.7248	0.8722	0.0585	1.7969
MoLEx	1.7454	0.4391	6.0796	0.4881	0.2056	6.5856
KuEx	2.8709	7.3161	0.1941	0.5778	6.9489	0.1073

Emissions data

The second dataset provides the proportion of global CO_2 emissions the year 2020 across the 211 countries. CO_2 emissions, primarily generated by the burning of fossil fuels, industrial processes and deforestation, are a major contributors to global warming and climate change. This dataset was recently used by Hussam et al. [34]. The data values are : 0.18, 1.88, 0.58, 3.53, 20.32, 5.39, 7.41, 0.11, 0.68, 2.09, 0.71, 0.26, 0.26, 0.21, 3.80, 0.73, 3.780, 0.99, 0.31, 2.16, 1.76, 5.01, 11.47, 6.53, 0.94, 3.37, 1.93, 6.08, 7.69, 0.67, 5, 0.04, 15.37, 0.56, 4.85, 14, 6.75, 4.66, 9.06, 1.68, 2.62, 2.56, 0.36, 15.52, 1.36, 0.57, 1.75, 0.08, 6.04, 1.75, 3.32, 8.6, 2.5, 2.56, 6.26, 0.92, 0.03, 7.62, 17.97, 0.59, 1.99, 1.53, 1.06, 0.4, 5.63, 5.24, 8.42, 6.94, 0.43, 4.89, 7.09, 3.47, 13.06, 0.64, 8.15, 1.02, 0.13, 3.99, 12.12, 0.43, 5.07. The Table 12 and Figure 9 together reveal the right-skewed nature of the emissions data, while the kurtosis value suggests a moderately peaked shape with a lighter tail. The histogram and violin plot show the skewed pattern of the data, and the TTT plot presents a likely bathtub hazard trend.

Table 12. Summary of emissions data

Mean	Q_1	Median	Q_3	Var.	Sk	Kur.	Min.	Max.
4.166	0.680	2.56	6.080	20.1696	1.5626	2.1989	0.03	20.32

The estimated *pdf* plot of the emissions data in Figure 10 displays a clear decreasing trend capturing both the high peak and long tail, indicating that the model is well-suited for decreasing data. Furthermore, the estimated hazard function plot indicated a bathtub-shaped trend, which aligns with the pattern suggested by the TTT-plot in Figure 9. The estimated distribution function plot follows an expected increasing pattern, which is a basic feature of a distribution function, and the PP-plot also supports the model's good fit to the emissions data by closely following the diagonal line.

The Table 13 presents a comparison of the different models applied to the emissions data. The results highlight that the *TLMkE* exhibited the best performance. Based on the various adequacy measures presented in the table, it can be concluded that the proposed *TLMkE* model demonstrates

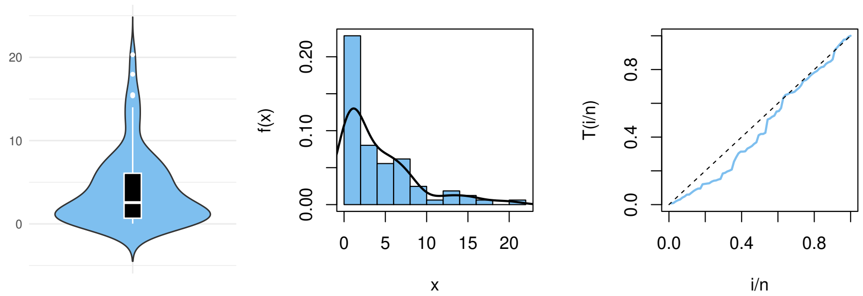

Figure 9. Violin plot, Histogram and TTT-plot of emissions data

superior performance for the emissions dataset. This better performance is due to the *TLMkE* ability of the model to effectively capture the right skewness and heavy-tail. This flexibility allows the *TLMkE* model to accurately represent the data's high peak and long-tail due to few countries with very high emissions. A higher p-value adds support for its better fit. Overall, the results suggest that *TLMkE* model is appropriate for modelling the emissions dataset. The Table 14 shows the MLE estimates and standard errors of the *TLMkE* model for emissions data.

