Computational Journal of Mathematical and Statistical Sciences
5(1), 1–34
DOI:10.21608/cjmss.2025.407822.1238
https://cjmss.journals.ekb.eg/

Research article

New Strategies for Finite Population Mean Estimation under Non-Response: Theory and Application to Education Data

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Abstract: The survey sampling designs are generally affected by non-response, which causes bias and inefficiency in the estimators of population parameters. To overcome this problem, in the present study, we construct a new class of estimators for the finite population mean under a stratified random sampling scheme when there is non-response. One of the salient dimensions of our method is the explicit use of dual auxiliary information within a stratified design framework to account for population heterogeneity within strata, which has not been systematically highlighted earlier. Two types of non-response are studied: one where the missing values arise only in the study variable, and another where both the study and auxiliary variables are subject to this response process. The bias, mean squared error (MSE) and efficiency of the proposed estimators are obtained under theoretical properties at the first-order approximation. Their finite performance is also examined via a real education survey and Monte Carlo simulation studies. The results indicate that the suggested estimators have smaller MSE and higher per cent relative efficiency (PRE) than existing ones. Graphical investigations at various threshold values of k indicate the effectiveness of the proposed estimators in practice, with a slight trade-off in efficiency as k increases. In the general context, this demonstrates that assessing more than one auxiliary with stratified sampling under non-response allows us to achieve a more precise and efficient mean estimate. This makes the proposed estimators a convenient and efficient tool for large-scale surveys where complete response data is seldom available.

Keywords: Auxiliary variable, non-response, stratified random sampling, MSE, PRE.
Mathematics Subject Classification: 60E05, 62E10, 62F10.
Received: 26 July 2025; Revised: 11 October 2025; Accepted: 14 October 2025; Online: 15 October 2025.

1. Introduction

The ideal in a sample survey is to collect data from each of the few chosen units in the sample, but because of non-response, this is generally not possible. Some questionnaires are difficult to administer because participants are hesitant to share personal information such as drug use, abortion, and STD status. Individual poll questions might not get. Some units may also not respond or identify themselves to the poll. If nonrespondents differ from respondents in the observed variables and the adequate sample size is less than the rate-based minimum, the sampling variance of the estimates will be inflated. It also introduces bias into the estimates. Hansen and Hurwitz [1] addressed the problem of nonresponse in mail surveys by proposing a procedure that allows for estimating a in the presence of non-response based on subsampling of the nonrespondents. Where the covariate of interest is a stigmatizing one, survey data are often not obtained in each unit. Several researchers addressed the non-response problem and considered in the case of simple random sampling, when two variables are correlated at a time, Sajjad and Ismail [2] presented an efficient modified estimator of the population mean due to measurement error and non-response, which was also suggested Hussain et al. [3] proposed an improved exponential type mean estimator including response information model and Singh et al. [4] under the non-response setup constructed an almost unbiased optimal ratio-type estimator for the population mean in stratified sampling. The non-response problem in survey sampling has long been recognized, and various techniques have been developed to estimate population parameters when the sample/measurement fraction is not 1. Early efforts were directed towards refining the estimation of the population mean. For instance, Bhushan and Pandey [5] designed the best random non-response design for mean estimation, and Jaiswal et al. [6] extended this work by considering regression and factor-type estimators to improve efficiency for the presence of not-at-random response. Other authors also continued such a line of work by dealing with the estimation of variances. In this respect, Pandey et al. [7] and Garg et al. [8] introduced a broad class of improved population variance estimators to address non-sampling errors, using calibrated weights in stratified sampling to minimize bias and achieve greater accuracy.

Besides using means and variances to impute non-response, one can consider more sophisticated methods suggested to deal with particular problems related to non-response. For example, Jan et al. [9] designed new dual exponential estimators for more flexible handling of non-response, and Bhushan and Pandey [10] proposed an optimum procedure for estimating the mean in a two-phase design with random non-response. Additional features included the implementation of more robust alternatives, such as the one proposed by Hussain et al. [11] (an alternative to mean-based methods for non-response situations). Likewise, Tiwari et al. [12] extended the horizons by proposing a general class of estimators that simultaneously incorporates non-response and measurement errors, addressing multiple sources of inaccuracy.

More recent research has focused on how to utilize auxiliary information effectively. Specifically, Aatizaz et al. [13], Hussain et al. [14], and Ahmad et al. [15] studied the dual purpose of auxiliary variables to enhance the estimation of the finite population cumulative distribution function (CDF) and mean in the context of non-response. These papers focused on the growing interest in using auxiliary information as a valuable resource for improving efficiency and reducing bias in survey sampling with incomplete responses.

If the target population is homogeneous, simple random sampling works fine. But if the population of interest is very homogeneous, stratified random sampling isn't a desirable alternative to regular random sampling. Under stratified random sampling, the entire population is grouped into distinct, non-overlapping clusters. Each stratum is represented by a particular sample, which is uniform across all strata. Values are essential for stratification to work at its best. Once the strata are known, a sample can be taken from each one, and all drawings must be redone separately. (Which is actually part of the more general principle of stratified random sampling, where in each stratum, there is simple random sampling) Several trials stratified the sample in multiple strata using different methods of population allocation. In this study, we use the proportional allocation method because it is better to apply an independent ratio estimate to each stratum when the sample size in all strata is large enough. Zaman et al. [17] using stratified random sampling to estimate the population mean. Yadav et al. [18] introduced the optimal strategy for maximizing the estimated population mean in stratified random sampling with linear cost function, Kumar et al. [19] developed new methods of imputation in stratified simple random sampling, Haq and Shabbir[20] proposed an upgraded ratio family estimators and were used in both simple and stratified random sampling, Jabeen et al. [21] worked on calibration approach for estimating domain means or totals when using stratified sampling method and also considered best linear unbiased estimator (BLUE) by Bhushan et al. [22], who have shown both some empirical results from real data based on SRS. These scientists investigate region means under stratified random sampling using auxiliary variables that are in the forms of single, two-auxiliary and dual auxiliary variables. Several recent studies have aimed to enhance estimation when non-response is used by utilizing auxiliary information. Hussain et al. [23] suggested estimators of the finite population distribution function (DF) of simple random sampling (SRS) under non-response with the aid of two auxiliary variables and demonstrated significant efficiency improvements. On this concept, Hussain et al. [24] constructed DF estimators that use dual auxiliary information in both SRS and stratified random sampling, demonstrating that stratification is efficacious in improving the efficiency of estimators in heterogeneous populations. More recently, Junaid et al. [25] proposed an ideal estimation of the DF with non-response in SRS and developed an estimator that minimises the mean square error (MSE) and ensures robustness through simulation experiments. In the same manner, Hussain et al. [26] suggested adjusted estimators of the DF using dual auxiliary information under stratified random sampling, demonstrating that their design is much better than traditional estimators when there are high correlations between the study and auxiliary variables. Simultaneously, Ahmad et al. [27] did not focus on the distribution function but on the mean. They introduced new classes of estimators for the finite population mean when non-response occurs using dual auxiliary information, demonstrating that these new estimators are more effective and have a smaller MSE compared to traditional ones. Latest developments in survey sampling are effective for estimation under non-response and missing information by applying stratified or ranked set designs. Kumar et al. [28] presented new imputation techniques in the context of stratified sampling, and Shahzad et al. [29] used calibrated estimators of the coefficient of variation for double stratified sampling, while Bhushan et al. [[30], [31]] introduced a class of efficient and log-type estimators for the stratified models. Bhushan et al. [[32], [33]] recommended logarithmic, minimum order statistics, and ratio type imputation methods, while Bhushan and Kumar [34] and Kumar et al. [35] proposed multi-auxiliary and logarithmic imputation methods. Together, these studies highlight the importance of auxiliary data, imputation methods and transformations in improving the efficiency of estimation in the presence of non-response and complex sampling.

Motivation and limitations:

The impetus for this study comes from the well-documented issue of non-response in sample surveys, which might lead to serious compromises of statistical inference. When the units in a sample do not report the required information, the adequate sample size is reduced, leading to greater sampling variability. More importantly, non-response can also lead to systematic bias, especially for the estimation of population means, which underlie most survey objectives. The problem becomes severe when mass non-response occurs among correlated units, often in surveys where sensitive information (such as income, health status, or behaviour) must be collected, and respondents are unwilling to provide accurate information. In these cases, classical estimators may not be sufficient, and one has to develop robust estimators that can take into account the incomplete information.

Previous studies have examined the use of auxiliary variables in non-response adjustment, and recent studies have investigated the effectiveness of dual auxiliary variables for enhancing estimator efficiency. However, a large part of such previous work has been limited to situations where population demography involves simple random sampling and did not provide satisfactory treatments for complications generated by population heterogeneity. In the present investigation, a new class of estimators based on stratified random sampling of auxiliary characteristics is proposed. Its uniqueness lies in the fact that it not only utilises variations among strata in the estimation process. The introduced estimators utilise both the between-stratum and within-measurement-unit information involving dual auxiliary variables, which we consider a better option, especially when non-response varies. The theoretical properties of the estimators are validated using analytical expressions for bias and mean squared error, as well as numerical examples. While the results show a significant advantage of the presented method, additional real-world applications to other empirical data are desired to reinforce the practical evidence about our methodology.

Variations from earlier studies:

1. This study presents a dual auxiliary variable technology that increases the effectiveness of estimates in non-response contexts, in contrast to earlier work that was primarily based on one or two auxiliary variables.
2. Many previous studies have only examined one type of non-response, either on the study variable alone or on the survey and auxiliary variables in combination. In this research, two non-response scenarios are explicitly modelled.
3. It performs better than conventional estimators (such as ratio, product, regression, and exponential-type estimators) by achieving a higher PRE and a lower MSE.
4. In contrast to many earlier studies that merely offer theoretical explanations, this study shows efficiency comparisons for a range of k values (k = 2, 3, 4, 5).
5. This study validates the suggested methodology's practical usefulness by applying it to actual educational survey data, in contrast to general estimators that concentrate on fictitious or simulated data.