Table 13. Goodness-of-fit and adequacy measures of emissions data

Distribution	κ_1	κ_2	κ_3	p-value	κ_4	κ_5	κ_6	κ_7
TLMkE	0.0349	0.2248	0.0547	0.9685	393.8612	394.1729	401.0446	396.7433
MoLEx	0.0481	0.2823	0.0645	0.8894	395.1435	395.4552	402.3269	398.0256
PMKEx	0.0570	0.3448	0.0675	0.8547	396.2146	396.5263	403.3979	399.0966
HLMkEx	0.0385	0.2544	0.0655	0.8777	394.3738	394.6855	401.5571	397.2559
KuEx	0.0578	0.3297	0.0682	0.8459	395.8591	396.1707	403.0424	398.7411

Table 14. MLE estimates and standard errors of emissions data

Distribution	$\hat{a}$	$\hat{b}$	$\hat{c}$	E_1	E_2	E_3
TLMkE	0.3448	3.1198	0.1617	0.0993	1.3265	0.0605
MoLEx	0.8163	0.2684	0.9842	0.1292	0.0952	0.481
PMKEx	2.3666	0.4773	0.2670	3.1824	0.2509	0.3623
HLMkEx	0.1322	0.6094	1.3839	0.1705	0.7302	2.0043
KuEx	0.7940	1.7458	0.1091	0.1027	1.4142	0.0967

Turbocharger suits data

The third dataset provides the 40 failure times (10^3 hours) for turbocharger units of one engine. A turbocharger is a device that pushes more air into an engine cylinder, helping it burn fuel and produce power. This dataset was recently used by Swetha and Nagarjuna [55]. The data is: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