Novelty and Importance:

1. The paper covers prevailing non-response estimators by merging manifold auxiliary variables in a stratified random sampling background.
2. Even when non-response occurs in auxiliary variables, the planned estimator can adapt to shifting levels of non-response and remnants operative. This scenario has not been extensively covered in prior research.

3. We used stratified random sampling because it preforms better in terms of MSEs and PREs as compared to simple random sampling.

Implementation of the proposed work in real-world survey designs:
The proposed estimator of the population mean, which is used to estimate the population mean when there are non-responses and stratified random sampling, can be used effectively when used in survey design by first dividing the population into clearly defined strata using auxiliary variables like region, age group, or socioeconomic status and then selecting a probability sample within a stratum. As non-response is inevitable in practice, whether at the unit or item level, the estimator corrects for missing data by incorporating known values of auxiliary parameters, such as means or distributional characteristics of census or administrative data, thereby making the estimator less biased and more efficient. In large-scale surveys such as national health surveys, education surveys, or agricultural surveys, this method provides reliability of the estimates even when there are dissimilar response rates between the strata. The operationalization of the technique depends on standard statistical software, where the response-adjusted stratum means are combined with auxiliary information to produce the overall estimate, its variance, and measures of efficiency. This makes the estimator very convenient for official statistics agencies, as it helps them minimize the negative impact of non-response, enhance cost-effectiveness, and produce more valid and policy-instrumental survey results.
The remaining sections of the paper are structured as follows:

The other parts of this paper will be structured as follows. In Section 2, the methodology and notations of the study are provided. Section 3 will consider the available estimators of the mean that have been interpreted to suit the stratified random sampling in the absence of non-response. Section 4 builds on the previous research by creating a new type of estimators that can better handle non-response situations. Theoretical properties of the proposed estimators, such as bias, mean square error (MSE), and efficiency, are calculated up to the first-order approximation. Section 5 will use the proposed and existing estimators on real data on education, whereas Section 6 will assess the performance of the estimators using a Monte Carlo simulation study. The comparative values, which are presented as MSE and percentage relative efficiency (PRE) at varying values of the threshold parameter. The analyses of k are made in Section 7, and the effects of increasing k from 2 to 5 are considered. Section 8 summarises the main contributions and findings, which are used to conclude the study. Section 9 outlines potential extensions and future research directions, while Section 10 focuses on the practical consequences of the suggested work for survey sampling and applied statistics.

2. Methodology and Notations

Consider a population Ω=(Ω_1, $\Omega_2, \ldots, \Omega_N$) having N distinct and identifiable units. Let (y_1, $y_2, \ldots, y_N$) be the values of the study variable Y. Let the population of size N be stratified into L strata with the h^{th}stratum containing N_h units, where h=1,2,...L. Such that $\sum_{h=1}^{L} N_h$=N. A random sample of size n_h, from this population is drawn by simple random sampling with-out replacement (SRSWOR) from the h^{th}stratum such that $\sum_{h=1}^{L} n_h$=n, out of n_h selected units, n_{hr} units respond, the remaining n_{hm}=$n_h - n_{hr}$ units do not respond. Let y_{ih}, x_{ih} be the observed values of the study variable Y and the auxiliary variable X respectively, on the i^{th} unit of the h^{th} stratum.
Y=the study variable, X= the auxiliary variable,

$W_{hi}=\frac{N_{hi}}{N_h}$, weight of the h^{th} stratum and for i=1,2 (1=full response, and 2=non-response)
$\overline{Y}_h=\frac{\sum_{i=1}^{N_h} Y_{ih}}{N_h}$, $\overline{X}_h=\frac{\sum_{i=1}^{N_h} X_{ih}}{N_h}$ and $\overline{R}x_h=\frac{\sum_{i=1}^{N_h} R_{xih}}{N_h}$ the population means of Y, X and rank of X for the h^{th} stratum.
$\overline{Y}_{st}=\sum_{i=1}^{N_h} W_h\overline{Y}_h$, $\overline{X}_{st}=\sum_{i=1}^{N_h} W_h\overline{X}_h$ and $\overline{R}x_{(st)}=\sum_{i=1}^{N_h} W_h\overline{R}x_h$ the population means of Y, X and rank of X.
$\widehat{\overline{Y}}_h=\frac{\sum_{i=1}^{n_h} Y_{ih}}{n_h}$, $\widehat{\overline{X}}_h=\frac{\sum_{i=1}^{n_h} X_{ih}}{n_h}$ and $\widehat{\overline{R}}x_h=\frac{\sum_{i=1}^{n_h} Rx_{ih}}{n_h}$ the sample means of Y, X and rank of X for the h^{th} stratum.
$\widehat{\overline{Y}}_{st}=\sum_{h=1}^{L} W_h\widehat{\overline{Y}}_h$, $\widehat{\overline{X}}_{st}=\sum_{h=1}^{L} W_h\widehat{\overline{X}}_h$ and $\widehat{\overline{R}}x_{(st)}=\sum_{h=1}^{L} W_h\widehat{\overline{R}}x_h$ the sample means Y, X and rank of X.
$\overline{Y}_{h(2)}=\sum_{i=1}^{N_{h2}} \frac{Y_{ih}}{N_{h2}}$, $\overline{X}_{h(2)}=\sum_{i=1}^{N_{h2}} \frac{X_{ih}}{N_{h2}}$ and $\overline{R}x_{h(2)}=\sum_{i=1}^{N_{h2}} \frac{Rx_{ih}}{N_{h2}}$ the population mean of Y, X and rank of X, for non-response group.
$\overline{Y}_{h(1)}=\sum_{i=1}^{n_{h1}} \frac{Y_{ih}}{n_1}$, $\overline{X}_{h(1)}=\sum_{i=1}^{n_{h1}} \frac{X_{ih}}{n_1}$ and $\overline{R}x_{h(1)}=\sum_{i=1}^{n_{h1}} \frac{Rx_{ih}}{n_1}$ the sample means of Y, X and rank of X, based on n_{h1}, responding units of n_h units.
$\overline{Y}_{h(2rh)}=\sum_{i=1}^{r_h} \frac{Y_{ih}}{r_h}$, $\overline{X}_{h(2rh)}=\sum_{i=1}^{r_h} \frac{X_{ih}}{r_h}$ and $\overline{R}x_{h(2rh)}=\sum_{i=1}^{r_h} \frac{Rx_{ih}}{r_h}$ the means of Y, X and rank of X, based on r_h, responding units of n_{h2} units' non-response .
$\sigma^2_{y(1)h}=\sum_{i=1}^{N_h} \frac{\left(Y_{ih}-\overline{Y}_{h(1)}\right)^2}{N_h-1}$, $\sigma^2_{x(1)h}=\sum_{i=1}^{N_h} \frac{\left(X_{ih}-\overline{X}_{h(1)}\right)^2}{N_h-1}$, and $\sigma^2_{rx(1)h}=\sum_{i=1}^{N_h} \frac{\left(RX_{ih}-\overline{R}x_{h(1)}\right)^2}{N_h-1}$ the population variances of Y, X and rank of X for the h^{th} stratum.
$\sigma^2_{y(2)h}=\sum_{i=1}^{N_{h2}} \frac{\left(Y_{ih}-\overline{Y}_{h(2)}\right)^2}{N_{h2}-1}$, $\sigma^2_{x(2)h}=\sum_{i=1}^{N_{h2}} \frac{\left(X_{ih}-\overline{X}_{h(2)}\right)^2}{N_{h2}-1}$, and $\sigma^2_{rx(2)h}=\sum_{i=1}^{N_{h2}} \frac{\left(RX_{ih}-\overline{R}x_{h(2)}\right)^2}{N_{h2}-1}$ the population variances of Y, X and rank of X for the h^{th} stratum non-response group,
$C_{y1h}=\frac{\sigma_{y1h}}{\overline{Y}_{h(1)}}$, $C_{x1h}=\frac{\sigma_{x1h}}{\overline{X}_{h(1)}}$ and $C_{rx1h}=\frac{\sigma_{rx1h}}{\overline{R}x_{h(1)}}$ the population coefficient of variations of Y, X and rank of X, for the h^{th} stratum,
$C_{y2h}=\frac{\sigma_{y2h}}{\overline{Y}_{h(2)}}$, $C_{x2h}=\frac{\sigma_{x2h}}{\overline{X}_{h(2)}}$ and $C_{rx2h}=\frac{\sigma_{rx2h}}{\overline{R}x_{h(2)}}$ the population coefficient of variations of Y, X and rank of X, for the h^{th} stratum non-response group,
$\sigma_{\left(y_{(1)h}x_{(1)h}\right)}=\sum_{i=1}^{N_h} \frac{\left(Y_{ih}-\overline{Y}_{h(1)}\right)}{N_h-1}+\frac{\left(X_{ih}-\overline{X}_{h(1)}\right)}{N_h-1}$, $\sigma_{\left(y_{(1)h}rx_{(1)h}\right)}=\sum_{i=1}^{N_h} \frac{\left(Y_{ih}-\overline{Y}_{h(1)}\right)}{N_h-1} + \frac{\left(R_{xih}-\overline{R}x_{h(1)}\right)}{N_h-1}$, and $\sigma_{\left(x_{(1)h}rx_{(1)h}\right)}=\sum_{i=1}^{N_h} \frac{\left(X_{ih}-\overline{X}_{h(1)}\right)}{N_h-1} + \frac{\left(R_{xih}-\overline{R}x_{h(1)}\right)}{N_h-1}$ the population covariances between Y, X and rank of X, for the h^{th} stratum.
$\sigma_{\left(y_{(2)h}x_{(2)h}\right)}=\sum_{i=1}^{N_{h2}} \frac{\left(Y_{ih}-\overline{Y}_{h(2)}\right)}{N_h-1} + \frac{\left(X_{ih}-\overline{X}_{h(2)}\right)}{N_h-1}$, $\sigma_{\left(y_{(2)h}rx_{(2)h}\right)}=\sum_{i=1}^{N_{h2}} \frac{\left(Y_{ih}-\overline{Y}_{h(2)}\right)}{N_h-1} + \frac{\left(R_{xih}-\overline{R}x_{h(2)}\right)}{N_h-1}$ and $\sigma_{\left(x_{(2)h}rx_{(2)h}\right)}=\sum_{i=1}^{N_{h2}} \frac{\left(X_{ih}-\overline{X}_{h(2)}\right)}{N_h-1} + \frac{\left(R_{xih}-\overline{R}x_{h(2)}\right)}{N_h-1}$ the population covariances between Y, X and rank of X, for the h^{th} stratum non-response group.
$\varrho_{\left(y_{(1)h}x_{(1)h}\right)}=\frac{\sigma\left(y_{(1)h}x_{(1)h}\right)}{\sigma_{y_{(1)h}}\sigma_{x_{(1)h}}}$, $\varrho_{\left(y_{(1)h}rx_{(1)h}\right)}=\frac{\sigma\left(y_{(1)h}rx_{(1)h}\right)}{\sigma_{y_{(1)h}}\sigma_{rx_{(1)h}}}$ and $\varrho_{\left(x_{(1)h}rx_{(1)h}\right)}=\frac{\sigma\left(x_{(1)h}rx_{(1)h}\right)}{\sigma_{x_{(1)h}}\sigma_{rx_{(1)h}}}$ the population correlation coefficients between Y, X and rank of X, for the h^{th} stratum.
$\varrho_{\left(y_{(2)h}x_{(2)h}\right)}=\frac{\sigma\left(y_{(2)h}x_{(2)h}\right)}{\sigma_{y_{(2)h}}\sigma_{x_{(2)h}}}$, $\varrho_{\left(y_{(2)h}rx_{(2)h}\right)}=\frac{\sigma\left(y_{(2)h}rx_{(2)h}\right)}{\sigma_{y_{(2)h}}\sigma_{rx_{(2)h}}}$ and $\varrho_{\left(x_{(2)h}rx_{(2)h}\right)}=\frac{\sigma\left(x_{(2)h}rx_{(2)h}\right)}{\sigma_{x_{(2)h}}\sigma_{rx_{(2)h}}}$ the population correlation coefficient between Y, X and rank of X, for the h^{th} stratum non-response group.
We can write

$$\overline{Y} = W_{1h}\overline{Y}_{1(h)} + W_{2h}\overline{Y}_{2(h)} \tag{2.1}$$

$$\overline{X} = W_{1h}\overline{X}_{1(h)} + W_{2h}\overline{X}_{2(h)} \tag{2.2}$$

where
$W_{jh}=\frac{N_{jh}}{N}$, $\overline{Y}_{j(h)}=\sum_{i=1}^{N_{jh}} \frac{Y_{hi}}{N_{jh}}$ and $\overline{X}_{j(h)}=\sum_{i=1}^{N_{jh}} \frac{X_{hi}}{N_{jh}}$, for j=1,2
On the lines of Hansen and Hurwitz [1], the estimator $\overline{Y}_h$ under non-response is given by:

$$\widehat{\overline{Y}}^{*} = w_{1h}\widehat{\overline{Y}}_{(1)h} + w_{2h}\widehat{\overline{Y}}_{(2r)h}$$

$$Var\left(\widehat{\overline{Y}}^{*}\right) = \lambda_{1h}\sigma^{2}_{y(1)h} + \lambda_{2h}\sigma^{2}_{y(2)h} \tag{2.3}$$

Similarly

$$\widehat{\overline{X}}^{*} = w_{1h}\widehat{\overline{X}}_{(1)h} + w_{2h}\widehat{\overline{X}}_{(2r)h} \tag{2.4}$$

$$Var\left(\widehat{\overline{X}}^{*}\right) = \lambda_{1h}\sigma^{2}_{x(1)h} + \lambda_{2h}\sigma^{2}_{x(2)h} \tag{2.5}$$

Where $\lambda_{1h} = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)$, $\lambda_{2h} = \left(\frac{k-1}{n_h}\right) w_{2(h)}$ and k= 2, 3, 4, and 5.
To derive the bias and MSE of the suggested and existing estimators, we discuss the following error term, which is given by:

$$\varepsilon^{*}_{o(st)} = \frac{\widehat{\overline{Y}}^{*}_{st} - \overline{Y}_{st}}{\overline{Y}_{st}}, \varepsilon^{*}_{1(st)} = \frac{\widehat{\overline{X}}^{*}_{st} - \overline{X}_{st}}{\overline{X}_{st}}, \varepsilon_{1(st)} = \frac{\widehat{\overline{X}}_{st} - \overline{X}_{st}}{\overline{X}_{st}}, \varepsilon^{*}_{2(st)} = \frac{\widehat{\overline{R}x}^{*}_{st} - \overline{R}x_{st}}{\overline{R}x_{st}}, \varepsilon_{2(st)} = \frac{\widehat{\overline{R}}x_{(st)} - \overline{R}x_{(st)}}{\overline{R}x_{(st)}}$$

Such that
$E\left(\varepsilon^{*}_{i(st)}\right)$=$E(\varepsilon_{i(st)})$=0, ($i^{*}$=0,1,2 when the non-response on both study and auxiliary variables) and (i=0,1 when the non-response on auxiliary variables)
$\mathcal{A}_{rst}$=$E\left[\varepsilon^{*r}_{0(st)}\varepsilon^{*s}_{1(st)}\varepsilon^{*t}_{2(st)}\right]$ and $\mathcal{B}_{rs}$=$E\left[\varepsilon^{s}_{1(st)}\varepsilon^{t}_{2(st)}\right]$
Where r, s, t=0,1, 2
$E\left(\varepsilon^{*2}_{0(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{y(1)h} + W^{2}_{2h}\lambda^{2}_{2h}C^{2}_{y(2)h}\right)=\mathcal{A}^{2}_{200}$ for situation-I
$E\left(\varepsilon^{*2}_{1(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{x(1)h} + W^{2}_{2h}\lambda^{2}_{2h}C^{2}_{x(2)h}\right)=\mathcal{A}^{2}_{020}$
$E\left(\varepsilon^{*2}_{2(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{rx(1)h} + W^{2}_{2h}\lambda^{2}_{2h}C^{2}_{rx(2)h}\right)=\mathcal{A}^{2}_{002}$
$E\left(\varepsilon^{*2}_{0(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{y(1)h} + W^{2}_{2h}\lambda^{2}_{2h}C^{2}_{y(2)h}\right)=\mathcal{B}^{2}_{200}$ for situation-II
$E\left(\varepsilon^{2}_{1(st)}\right)=\sum_{h=1}^{L} W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{x(1)h}=\mathcal{B}^{2}_{020}$
$E\left(\varepsilon^{2}_{2(st)}\right)=\sum_{h=1}^{L} W^{2}_{1h}\lambda^{2}_{1h}C^{2}_{rx(1)h}=\mathcal{B}^{2}_{002}$
$E\left(\varepsilon^{*}_{0(st)}\varepsilon^{*}_{1(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(y_{(1)h}x_{(1)h}\right)}C_{y1h}C_{x1h} + W^{2}_{2h}\lambda^{2}_{2h}\varrho_{\left(y_{(2)h}x_{(2)h}\right)}C_{y2h}C_{x2h}\right)=\mathcal{A}_{110}$,
$E\left(\varepsilon^{*}_{0(st)}\varepsilon^{*}_{2(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(y_{(1)h}rx_{(1)h}\right)}C_{y1h}C_{rx1h} + W^{2}_{2h}\lambda^{2}_{2h}\varrho_{\left(y_{(2)h}rx_{(2)h}\right)}C_{y2h}C_{rx2h}\right)=\mathcal{A}_{101}$,
$E\left(\varepsilon^{*}_{1(st)}\varepsilon^{*}_{2(st)}\right)=\sum_{h=1}^{L}\left(W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(x_{(1)h}rx_{(1)h}\right)}C_{x1h}C_{rx1h} + W^{2}_{2h}\lambda^{2}_{2h}\varrho_{\left(x_{(2)h}rx_{(2)h}\right)}C_{x2h}C_{rx2h}\right)=\mathcal{A}_{011}$,
$E\left(\varepsilon^{*}_{0(st)}\varepsilon_{1(st)}\right)=\sum_{h=1}^{L} W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(y_{(1)h}x_{(1)h}\right)}C_{y1h}C_{x1h}=\mathcal{B}_{110}$
$E\left(\varepsilon^{*}_{0(st)}\varepsilon_{2(st)}\right)=\sum_{h=1}^{L} W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(y_{(1)h}Rx_{(1)h}\right)}C_{y1h}C_{rx1h}=\mathcal{B}_{101}$
$E(\varepsilon_{1(st)}\varepsilon_{(st)})=\sum_{h=1}^{L} W^{2}_{1h}\lambda^{2}_{1h}\varrho_{\left(x_{(1)h}Rx_{(1)h}\right)}C_{x1h}C_{rx1h}=\mathcal{B}_{011}$
Three situations were employed under non-response. However, in this paper, we looked at two scenarios. In condition I, the non-response affects both the study and auxiliary variables, but in situation II, it affects only the study variable. The notations listed in Table 1 are used for notational convenience.

3. Adapted estimators

In this section, specific estimators for population mean based on non-response are discussed.