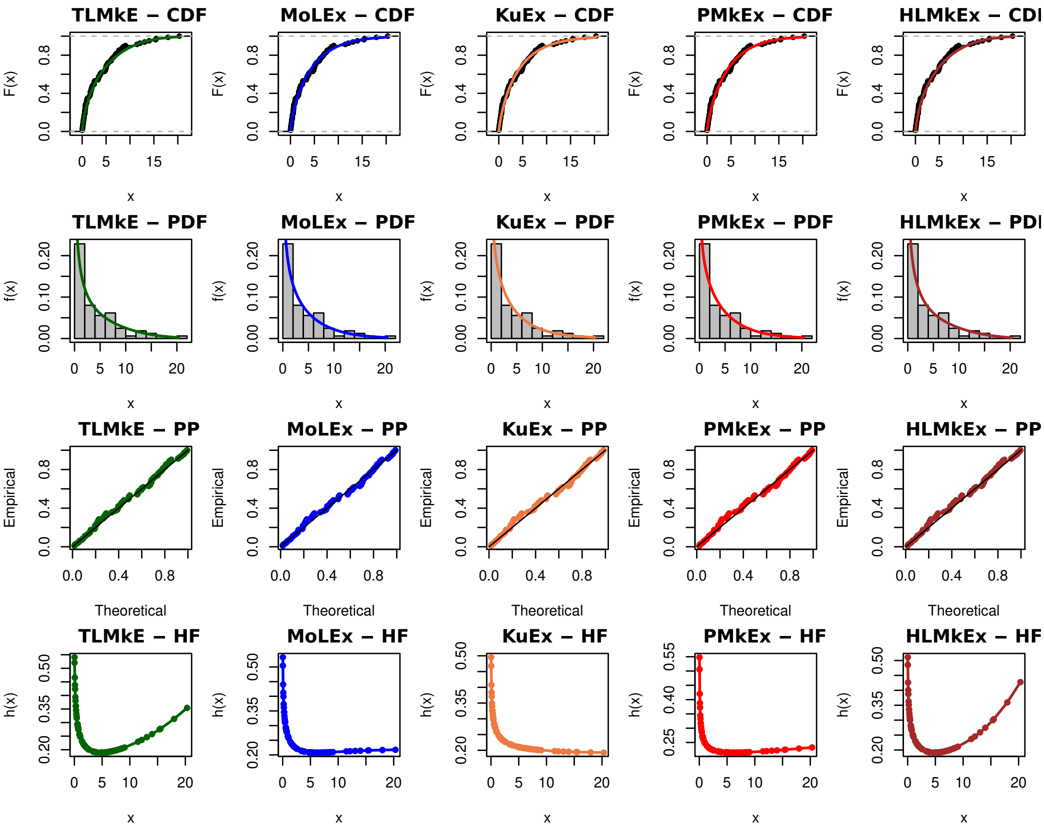

Figure 10. Estimated cdf, pdf and PP-plots of emissions data

The Table 16 and Figure 11 present descriptive measures of the turbocharger suits data. The observed negative skewness shows asymmetry in the data, with a longer tail on the left side. Further, the negative kurtosis suggests a platykurtic distribution. These characteristics are reflected well in the estimated *pdf* plot shown in Figure 12.

Table 15. Goodness-of-fit and adequacy measures of turbocharger suits data

Distribution	κ_1	κ_2	κ_3	p-value	κ_4	κ_5	κ_6	κ_7
TLMkE	0.0351	0.2773	0.0993	0.8252	166.321	166.9876	171.3876	168.1529
PMKEx	0.0541	0.4156	0.1003	0.8162	168.4941	169.1608	173.5608	170.3261
MoLEx	0.1494	1.024	0.1172	0.6416	178.0103	178.677	183.0769	179.8422
KuEx	0.1523	1.0483	0.1215	0.5967	177.0059	177.6726	182.0726	178.8379
HLMkEx	0.1067	0.7618	0.1329	0.4803	179.8848	180.5515	184.9514	181.7167

The estimated PDF plot of the *TLMkE* distribution in Figure 12 illustrates a visual representation of the turbocharger suits data. It exhibits an inverted J-shaped plot with more concentration towards the left, accurately capturing the data's negative skewness of the data. The estimated hazard function plot for the *TLMkE* model shows an increasing trend, which is expected for aging components. The estimated distribution function plot shows the usual increasing pattern, and the PP-plot displays the model's good fit to the turbocharger suits data.

The Table 15 presents the best performance of *TLMkE* model over the competing models as shown

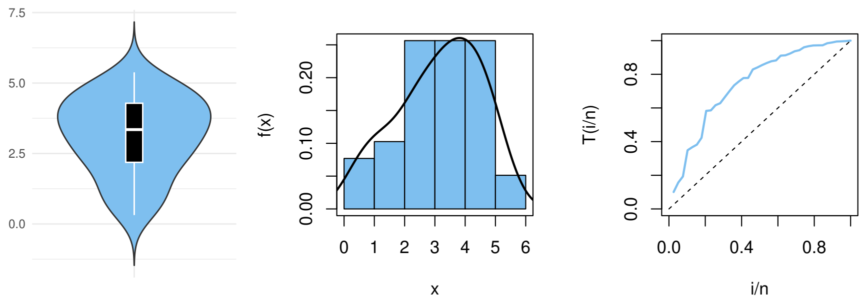

Figure 11. Violin plot, Histogram and TTT-plot of Turbocharger suits data

Table 16. Summary of turbocharger suits data

Mean	Q_1	Median	Q_3	Var.	Sk	Kur.	Min.	Max.
6.253	5.075	6.5	7.825	3.82409	-0.66255	-0.35898	1.6	9

by a high p-value and lower goodness-of-fit and information measures and Table 17 presents the MLE estimates and standard errors of the models, respectively.