3.1. Existing estimators under situation-I of non-response.

Situation I involves survey sampling, where non-response is only encountered for the study variable Y; however, auxiliary information is available for the entire chosen sample. The [1] sub-sampling method is frequently used in this type of case. A sub-sample of non-respondents is chosen, and special efforts are exerted to get their responses. The information from the respondents and the non-respondent sub-sample is then pooled together to form an estimator of the population mean. Given that the non-response does not affect the auxiliary information, it can be successfully used to enhance the efficiency of the estimators.

(1) Ratio estimator $\overline{Y}_{(st)(s-i-of\ non-r)}$ of the stratified population mean in Situation -I non-response is by (3.1).

$$\widehat{\overline{\overline{Y}}}_{R(st)H(s-i-of\ non-r)}=\widehat{\overline{\overline{Y}}}_{(st)H(s-i-of\ non-r)}\left(\frac{\overline{X}_{(st)}}{\widehat{\overline{\overline{X}}}_{(st)H(s-i-of\ non-r)}}\right). \tag{3.1}$$

The properties of $\widehat{\overline{\overline{Y}}}_{R(st)H(s-i-of\ non-r)}$:

B($\widehat{\overline{\overline{Y}}}_{R(st)H(s-i-of\ non-r)}$) $\cong \overline{Y}_{(st)}\,(\mathcal{A}_{020}-\mathcal{A}_{110})$, and MSE is given by (3.2)

$$\mathrm{MSE}(\widehat{\overline{\overline{Y}}}_{R(st)H}) \cong \overline{Y}^{2}_{(st)}\,(\mathcal{A}_{200}+\mathcal{A}_{020}-2\mathcal{A}_{110})\,. \tag{3.2}$$

(2) The product estimator for $\overline{Y}_{(st)(s-i-of\ non-r)}$, are given by by (3.3)

$$\widehat{\overline{\overline{Y}}}_{P(st)H(s-i-of\ non-r)}=\widehat{\overline{\overline{Y}}}_{(st)H(s-i-of\ non-r)}\left(\frac{\widehat{\overline{\overline{X}}}_{(st)H(s-i-of\ non-r)}}{\overline{X}_{(st)}}\right). \tag{3.3}$$

The properties of $\widehat{\overline{\overline{Y}}}_{P(st)H(s-i-of\ non-r)}$ respectively:

B($\widehat{\overline{\overline{Y}}}_{P(st)H}$) $=\overline{Y}_{(st)}\mathcal{A}_{110}$, and MSE is given by (3.4)

$$\mathrm{MSE}(\widehat{\overline{\overline{Y}}}_{P(st)H}) \cong \overline{Y}^{2}_{(st)}\,(\mathcal{A}_{200}+\mathcal{A}_{020}+2\mathcal{A}_{110})\,. \tag{3.4}$$

(3) The conventional regression estimator of $\overline{Y}_{(st)(s-i-of\ non-r)}$ given by (3.5)

$$\widehat{\overline{\overline{Y}}}_{Reg(st)H(s-i-of\ non-r)}=\widehat{\overline{\overline{Y}}}_{(st)H(s-i-of\ non-r)}+w(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s-i-of\ non-r)}), \tag{3.5}$$

however, w is constant.

The minimum variance of $\widehat{\overline{\overline{Y}}}_{Reg(st)H(s-i-of\ non-r)}$ at $w_{(\mathrm{opt})}=\frac{(\overline{Y}_{(st)}\ \mathcal{A}_{110})}{(\overline{X}_{(st)}\ \mathcal{A}_{020})}$ is given by (3.6)

$$\mathrm{Var}_{\min}(\widehat{\overline{\overline{Y}}}_{Reg(st)H}) = \frac{\overline{Y}^{2}_{(st)}\left(\mathcal{A}_{200}\mathcal{A}_{020}-\mathcal{A}_{110}{}^{2}\right)}{\mathcal{A}_{020}}\,. \tag{3.6}$$

(4) Zaman and Bulut [17], proposed a difference estimator, which is intended to enhance the ability to estimate the population mean. This estimator utilises the difference between the study and auxiliary

variables, hence minimising the bias and increasing efficiency over traditional estimators is given by (3.7).

$$\widehat{\overline{\overline{Y}}}_{R,D(st)H(s-i-of\ non-r)}=w_1\widehat{\overline{\overline{X}}}_{(st)H(s-i-of\ non-r)}+w_2(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s-i-of\ non-r)}), \tag{3.7}$$

The bias and MSE of $\widehat{\overline{\overline{Y}}}_{R,D(st)H}$, are given by (3.8)

$$B(\widehat{\overline{\overline{Y}}}_{R,D(st)H(s-i-of\ non-r)})=\overline{Y}_{(st)}(w_1-1),$$

$$MSE(\widehat{\overline{\overline{Y}}}_{R,D(st)H})\cong\overline{Y}^2_{(st)}(w_1-1)^2+\overline{Y}^2_{(st)}\mathcal{A}_{200}w_1^2+\overline{X}^2_{(st)}\mathcal{A}_{020}w_2^2-2\overline{Y}_{(st)}\overline{X}_{(st)}\mathcal{A}_{110}w_1w_2. \tag{3.8}$$

The estimates of w_1 and w_2 are derived by simplifying (3.8) using the weights that minimize MSE, which result in making the proposed estimator more efficient.

$$w_{1(\text{opt})}=\frac{\mathcal{A}_{020}}{\left\{\mathcal{A}_{020}\left(1+\mathcal{A}_{200}\right)-\mathcal{A}_{110}{}^2\right\}}\ \text{and}\ w_{2(\text{opt})}=\frac{\mathcal{A}_{110}}{\overline{X}_{(st)}\left\{\mathcal{A}_{020}\left(1+\mathcal{A}_{200}\right)-\mathcal{A}_{110}{}^2\right\}},$$

respectively.

$$MSE_{\min}(\widehat{\overline{\overline{Y}}}_{R,D(st)H(s-i-of\ non-r)})\cong\frac{\overline{Y}^2_{(st)}\left(\mathcal{A}_{200}\mathcal{A}_{020}-\mathcal{A}_{110}{}^2\right)}{\left\{\mathcal{A}_{020}\left(1+\mathcal{A}_{200}\right)-\mathcal{A}_{110}{}^2\right\}}. \tag{3.9}$$

(5) Yadav et al. [18] proposed a ratio-type exponential estimator, which makes use of the auxiliary information by taking an exponential modification. This estimator was suggested for improving the estimator of the population mean to decrease bias and MSE relative to the conventional ratio estimator is given by (3.10)

$$\widehat{\overline{\overline{Y}}}_{S(st)H(s-i-of\ non-r)}=\widehat{\overline{\overline{Y}}}_{(st)H(s-i-of\ non-r)}\exp\left(\frac{a(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H}(s-i-of\ non-r))}{a(\overline{X}_{(st)}+\widehat{\overline{\overline{X}}}_{(st)H}(s-i-of\ non-r))+2b}\right), \tag{3.10}$$

$$B(\widehat{\overline{\overline{Y}}}_{S(st)})\cong\overline{Y}_{(st)}\left(\frac{3\Theta^2\mathcal{A}_{200}}{8}-\frac{\Theta\mathcal{A}_{110}}{2}\right),$$

The mean square errors is given by (3.11)

$$MSE(\widehat{\overline{\overline{Y}}}_{S(st)H(s-i-of\ non-r)})\cong\frac{\overline{Y}^2_{(st)}}{4}(4\mathcal{A}_{200}+\Theta^2\mathcal{A}_{020}-4\Theta\mathcal{A}_{110}), \tag{3.11}$$

where $\Theta=a\overline{X}_{(st)}/(a\overline{X}_{(st)}+b)$.
When we use $a=1$ and $b=0$ we get the value of $\Theta=1$

(6) In the literature, Kumar et al. [19] introduced a general ratio-type exponential estimator by merging the properties of two estimators (ratio and exponential) given in (3.12). The formulation was intended to increase the efficiency of population mean estimation based on reduced bias and Mean Squared Error (MSE) under heterogeneity between study and auxiliary variables.

$$\begin{aligned}\widehat{\overline{\overline{Y}}}_{GK(st)H(s\text{-}i\text{-}of\ non\text{-}r)}&=\left\{w_3\widehat{\overline{\overline{Y}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}+w_4\left(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right)\right\}\\&\times\exp\left(\frac{a\left(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right)}{a\left(\overline{X}_{(st)}+\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right)+2b}\right)\end{aligned} \tag{3.12}$$

where w_3 and w_4 are constants.
The bias of $\widehat{\overline{Y}}_{GK(st)H(s-i-of\ non-r)}$, respectively, is given by

$$B(\widehat{\overline{Y}}_{GK(st)H(s-i-of\ non-r)}) \cong \overline{Y}_{(st)} - w_3\overline{Y}_{(st)} + \frac{3}{8}w_3\Theta^2\overline{Y}_{(st)}\mathcal{A}_{200} + \frac{1}{2}w_4\Theta\overline{X}_{(st)}\mathcal{A}_{020} - \frac{1}{2}w_3\Theta\overline{Y}_{(st)}\mathcal{A}_{110},$$

The MSE of $\widehat{\overline{Y}}_{GK(st)H(s-i-of\ non-r)}$, respectively, is given by (3.13)

$$\begin{aligned} MSE(\widehat{\overline{Y}}_{GK(st)H(s-i-of\ non-r)}) \cong\ & \overline{Y}^2_{(st)}(w_3-1)^2 + w_3^2\,\overline{Y}^2_{(st)}\,\mathcal{A}_{200} + w_4^2\,\overline{X}^2_{(st)}\,\mathcal{A}_{020} + \Theta^2\,w_3^2\,\overline{Y}^2_{(st)}\,\mathcal{A}_{020} \\ & + 2\,w_3w_4\,\Theta\,\overline{Y}_{(st)}\overline{X}_{(st)}\,\mathcal{A}_{020} - \tfrac{3}{4}\,w_3\,\Theta^2\,\overline{Y}^2_{(st)}\,\mathcal{A}_{020} - w_4\,\Theta\,\overline{Y}_{(st)}\overline{X}_{(st)}\,\mathcal{A}_{020} \\ & + w_3\,\Theta\,\overline{Y}^2_{(st)}\,\mathcal{A}_{110} - 2\,w_3^2\,\Theta\,\overline{Y}^2_{(st)}\,\mathcal{A}_{110} - 2\,w_3w_4\,\overline{Y}_{(st)}\overline{X}_{(st)}\,\mathcal{A}_{110}. \end{aligned} \tag{3.13}$$