Table 17. MLE estimates and standard errors of turbocharger suits data

Distribution	$\hat{a}$	$\hat{b}$	$\hat{c}$	E_1	E_2	E_3
TLMkE	4.4126	0.5211	0.0795	1.3157	0.2053	0.0034
PMKEx	2.7928	0.0933	1.0325	1.0967	0.0727	0.3998
MoLEx	3.2202	0.1595	5.6257	0.7298	0.0591	8.7652
KuEx	5.6564	8.0716	0.1701	1.5287	9.1227	0.0860
HLMkEx	2.7280	0.0545	11.8217	0.9169	0.0183	1.9380

To evaluate how well the proposed model fits real-world data, we used two key measures: the Vuong test [60] and Leave-One-Out Log-Likelihood (LOL). The Vuong test statistically compares two non-nested models by examining their log-likelihood ratios, producing a Z-score that indicates whether one model fits the data significantly better or if they're equivalent. A positive Z-score ($Z > 1.96$ at 5% level of significance) and a small p-value show that first model performs better. Meanwhile, the LOL evaluates predictive accuracy by iteratively fitting the model to all but one data point and testing on the excluded observation, indicating the reliability of the model in real-world applications. In general, the Mean absolute error (MAE) represents the average magnitude of prediction errors, with lower values indicating better model accuracy. Both LOL and MAE provide a detailed view of statistical reliability and practical accuracy.

The Tables 18 and 19 demonstrate the *TLMkE* model's strong statistical and predictive performance across multiple datasets, establishing itself as a reliable choice for modelling complex distributions.

- In the life expectancy and Turbocharger suits datasets, it significantly dominates competing models including MoLEx, KuEx and HLMkEX in both Vuong tests and predictive accuracy metrics, while the PMkEx is the closest competitor to the model. This is expected because the PMkEx also shares the MkEx baseline, but the addition of the Topp-Leone model in TLMkE provides a