, after simplification we get minimum MSE in (3.14)

$$MSE_{\min}\left(\widehat{\overline{Y}}_{GK(st)H(s-i-of\ non-r)}\right) \cong \frac{\overline{Y}^2_{(st)}}{64}\left(64 - 16\mathcal{A}_{020} - \frac{\mathcal{A}_{020}\,(-8+\mathcal{A}_{020})^2}{\mathcal{A}_{020}\,(1+\mathcal{A}_{200}) - \mathcal{A}_{110}}\right). \tag{3.14}$$

(7) Haq and Shabbir [20] proposed an efficient estimator on the population mean that uses auxiliary information for a reduction in MSE compared to those of typical estimators, is give by (3.15)

$$\begin{aligned} \widehat{\overline{Y}}_{H.S(st)H(s-i-of\ non-r)} \ =\ & \left[w_5\widehat{\overline{Y}}_{(st)H(s-i-of\ non-r)} + w_6\left(\overline{X}_{(st)} - \widehat{\overline{X}}_{(st)H(s-i-of\ non-r)}\right)\right] \\ & \times \exp\left(\frac{\overline{X}_{(st)}+b}{\alpha(a\widehat{\overline{X}}_{(st)H(s-i-of\ non-r)}+b) + (1-\alpha)(a\overline{X}_{(st)}+b)} - 1\right). \end{aligned} \tag{3.15}$$

Where

$$\Theta = (\alpha * (a\overline{X}_{(st)}))/(a\overline{X}_{(st)}+b).$$

For particular values of the family parameters, alternative well-known estimators may be obtained. Note that when ($\alpha = 1,\ a = 1$ and $b = 0$), we get Θ=1 and the family becomes one: hence it can be analysed in its own right by considering its bias, MSE and efficiency. When α=0, a=1, and b=0, the auxiliary variable does not affect the estimate, and we thus recover the classical unbiased estimator of the population mean in stratified sampling. In another example, if α=1,a=0, and b=1, the estimator involves the auxiliary variable differently by obtaining a type form under just one member. There are still both ratio- and product-type corrections left, and the family remains wide for finite values of a and b. The general family links with some existing estimators in the literature, given by different parameter constraints, and is also flexible enough to generate new ones.
Using ((3.15)), the bias and MSE of $\widehat{\overline{Y}}_{H.S(st)H(s-i-of\ non-r)}$, are given by:

$$B(\widehat{\overline{Y}}_{H.S(st)H(s-i-of\ non-r)}) = -\ \overline{Y}_{(st)} + \Theta w_6\overline{X}_{(st)}\mathcal{A}_{020} + w_5\left(\overline{Y}_{st} + \frac{3\overline{Y}_{(st)}\Theta^2\mathcal{A}_{020}}{2} - \Theta\overline{Y}_{(st)}\mathcal{A}_{110}\right)$$

$$\mathrm{MSE}\left(\widehat{\overline{Y}}_{H.S(st)H}\right) = \overline{Y}_{(st)}^{\,2} + w_6\overline{X}_{(st)}\left(-2\Theta\overline{Y}_{(st)} + w_6\overline{X}_{(st)}\right)\mathcal{A}_{020} + w_5\overline{Y}_{(st)}\left[-2\overline{Y}_{(st)} + \Theta\left(-3\Theta\overline{Y}_{(st)} + 4w_6\overline{X}_{(st)}\right)\mathcal{A}_{020} + 2\left(\Theta\overline{Y}_{(st)} - w_6\overline{Y}_{(st)}\right)\mathcal{A}_{110}\right] + \overline{Y}_{(st)}^{\,2} w_6^2\left(1 + 4\Theta^2\mathcal{A}_{020} - 4\Theta\mathcal{A}_{110} + \mathcal{A}_{200}\right) \quad (3.16)$$

using the optimum values of constants in equation (3.16), we have (3.17)

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{H.S(st)H(s-i-of\ non-r)}\right) \cong \frac{\overline{Y}_{(st)}^{\,2}\left[4\mathcal{A}_{110}^{\,2} + \mathcal{A}_{020}\left\{\Theta^4\mathcal{A}_{020}^{\,2} - \Theta^2\mathcal{A}_{110}^{\,2} + 4\left(-1 + \Theta^2\mathcal{A}_{020}\right)\mathcal{A}_{020}\right\}\right]}{4\left\{\mathcal{A}_{110}^{\,2} - \mathcal{A}_{020}\left(1 + \mathcal{A}_{200}\right)\right\}} \quad (3.17)$$

(8) Ahmad et al. [15] suggested a family of estimators given by (3.18)

$$\widehat{\overline{Y}}^{*}_{A.H(st)H(s-i-of\ non-r)} = \left\{w_7\,\widehat{\overline{Y}}_{(st)H(s-i-of\ non-r)} + w_8\left(\overline{X}_{(st)} - \widehat{\overline{X}}_{(st)H(s-i-of\ non-r)}\right) + w_9\left(\overline{Rx}_{(st)} - \widehat{\overline{Rx}}_{(st)H(s-i-of\ non-r)}\right)\right\}\exp\left[\frac{a\left(\overline{X}_{(st)} - \widehat{\overline{X}}_{(st)H(s-i-of\ non-r)}\right)}{a\left(\overline{X}_{(st)} + \widehat{\overline{X}}_{(st)H(s-i-of\ non-r)}\right) + 2b}\right]. \quad (3.18)$$

$$\mathrm{Bias}\left(\widehat{\overline{Y}}^{*}_{A.H(st)H(s-i-of\ non-r)}\right) \cong \overline{Y}_{(st)}\left(w_7 - 1\right) + \frac{3}{8}\Theta^2 w_7\overline{Y}_{(st)}\mathcal{A}_{020} + \frac{1}{2}\Theta\overline{X}_{(st)}w_8\mathcal{A}_{020} - \frac{1}{2}\Theta w_7\overline{Y}_{(st)}\mathcal{A}_{110} + \frac{1}{2}\Theta\overline{Rx}_{(st)}\mathcal{A}_{011},$$

and the mean square error is give in (3.19)

$$\mathrm{MSE}\left(\widehat{\overline{Y}}^{*}_{A.H(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right) \cong \overline{Y}_{(st)}^{\,2}(w_7 - 1)^2 + w_7^2\overline{Y}_{(st)}^{\,2}\mathcal{A}_{200} + w_8\overline{X}_{(st)}^{\,2}\mathcal{A}_{200} + \overline{Rx}_{(st)}^{\,2}w_9\mathcal{A}_{002} + \Theta^2 w_7^2\overline{Y}_{(st)}^{\,2}\mathcal{A}_{200} - \Theta w_8\overline{Y}_{(st)}\mathcal{A}_{200} + 2\Theta\overline{Y}_{(st)}\mathcal{A}_{200} - \tfrac{3}{4}\Theta^2 w_7\overline{Y}_{(st)}^{\,2}\mathcal{A}_{200} + \Theta w_7\overline{Y}_{(st)}^{\,2}\mathcal{A}_{110} - 2\Theta w_7^2\overline{Y}_{(st)}^{\,2}\mathcal{A}_{110} - 2w_7w_8\overline{Y}_{(st)}\overline{X}_{(st)}\mathcal{A}_{110} - 2w_7w_8\overline{Y}_{(st)}\overline{Rx}_{(st)}\mathcal{A}_{101} - \Theta w_9\overline{Y}_{(st)}\overline{Rx}_{(st)}\mathcal{A}_{011} + 2\Theta w_7w_9\overline{Y}_{(st)}\overline{Rx}_{(st)}\mathcal{A}_{011} - 2w_8w_9\overline{Y}_{(st)}\overline{Rx}_{(st)}\mathcal{A}_{011}. \quad (3.19)$$

, after simplification we have (3.20)

$$MSE\left(\widehat{\overline{Y}}^{*}_{A.H(st)H(s-i-of\ non-r)}\right) \cong \frac{\overline{Y}_{(st)}^{\,2}}{64}\left[64 - 16\Theta^2\mathcal{A}_{011} + \frac{\left(\Theta^2\mathcal{A}_{020-8}\right)^2\left(\mathcal{A}_{101}^2 - \mathcal{A}_{002}\mathcal{A}_{020}\right)}{\left[\begin{array}{c}-\mathcal{A}_{020}\mathcal{A}_{101}^2 + 2\mathcal{A}_{011}\mathcal{A}_{101}\mathcal{A}_{110} - \mathcal{A}_{011}^2\left(1 + \mathcal{A}_{200}^2\right)\\ +\mathcal{A}_{020}\left\{-\mathcal{A}_{110}^2 + \mathcal{A}_{020}\left(1 + \mathcal{A}_{200}\right)\right\}\end{array}\right]}\right] \quad (3.20)$$

3.2. Existing estimators under situation-II of non-response.

The situation II is more demanding and happens when both the study variable have non-response.
Y and the auxiliary variable X. Such a scenario means that not all the sampled units have auxiliary information that is necessary to perform the estimation process. Sub-sampling of non-respondents has to be conducted in both to overcome this problem.
Y and X, and the obtained data are then utilized in creating estimators of the population mean. Nevertheless, due to the impact of missing values on both the study and auxiliary variables, the estimators obtained are generally less efficient compared to those in Situation I.
(1) The ratio estimator for estimating is give by (3.21)

$$\overline{Y}_{(st)(s-ii-of\ non-r)}, given by \widehat{\overline{Y}}_{R(st)H(s-ii-of\ non-r)} = \widehat{\overline{Y}}_{(st)H(s-ii-of\ non-r)} \left(\frac{\overline{X}_{(st)}}{\widehat{\overline{X}}_{(st)H(s-ii-of\ non-r)}} \right). \tag{3.21}$$