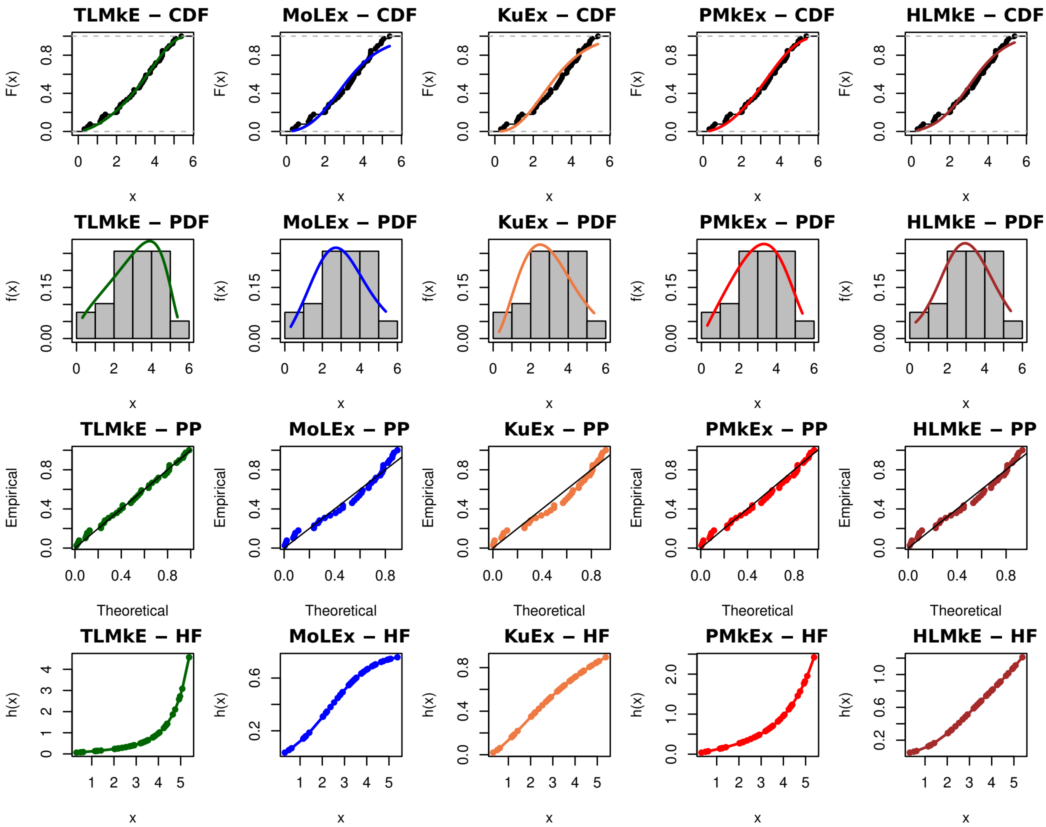

Figure 12. Estimated cdf, pdf and PP-plots of turbocharger suits data

greater flexibility in capturing skewness and tail behaviour.

- Although the emissions data show less statistical performance in the Vuong test, TLMkE still achieves the best predictive performance, maintaining the highest log-likelihood and lowest MAE among all models.
- The *TLMkE* distribution achieves higher LOL values, demonstrating that the model makes more accurate predictions than other models across the three datasets
- Lower MAE values indicate that the model's predictions are closer to the observed values than those of the competing models.

This suggests that although the model's advantage may vary slightly depending on the data, it consistently delivers good results.

Table 18. Vuong Statistic values for TLMkE vs comparing models

Model	Blood Cancer data		Emissions data		Turbocharger suits data	
	Z	p	Z	p	Z	p
TLMkE vs MoLEx	3.3028	0.001	0.9656	0.3342	3.2662	0.0011
TLMkE vs KuEx	2.9491	0.0032	1.1689	0.2425	2.7622	0.0057
TLMkE vs PMkEx	1.2429	0.2139	1.3918	0.164	1.52	0.1285
TLMkE vs HLMkEx	3.9823	< 0.0001	0.7999	0.4238	2.5634	0.0104

Table 19. Predictive Analysis of proposed and comparing models

Model	Blood Cancer data		Emissions data		Turbocharger suits data	
	LOL	MAE	LOL	MAE	LOL	MAE
TLMkE	**-80.1605**	**0.1901**	**-193.9306**	**0.1745**	**-63.7751**	**0.0491**
MoLEx	-86.0052	0.2579	-194.5718	0.2219	-69.8393	0.1927
KuEx	-85.503	0.3411	-194.9295	0.1871	-70.3368	0.2425
PMkEx	-81.2471	0.1908	-195.1073	0.2324	-65.3051	0.1204
HLMkEx	-86.9424	0.4913	-194.1869	0.2183	-67.4871	0.1552

7. Conclusion

This study introduced a new two-parameter generalized family of distributions, the Topp-Leone Modified Kies-G family, which demonstrates considerable flexibility in modelling diverse data patterns. Key statistical properties, which include moments, generating function, order statistics, entropy, mean deviation, inequality and actuarial measures, were derived. The proposed family is extended using an exponential distribution for which properties, including moments, stochastic ordering, inequality and actuarial measures, are investigated. A simulation study demonstrated the reliability of certain estimation methods. Furthermore, the superior performance of the *TLMkE* model over MoLEx, KuEx, PMkEx and HLMkEx distributions was demonstrated using various goodness-of-fit measures, information measures and Voungs statistic and the predictive performance of the model was shown through Leave-one-out log likelihood. The practical applicability of the TLMkE model is demonstrated through three real datasets: Life expectancy, turbocharger suits and emissions data. Overall, the *TLMkE* model offers a flexible tool for modelling data for reliability and survival analysis.

8. Future scope of research

The proposed Topp-Leone Modified Kies-G family has several ways for future research. We plan to extend the model using different baseline distributions, including the Lomax and Weibull distributions. Furthermore, we aim to apply the model to other areas such as deriving and analysing the stress-strength reliability property of the model and parameter estimation using Bayesian estimation techniques, such as the Markov Chain Monte Carlo method and application to different censoring schemes.

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References

1.Abdelall, Y. Y., Hassan, A. S., and Almetwally, E. M. (2024). A new extention of the odd inverse weibull-g family of distributions: Bayesian and non-bayesian estimation with engineering applications. *Computational Journal of Mathematical and Statistical Sciences*, 3(2):359–388.

2.Abushal, T. (2021). Power inverted topp-leone distribution in acceptance sampling plans. *Computers, Materials & Continua*.

3.Adepoju, K. and Chukwu, O. (2015). Maximum likelihood estimation of the kumaraswamy exponential distribution with applications. *Journal of Modern Applied Statistical Methods*, 14(1):18.

4.Afify, A. Z., Al-Mofleh, H., Aljohani, H. M., and Cordeiro, G. M. (2022a). The marshall–olkin–weibull-h family: estimation, simulations, and applications to covid-19 data. *Journal of King Saud University-Science*, 34(5):102115.

5.Afify, A. Z., Gemeay, A. M., Alfaer, N. M., Cordeiro, G. M., and Hafez, E. H. (2022b). Power-modified kies-exponential distribution: Properties, classical and bayesian inference with an application to engineering data. *Entropy*, 24(7):883.

6.Ahmad, Z., Hamedani, G. G., and Butt, N. S. (2019). Recent developments in distribution theory: a brief survey and some new generalized classes of distributions. *Pakistan Journal of Statistics and Operation Research*.

7.Al-Babtain, A. A., Shakhatreh, M. K., Nassar, M., and Afify, A. Z. (2020). A new modified kies family: Properties, estimation under complete and type-ii censored samples, and engineering applications. *Mathematics*, 8(8):1345.

8.Al-Marzouki, S., Jamal, F., Chesneau, C., and Elgarhy, M. (2019). Type ii topp leone power lomax distribution with applications. *Mathematics*, 8(1):4.

9.Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., and Shahbaz, M. Q. (2016). Toppâ€“leone family of distributions: Some properties and application. *Pakistan Journal of Statistics and Operation Research*, pages 443–451.

10.Alghamdi, S. M., Shrahili, M., Hassan, A. S., Gemeay, A. M., Elbatal, I., and Elgarhy, M. (2023). Statistical inference of the half logistic modified kies exponential model with modeling to engineering data. *Symmetry*, 15(3):586.

11.Alizadeh, M., Tahir, M., Cordeiro, G. M., Mansoor, M., Zubair, M., and Hamedani, G. (2015). The kumaraswamy marshal-olkin family of distributions. *Journal of the Egyptian Mathematical Society*, 23(3):546–557.

12.Aljohani, H. M., Almetwally, E. M., Alghamdi, A. S., and Hafez, E. (2021). Ranked set sampling with application of modified kies exponential distribution. *Alexandria Engineering Journal*, 60(4):4041–4046.

13.Almuqrin, M. A., Gemeay, A. M., El-Raouf, M. A., Kilai, M., Aldallal, R., and Hussam, E. (2022). A flexible extension of reduced kies distribution: Properties, inference, and applications in biology. *Complexity*, 2022(1):6078567.

14.Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1):63–79.

15.Artzner, P. (1999). Application of coherent risk measures to capital requirements in insurance. *North American Actuarial Journal*, 3(2):11–25.

16.Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, 9(3):203–228.

17.Atchadé, M. N., N'bouké, M. A., Djibril, A. M., Al Mutairi, A., Mustafa, M. S., Hussam, E., Alsuhabi, H., and Nassr, S. G. (2024). A new topp-leone kumaraswamy marshall-olkin generated family of distributions with applications. *Heliyon*, 10(2).

18.Atchadé, M. N., N'bouké, M., Djibril, A. M., Shahzadi, S., Hussam, E., Aldallal, R., Alshanbari, H. M., Gemeay, A. M., and El-Bagoury, A.-A. H. (2023). A new power topp–leone distribution with applications to engineering and industry data. *PLoS one*, 18(1):e0278225.