The properties of $\widehat{\overline{Y}}_{R(st)H(s-ii-of\ non-r)}$:
$B(\widehat{\overline{Y}}_{R(st)H(s-ii-of\ non-r)}) \cong \overline{Y}_{(st)}\ (\mathcal{B}_{020} - \mathcal{B}_{110})$, and MSE is give by (3.22)

$$\mathrm{MSE}(\widehat{\overline{Y}}_{R(st)H(s-ii-of\ non-r)}) \cong \overline{Y}^2_{(st)}\ (\mathcal{B}_{200} + \mathcal{B}_{020} - 2\mathcal{B}_{110})\,. \tag{3.22}$$

(2) The product estimator for $\overline{Y}_{(st)(s-ii-of\ non-r)}$, with bias and mse is given by (3.23)

$$\widehat{\overline{Y}}_{P(st)H(s-ii-of\ non-r)} = \widehat{\overline{Y}}_{(st)H(s-ii-of\ non-r)} \left(\frac{\widehat{\overline{X}}_{(st)H(s-ii-of\ non-r)}}{\overline{X}_{(st)}} \right). \tag{3.23}$$

$B(\widehat{\overline{Y}}_{P(st)H(s-ii-of\ non-r)}) = \overline{Y}_{(st)}\mathcal{B}_{110}$ and MSE is given by (3.24)

$$\mathrm{MSE}(\widehat{\overline{Y}}_{P(st)H(s-ii-of\ non-r)}) \cong \overline{Y}^2_{(st)}\ (\mathcal{B}_{200} + \mathcal{B}_{020} + 2\mathcal{B}_{110})\,. \tag{3.24}$$

(3) The conventional regression estimator of $\overline{Y}_{(st)}$ given by (3.25)

$$\widehat{\overline{Y}}_{Reg(st)H(s-ii-of\ non-r)} = \widehat{\overline{Y}}_{(st)H(s-ii-of\ non-r)} + k(\overline{X}_{(st)} - \widehat{\overline{X}}_{(st)H(s-ii-of\ non-r)}), \tag{3.25}$$

however, k is constant.
The minimum variance of $\widehat{\overline{Y}}_{Reg(st)H(s-ii-of\ non-r)}$ at $k_{(\mathrm{opt})} = \frac{(\overline{Y}_{(st)}\ \mathcal{B}_{110})}{(\overline{X}_{(st)}\ \mathcal{B}_{020})}$ is given by (3.26):

$$\mathrm{Var}_{\min}(\widehat{\overline{Y}}_{Reg(st)H(s-ii-of\ non-r)}) = \frac{\overline{Y}^2_{(st)} \left(\mathcal{B}_{200}\mathcal{B}_{020} - \mathcal{B}_{110}{}^2 \right)}{\mathcal{B}_{020}}\,. \tag{3.26}$$

(4) Zaman and Bulut [17] suggested a difference estimator, given by (3.27)

$$\widehat{\overline{Y}}_{R,D(st)H(s-ii-of\ non-r)} = k_1\widehat{\overline{X}}_{(st)H} + k_2(\overline{X}_{(st)} - \widehat{\overline{X}}_{(st)H}), \tag{3.27}$$

$$B(\widehat{\overline{Y}}_{R,D(st)H(s-ii-of\ non-r)}) = \overline{Y}_{(st)}(k_1 - 1),$$

and MSE is given by (3.28)

$$\mathrm{MSE}\left(\widehat{\overline{\overline{Y}}}_{R,D(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right) \cong \overline{Y}^2_{(st)}(k_1-1)^2 + \overline{Y}^2_{(st)}\,\mathcal{B}_{200}\,k_1^2 + \overline{X}^2_{(st)}\,\mathcal{B}_{020}\,k_2^2 - 2\,\overline{Y}_{(st)}\,\overline{X}_{(st)}\,\mathcal{B}_{110}\,k_1k_2. \tag{3.28}$$

The MSE of $\widehat{\overline{\overline{Y}}}_{R,D(st)H(s-ii-of\ non-r)}$ at the values of k_1 and k_2 is given by (3.29)

$$\mathrm{MSE}_{\min}(\widehat{\overline{\overline{Y}}}_{R,D(st)H(s-ii-of\ non-r)}) \cong \frac{\overline{Y}^2_{(st)}\left(\mathcal{B}_{200}\mathcal{B}_{020}-\mathcal{B}_{110}{}^2\right)}{\left\{\mathcal{B}_{020}\,(1+\mathcal{B}_{200})-\mathcal{B}_{110}{}^2\right\}}. \tag{3.29}$$

(5) Yadav et al. [18] suggested a ratio-type exponential estimator, given by (3.30)

$$\widehat{\overline{\overline{Y}}}_{S(st)H(s-ii-of\ non-r)}=\widehat{\overline{\overline{Y}}}_{(st)H(s-ii-of\ non-r)}\exp\left(\frac{a(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H})}{a(\overline{X}_{(st)}+\widehat{\overline{\overline{X}}}_{(st)H})+2b}\right), \tag{3.30}$$

$$\mathrm{B}(\widehat{\overline{\overline{Y}}}_{S(st)H(s-ii-of\ non-r)}) \cong \overline{Y}_{(st)}\left(\frac{3\Theta^2\mathcal{B}_{200}}{8}-\frac{\Theta\mathcal{B}_{110}}{2}\right),$$

and MSE is give in (3.31)

$$\mathrm{MSE}(\widehat{\overline{\overline{Y}}}_{S(st)H(s-ii-of\ non-r)}) \cong \frac{\overline{Y}^2_{(st)}}{4}(4\mathcal{B}_{200}+\Theta^2\mathcal{B}_{020}-4\Theta\mathcal{B}_{110}), \tag{3.31}$$

(6) Kumar et al. [19] developed a generalized ratio-type exponential estimator, given by (3.32)

$$\begin{aligned}\widehat{\overline{\overline{Y}}}_{GK(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} &= \left[k_3\widehat{\overline{\overline{Y}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}+k_4\left(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)\right] \\ &\times\exp\left(\frac{a\left(\overline{X}_{(st)}-\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)}{a\left(\overline{X}_{(st)}+\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)+2b}\right).\end{aligned} \tag{3.32}$$

The properties of $\widehat{\overline{\overline{Y}}}_{GK(st)H(s-ii-of\ non-r)}$, respectively, are given by

$$\mathrm{B}(\widehat{\overline{\overline{Y}}}_{GK(st)H(s-ii-of\ non-r)}) \cong \overline{Y}_{(st)}-k_3\overline{Y}_{(st)}+\frac{3}{8}k_3\Theta^2\overline{Y}_{(st)}\mathcal{B}_{200}+\frac{1}{2}k_4\Theta\overline{X}_{(st)}\mathcal{B}_{020}-\frac{1}{2}k_3\Theta\overline{Y}_{(st)}\mathcal{B}_{110},$$

, and MSE is given by (3.33)

$$\begin{aligned}\mathrm{MSE}\left(\widehat{\overline{\overline{Y}}}_{GK(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right) &\cong \overline{Y}_{(st)}{}^2(k_3-1)^2+k_3^2\overline{Y}_{(st)}{}^2\mathcal{B}_{200}+k_4^2\overline{X}_{(st)}{}^2\mathcal{B}_{020}+\Theta^2k_3^2\overline{Y}_{(st)}{}^2\mathcal{B}_{020} \\ &+ 2w_3k_4\Theta\overline{Y}_{(st)}\overline{X}_{(st)}\mathcal{B}_{020}-\tfrac{3}{4}k_3\Theta^2\overline{Y}_{(st)}{}^2\mathcal{B}_{020}-k_4\Theta\overline{Y}_{(st)}\overline{X}_{(st)}\mathcal{B}_{020} \\ &+ k_3\Theta\overline{Y}_{(st)}{}^2\mathcal{B}_{110}-2k_3^2\Theta\overline{Y}_{(st)}{}^2\mathcal{B}_{110}.\end{aligned} \tag{3.33}$$

after some simplification we get equation (3.34)

$$-2k_3k_4\overline{Y}_{(st)}\overline{X}_{(st)}\mathcal{B}_{110}.(38)\mathrm{MSE}_{\min}\left(\widehat{\overline{\overline{Y}}}_{GK(st)H(s-ii-of\ non-r)}\right) \cong \frac{\overline{Y}^2_{(st)}}{64}\left(64-16\mathcal{B}_{020}-\frac{\mathcal{B}_{020}\,(-8+\mathcal{B}_{020})^2}{\mathcal{B}_{020}\,(1+\mathcal{B}_{200})-\mathcal{B}_{110}}\right). \tag{3.34}$$

(7) Haq and Shabbir [20] developed the estimator given in (3.35).

$$\begin{aligned}\widehat{\overline{\overline{Y}}}_{H.S(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} &= \left\{w_5\widehat{\overline{\overline{Y}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} + w_6\left(\overline{X}_{(st)} - \widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)\right\}\\ &\times \exp\left[\frac{\overline{X}_{(st)} + b}{\alpha\left(a\widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} + b\right) + (1-\alpha)\left(a\overline{X}_{(st)} + b\right)} - 1\right].\end{aligned} \tag{3.35}$$