19.Bandar, S. A., Hussein, E. A., Yousof, H. M., Afify, A. Z., and Abdellatif, A. D. (2023). A novel extension of the reduced-kies family: Properties, inference, and applications to reliability engineering data. *Advanced Mathematical Models & Applications*, 8(1).

20.Bantan, R. A., Jamal, F., Chesneau, C., and Elgarhy, M. (2020). On a new result on the ratio exponentiated general family of distributions with applications. *Mathematics*, 8(4):598.

21.Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H., Gemeay, A. M., Elgarhy, M., and Nassr, S. G. (2023). A new sine family of generalized distributions: Statistical inference with applications. *Mathematical and Computational Applications*, 28(4):83.

22.Bonferroni, C. E. (1941). *Elementi di statistica generale*. Universitacommerciale Bocconi.

23.Chipepa, F., Oluyede, B., Makubate, B., et al. (2020). The topp-leone-marshall-olkin-g family of distributions with applications. *International Journal of Statistics and Probability*, 9(4):15–32.

24.Cordeiro, G. M. and De Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, 81(7):883–898.

25.Diab, L., Osi, A. A., Sabo, S. A., and Musa, I. Z. (2025). A new inverted modified kies family of distributions: Properties, estimation, simulations and applications. *Journal of Radiation Research and Applied Sciences*, 18(2):101542.

26.El-Saeed, A. R., Hassan, A. S., Elharoun, N. M., Al Mutairi, A., Khashab, R. H., and Nassr, S. G. (2023). A class of power inverted topp-leone distribution: Properties, different estimation methods & applications. *Journal of Radiation Research and Applied Sciences*, 16(4):100643.

27.Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4):497–512.

28.Furman, E. and Landsman, Z. (2006). Tail variance premium with applications for elliptical portfolio of risks. *ASTIN Bulletin: The Journal of the IAA*, 36(2):433–462.

29.Gabanakgosi, M. and Oluyede, B. (2024). The topp-leone-gompertz-g power series class of distributions with applications. *Pakistan Journal of Statistics and Operation Research*, pages 171–195.

30.Hassan, A. S. and Alharbi, R. S. (2023). Different estimation methods for the unit inverse exponentiated weibull distribution. *Communications for Statistical Applications and Methods*, 30(2):191–213.

31.Hassan, A. S. and Nassr, S. G. (2018). The inverse weibull generator of distributions: Properties and applications. *Journal of Data Science*, 16(4).

32.Hassan, A. S. and Nassr, S. G. (2019). Power lindley-g family of distributions. *Annals of Data Science*, 6(2):189–210.

33.Hassan, A. S. and Nassr, S. G. (2020). A new generalization of power function distribution: Properties and estimation based on censored samples. *Thailand Statistician*, 18(2):215–234.

34.Hussam, E., Habadi, M. I., Albayyat, R. H., and Mohammed, M. O. M. (2025). Estimation of parameters for a new model: Real data application and simulation. *Alexandria Engineering Journal*, 122:543–554.

35.Jamal, F., Chesneau, C., and Elgarhy, M. (2020). Type ii general inverse exponential family of distributions. *Journal of Statistics and Management Systems*, 23(3):617–641.

36.Kilai, M., Waititu, G. A., Kibira, W. A., Alshanbari, H. M., and El-Morshedy, M. (2022). A new generalization of gull alpha power family of distributions with application to modeling covid-19 mortality rates. *Results in Physics*, 36:105339.

37.Kumar, C. and Dharmaja, S. (2013). On reduced kies distribution. *Collection of recent statistical methods and applications*, pages 111–123.

38.Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of hydrology*, 46(1-2):79–88.

39.Landsman, Z. (2010). On the tail mean-variance optimal portfolio selection. *Insurance: Mathematics and Economics*, 46(3):547–553.

40.Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. *Publications of the American statistical association*, 9(70):209–219.

41.Mansoor, M., Tahir, M., Cordeiro, G. M., Provost, S. B., and Alzaatreh, A. (2019). The marshall-olkin logistic-exponential distribution. *Communications in Statistics-Theory and Methods*, 48(2):220–234.

42.Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika*, 84(3):641–652.

43.Mudholkar, G. and Srivastava, D. (1993). Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2):299–302.

44.Mulagala, V. and Nagarjuna, V. (2025). Sine modified-kies generalized family of distributions: properties & applications. *Physica Scripta*.

45.Nadarajah, S. and Kotz, S. (2003). Moments of some J-shaped distributions. *Journal of Applied Statistics*, 30(3):311–317.

46.Nassr, S. G., Hassan, A. S., Alsultan, R., and El-Saeed, A. R. (2022). Acceptance sampling plans for the three-parameter inverted topp–leone model. *Math. Biosci. Eng*, 19:13628–13659.

47.Oluyede, B., Chamunorwa, S., Chipepa, F., and Alizadeh, M. (2022). The topp-leone gompertz-g family of distributions with applications. *Journal of Statistics and Management Systems*, 25(6):1399–1423.

48.Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*, volume 4, pages 547–562. University of California Press.

49.Reyad, H., Alizadeh, M., Jamal, F., and Othman, S. (2018). The topp leone odd lindley-g family of distributions: Properties and applications. *Journal of Statistics and Management Systems*, 21(7):1273–1297.

50.S Hassan, A., Abdullah Khaleel, M., and G Nassr, S. (2021). Transmuted topp-leone power function distribution: Theory and application. *Journal of Statistics Applications & Probability*, 10(1):215–227.

51.Sakthivel, M. and Pandiyan, P. (2025). A new model of mixture distribution using a survival analysis of cancer patients. *Brazilian Journal of Biometrics*, 43(1):e43733–e43733.

52.Sule, I., Doguwa, S. I., Isah, A., and Jibril, H. M. (2020a). The topp leone kumaraswamy-g family of distributions with applications to cancer disease data. *Journal of Biostatistics and Epidemiology*, 6(1):40–51.

53.Sule, I., Sani, I. D., Audu, I., and Jibril, H. M. (2020b). On the topp leone exponentiated-g family of distributions: Properties and applications. *Asian Journal of Probability and Statistics*.

54.Swetha, P. S., Hamdani, H., Ouertani, M. N., Nagarjuna, V., Nasiru, S., and Elgarhy, M. (2025). Kumaraswamy modified kies power lomax distribution: Properties, actuarial measures, and applications. *Engineering Reports*, 7(9):e70390.

55.Swetha, P. S. and Nagarjuna, V. (2025). The kumaraswamy modified kies-g family of distributions: properties and applications. *Physica Scripta*, 100(3):035024.

56.Tahir, M., Zubair, M., Mansoor, M., Cordeiro, G. M., Alizadehk, M., and Hamedani, G. (2016). A new weibull-g family of distributions. *Hacettepe Journal of Mathematics and statistics*, 45(2):629–647.

57.Tahir, M. H. and Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well-established g families. *Anais da Academia Brasileira de Ciências*, 87(2):539–568.

58.Topp, C. W. and Leone, F. C. (1955). A family of j-shaped frequency functions. *Journal of the American Statistical Association*, 50(269):209–219.

59.Tsallis, C. (1988). Possible generalization of boltzmann-gibbs statistics. *Journal of statistical physics*, 52:479–487.

60.Vuong, Q. H. (1989). Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica: journal of the Econometric Society*, pages 307–333.

61.Zaidi, S. M., Sobhi, M., El-Morshedy, M., and Afify, A. Z. (2021). A new generalized family of distributions: Properties and applications. *Aims Math*, 6(1):456–476.

62.Zenga, M. et al. (2007). Inequality curve and inequality index based on the ratios between lower and upper arithmetic means.

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