Using (3.35), the bias and MSE of $\widehat{\overline{\overline{Y}}}_{H.S(st)H}$, are given by:

$$\mathrm{B}(\widehat{\overline{\overline{Y}}}_{H.S(st)H(s-ii-of\ non-r)}) = -\ \overline{Y}_{(st)} + \Theta k_6\overline{X}_{(st)}\mathcal{B}_{020} + k_5\left(\overline{Y}_{st} + \frac{3\overline{Y}_{(st)}\Theta^2\mathcal{B}_{020}}{2} - \Theta\overline{Y}_{(st)}\mathcal{B}_{110}\right)$$

and MSE in equation (3.36)

$$\begin{aligned}\mathrm{MSE}\left(\widehat{\overline{\overline{Y}}}_{H.S(st)H}\right) &= \overline{Y}_{(st)}^{\ 2} + k_6\overline{X}_{(st)}\left(-2\Theta\overline{Y}_{(st)} + k_6\overline{X}_{(st)}\right)\mathcal{B}_{020}\\ &+ k_5\overline{Y}_{(st)}\left[-2\overline{Y}_{(st)} + \Theta\left(-3\Theta\overline{Y}_{(st)} + 4k_6\overline{X}_{(st)}\right)\mathcal{B}_{020} + 2\left(\Theta\overline{Y}_{(st)} - k_6\overline{Y}_{(st)}\right)\mathcal{B}_{110}\right]\\ &+ \overline{Y}_{(st)}^{\ 2}k_6^2\left(1 + 4\Theta^2\mathcal{B}_{020} - 4\Theta\mathcal{B}_{110} + \mathcal{B}_{200}\right).\end{aligned} \tag{3.36}$$

The minimum MSE of $\widehat{\overline{\overline{Y}}}_{H.S(st)H(s-ii-of\ non-r)}$ at the values of k_5 and k_6 is given by (3.37)

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{\overline{Y}}}_{H.S(st)H(s-ii-of\ non-r)}\right) \cong \frac{\overline{Y}_{(st)}^{\ 2}\left[4\mathcal{B}_{110}^{\ 2} + \mathcal{B}_{020}\left\{\Theta^4\mathcal{B}_{020}^{\ 2} - \Theta^2\mathcal{B}_{110}^{\ 2} + 4\left(-1 + \Theta^2\mathcal{B}_{020}\right)\mathcal{B}_{020}\right\}\right]}{4\left\{\mathcal{B}_{110}^{\ 2} - \mathcal{B}_{020}\left(1 + \mathcal{B}_{200}\right)\right\}} \tag{3.37}$$

(8) Ahmad et al. [15] suggested a family of estimators given by (3.38)

$$\begin{aligned}\widehat{\overline{\overline{Y}}}^{*}_{A.H(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} =&\left\{k_7\widehat{\overline{\overline{Y}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} + k_8\left(\overline{X}_{(st)} - \widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)\right.\\ &\left. + w_9\left(\overline{R}x_{(st)} - \widehat{\overline{\overline{R}}}x_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)\right\}\\ &\times\exp\left(\frac{a\left(\overline{X}_{(st)} - \widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)}{a\left(\overline{X}_{(st)} + \widehat{\overline{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right) + 2b}\right)\end{aligned} \tag{3.38}$$

$$\mathrm{Bias}\left(\widehat{\overline{\overline{Y}}}^{*}_{A.H(st)H(s-ii-of\ non-r)}\right) \cong \overline{Y}_{(st)}\left(k_7 - 1\right) + \frac{3}{8}\Theta^2k_7\overline{Y}_{(st)}\mathcal{B}_{020} + \frac{1}{2}\Theta\overline{X}_{(st)}k_8\mathcal{B}_{020} - \frac{1}{2}\Theta k_7\overline{Y}_{(st)}\mathcal{B}_{110} + \frac{1}{2}\Theta\overline{R}x_{(st)}\mathcal{B}_{011},$$

and MSE is given in (3.39)

$$\mathrm{MSE}\left(\widehat{\overline{\overline{Y}}}^{*}_{A.H(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right) \cong \overline{Y}_{(st)}^{\ 2}(k_7 - 1)^2 + k_7^2\overline{Y}_{(st)}^{\ 2}\mathcal{B}_{200} + k_8\overline{X}_{(st)}^{\ 2}\mathcal{B}_{200} + k_9\overline{R}x_{(st)}^{\ 2}\mathcal{B}_{002}$$

$$
\begin{aligned}
&+ \quad \Theta^2 k_7^2 \overline{Y}_{(st)}^{\,2} \mathcal{B}_{200} - \Theta k_8 \overline{Y}_{(st)} \mathcal{B}_{200} + 2\Theta \overline{Y}_{(st)} \mathcal{B}_{200} - \tfrac{3}{4}\Theta^2 k_7 \overline{Y}_{(st)}^{\,2} \mathcal{B}_{200} \\
&+ \quad \Theta k_7 \overline{Y}_{(st)}^{\,2} \mathcal{B}_{110} - 2\Theta k_7^2 \overline{Y}_{(st)}^{\,2} \mathcal{B}_{110} - 2k_7 k_8 \overline{Y}_{(st)} \overline{X}_{(st)} \mathcal{B}_{110} \\
&- \quad 2k_7 k_8 \overline{Y}_{(st)} \overline{R}x_{(st)} \mathcal{B}_{101} - \Theta k_9 \overline{Y}_{(st)} \overline{R}x_{(st)} \mathcal{B}_{011} \\
&+ \quad 2\Theta k_7 k_9 \overline{Y}_{(st)} \overline{R}x_{(st)} \mathcal{B}_{011} - 2k_8 k_9 \overline{Y}_{(st)} \overline{R}x_{(st)} \mathcal{B}_{011}.
\end{aligned} \tag{3.39}
$$

The optimal values of k_7, k_8 and k_9, determined by reducing (3.39) , are given by:

$$
k_{7(\text{opt})} = \frac{\left(8 - \Theta^2 \mathcal{B}_{020}\right)\left(\mathcal{B}_{110}^2 - \mathcal{B}_{002}\mathcal{B}_{200}\right)}{8\left[-\mathcal{B}_{200}\mathcal{B}_{101}^2 + 2\mathcal{B}_{011}\mathcal{B}_{101}\mathcal{B}_{110} - \mathcal{B}_{101}^2\left(1 + \mathcal{B}_{200}\right) + \mathcal{B}_{002}\left\{-\mathcal{B}_{101}^2 + \mathcal{B}_{200}\left(1 + \mathcal{B}_{200}\right)\right\}\right]},
$$

$$
k_{8(\text{opt})} = \frac{\overline{Y}_{(st)} \left[\begin{array}{c} 4\Theta \mathcal{B}_{200}\mathcal{B}_{101}^2 - \mathcal{B}_{011}\mathcal{B}_{101}\left(-8 + \Theta^2 \mathcal{B}_{200} + 8\Theta \mathcal{B}_{110}\right) + \mathcal{B}_{002} \left\{ \begin{array}{c} -\Theta^3 \mathcal{B}_{020}^2 + 4\mathcal{B}_{110}\left(-2 + \Theta \mathcal{B}_{110}\right) \\ +\Theta \mathcal{B}_{020}\left(4 + \Theta \mathcal{B}_{110} - 4\mathcal{B}_{200}\right) \end{array} \right\} + \\ \Theta \mathcal{B}_{011}^2 \left(-4 + \Theta^2 \mathcal{B}_{020} + 4\mathcal{B}_{200}\right) \end{array} \right]}{8\overline{X}_{(st)}\left[-\mathcal{B}_{020}\mathcal{B}_{101}^2 + 2\mathcal{B}_{011}\mathcal{B}_{101}\mathcal{B}_{110} - \mathcal{B}_{101}^2\left(1 + \mathcal{B}_{200}\right) + \mathcal{B}_{002}\left\{-\mathcal{B}_{101}^2 + \mathcal{B}_{020}\left(1 + \mathcal{B}_{200}\right)\right\}\right]},
$$

$$
k_{9(\text{opt})} = \frac{\overline{Y}_{(st)}\left(8 - \Theta^2 \mathcal{B}_{020}\right)\left(\mathcal{B}_{020}\mathcal{B}_{101} - \mathcal{B}_{110}\mathcal{B}_{011}\right)}{8\overline{R}x_{(st)}\left[-\mathcal{B}_{020}\mathcal{B}_{101}^2 + 2\mathcal{B}_{011}\mathcal{B}_{101}\mathcal{B}_{110} - \mathcal{B}_{101}^2\left(1 + \mathcal{B}_{200}\right) + \mathcal{B}_{002}\left\{-\mathcal{B}_{101}^2 + \mathcal{B}_{020}\left(1 + \mathcal{B}_{200}\right)\right\}\right]}.
$$

Putting the value of $k_{7(\text{opt})}$, $k_{8(\text{opt})}$ and $k_{9(\text{opt})}$, we get the minimum MSE in (3.40)

$$
MSE\left(\widehat{\overline{Y}}^{*}_{A.H(st)H(s-i-of\ non-r)}\right) \cong \frac{\overline{Y}_{(st)}^{\,2}}{64}\left[64 - 16\Theta^2 \mathcal{B}_{011} + \frac{\left(\Theta^2 \mathcal{B}_{020-8}\right)^2\left(\mathcal{B}_{101}^2 - \mathcal{B}_{002}\mathcal{B}_{020}\right)}{\left[\begin{array}{c} -\mathcal{B}_{020}\mathcal{B}_{101}^2 + 2\mathcal{B}_{011}\mathcal{B}_{101}\mathcal{B}_{110} - \mathcal{B}_{011}^2\left(1 + \mathcal{B}_{200}^2\right) \\ +\mathcal{B}_{020}\left\{-\mathcal{B}_{110}^2 + \mathcal{B}_{020}\left(1 + \mathcal{B}_{200}\right)\right\} \end{array}\right]}\right] \tag{3.40}
$$

4. Proposed estimator

This article presents a new improved class of estimators for estimating the population mean in stratified random sampling in the presence of non-response. The suggested estimators are formulated using a suitable non-response adjustment class and efficiently utilize auxiliary variable information to improve the efficiency and precision of the estimates. In particular, two types of non-response are taken into account in the study. In the former situation, non-response only occurs for the study variable, but we have complete information regarding the auxiliary variable for all sampled units. This can be used to capture information from the auxiliary variable to minimize errors in estimation, even when some data is lost for the primary variable. In the latter case, a more complex problem arises where non-response occurs not only on the study variable but also on the auxiliary variable, requiring us to address a more complicated estimation issue. In both cases, the proposed estimators have been derived by extending or generalizing the methods introduced in Hussain et al. [14] and Sohaib et al. [15], providing an alternative principled and practical solution to the problem of estimating means when non-response occurs in stratified sampling.

4.1. Proposed estimator under situation-I of non-response.

$$\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}i\text{-}of\ non\text{-}r)} = \left\{\frac{w_{10}\widehat{\widehat{\overline{Y}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)} + w_{11}\left(\overline{X}_{(st)} - \widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right) + w_{12}\left(\overline{R}_{x(st)} - \widehat{\widehat{\overline{R}}}_{x(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right)}{4}\right\}$$
$$\times\left(\frac{\overline{X}_{(st)}}{\widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}} + \frac{\widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}}{\overline{X}_{(st)}}\right)\left(\frac{\overline{R}x_{(st)}}{\widehat{\widehat{\overline{R}}}x_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}} + \frac{\widehat{\widehat{\overline{R}}}x_{(st)H(s\text{-}i\text{-}of\ non\text{-}r)}}{\overline{R}_{x(st)}}\right). \tag{4.1}$$

Simplifying equation (4.1), we get equation (4.2)

$$\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}i\text{-}of\ non\text{-}r)} - \overline{Y}_{st} = \frac{w_{10}\overline{Y}_{(st)}(\varepsilon_{0(st)} - 1) - w_{11}\overline{X}_{(st)}\varepsilon_{1(st)} - w_{12}\overline{R}_{x(st)}\varepsilon_{2(st)}}{4}$$
$$\times\left(\frac{1}{\varepsilon_{1(st)}+1} + \varepsilon_{1(st)} + 1\right)\left(\frac{1}{\varepsilon_{2(st)}+1} + \varepsilon_{2(st)} + 1\right). \tag{4.2}$$

Expanding the right-hand side of (4.2), we get (4.3)

$$\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}i\text{-}of\ non\text{-}r)} = \frac{w_{10}\overline{Y}_{(st)}(\varepsilon_{0(st)} - 1) - w_{11}\overline{X}_{(st)}\varepsilon_{1(st)} - w_{12}\overline{R}_{x(st)}\varepsilon_{2(st)}}{4}$$
$$\times\left[\varepsilon_{1(st)} + 1 + \left(1 - \varepsilon_{1(st)} + \varepsilon^2_{1(st)} + \ldots\right)\right]$$
$$\times\left[\varepsilon_{2(st)} + 1 + \left(1 - \varepsilon_{2(st)} + \varepsilon^2_{2(st)} + \ldots\right)\right]. \tag{4.3}$$

In Eq. (4.3) we will neglect the terms of $\varepsilon_{(st)}$'s, power having greater than two, we get (4.4)

$$\widehat{\widehat{\overline{Y}}}_{Prop(st)(s\text{-}i\text{-}of\ non\text{-}r)} - \overline{Y}_{st} = \overline{Y}_{(st)}(w_{10} - 1) + w_{10}\overline{Y}_{(st)}\varepsilon_{0(st)} - w_{11}\overline{X}_{(st)}\varepsilon_{1(st)} - w_{12}\overline{R}_{x(st)}\varepsilon_{2(st)}$$
$$+\frac{1}{2}w_{10}\overline{Y}_{(st)}\varepsilon^2_{1(st)} + \frac{1}{2}w_{10}\overline{Y}_{(st)}\varepsilon^2_{2(st)}. \tag{4.4}$$

To get the bias of the proposed estimator, we need to take expectation on both the sides of (4.4), hence we will get the bias of the proposed estimator up to the first order of approximation

$$E\left(\widehat{\widehat{\overline{Y}}}_{Prop(st)(s\text{-}i\text{-}of\ non\text{-}r)} - \overline{Y}_{st}\right) = \overline{Y}_{(st)}(w_{10} - 1) + w_{10}\overline{Y}_{(st)}E(\varepsilon_{0(st)}) - w_{11}\overline{X}_{(st)}E(\varepsilon_{1(st)}) - w_{12}\overline{R}_{x(st)}E(\varepsilon_{2(st)})$$
$$+\tfrac{1}{2}w_{10}\overline{Y}_{(st)}E(\varepsilon^2_{1(st)}) + \tfrac{1}{2}w_{10}\overline{Y}_{(st)}E(\varepsilon^2_{2(st)}). \tag{4.5}$$

Using (4.5), the bias and MSE of $\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s-i-of\ non-r)}$, are given by:

$$\mathrm{B}(\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s-i-of\ non-r)}) = \overline{Y}_{st}[(w_{10} - 1) + \frac{1}{2}w_{10}\mathcal{A}_{020} + \frac{1}{2}w_{10}\mathcal{A}_{002}]$$

, and MSE is given by (4.6)

$$\mathrm{MSE}\left(\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}i\text{-}of\ non\text{-}r)}\right) \cong \overline{Y}_{(st)}^{\,2}\Big[(w_{10} - 1)^2 + w_{10}(\mathcal{A}_{020} + \mathcal{A}_{002}) + w_{10}(\mathcal{A}_{020} + \mathcal{A}_{002} + \mathcal{A}_{200})$$
$$- 2w_{10}\overline{Y}_{(st)}\overline{X}_{(st)}\overline{R}x_{(st)}\{w_{11}\mathcal{A}_{110} + w_{12}\mathcal{A}_{101}\} + 2\overline{X}_{(st)}\overline{R}x_{(st)}w_{11}w_{12}\mathcal{A}_{011}$$
$$+ \overline{X}_{(st)}^{\,2}w^2_{11}\mathcal{A}_{020} + w^2_{12}\overline{R}x_{(st)}^{\,2}\mathcal{A}_{002}\Big]. \tag{4.6}$$

The values of $w_{mathrm10}$, w_{11} and w_{12} by diminishing (4.6):

$$w_{10(\mathrm{opt})} = \frac{\beta[2 + \mathcal{A}_{020} + \mathcal{A}_{002}]}{2[\beta + \mathcal{A}_{200}\alpha + \beta(\mathcal{A}_{020} + \mathcal{A}_{002})]},$$

$$w_{11(\text{opt})}=\frac{\sqrt{\mathcal{A}_{200}}\ \overline{Y}_{(st)}\left(\frac{\mathcal{A}_{110}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{020}}}-\frac{\mathcal{A}_{011}}{\sqrt{\mathcal{A}_{020}}\ \sqrt{\mathcal{A}_{002}}}\frac{\mathcal{A}_{101}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{002}}}\right)[2+\mathcal{A}_{020}+\mathcal{A}_{002}]}{2\overline{X}_{st}\sqrt{\mathcal{A}_{020}}\,(\beta+\mathcal{A}_{200}\alpha+\beta[\mathcal{A}_{020}+\mathcal{A}_{002}])},$$

$$w_{12(\text{opt})}=\frac{\sqrt{\mathcal{A}_{200}}\ \overline{Y}_{st}\left(\frac{\mathcal{A}_{101}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{002}}}-\frac{\mathcal{A}_{011}}{\sqrt{\mathcal{A}_{020}}\ \sqrt{\mathcal{A}_{002}}}\frac{\mathcal{A}_{110}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{020}}}\right)[2+\mathcal{A}_{020}+\mathcal{A}_{002}]}{2\overline{R}x_{(st)}\sqrt{\mathcal{A}_{020}}\,(\beta+\mathcal{A}_{200}\alpha+\beta[\mathcal{A}_{020}+\mathcal{A}_{002}])},$$

the minimum MSE is given by (4.7)

$$\text{MSE}_{\min}\left(\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s-i-of\ non-r)}\right)\cong\frac{\overline{Y}_{(st)}^{\,2}\left[4\alpha\mathcal{A}_{200}-\beta(\mathcal{A}_{020}+\mathcal{A}_{002})^2\right]}{4[\beta+\mathcal{A}_{200}\alpha+\beta(\mathcal{A}_{020}+\mathcal{A}_{002})]}. \tag{4.7}$$

$\alpha=1-\frac{\mathcal{A}_{011}{}^2}{\mathcal{A}_{020}\mathcal{A}_{002}}-\frac{\mathcal{A}_{101}{}^2}{\mathcal{A}_{200}\mathcal{A}_{002}}-\frac{\mathcal{A}_{110}{}^2}{\mathcal{A}_{200}\mathcal{A}_{020}}+2\frac{\mathcal{A}_{101}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{002}}}\times\frac{\mathcal{A}_{011}}{\sqrt{\mathcal{A}_{020}}\ \sqrt{\mathcal{A}_{002}}}\times\frac{\mathcal{A}_{110}}{\sqrt{\mathcal{A}_{200}}\ \sqrt{\mathcal{A}_{020}}}$ and $\beta=1-\frac{\mathcal{A}_{011}{}^2}{\mathcal{A}_{020}\mathcal{A}_{002}}$

4.2. Proposed estimator under situation-II of non-response.

$$\begin{aligned}\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} &= \left\{\frac{k_{10}\widehat{\widehat{\overline{Y}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}+k_{11}\left(\overline{X}_{(st)}-\widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)+k_{12}\left(\overline{R}_{x(st)}-\widehat{\widehat{\overline{R}}}_{x(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}\right)}{4}\right\}\\ &\times\left(\frac{\overline{X}_{(st)}}{\widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}}+\frac{\widehat{\widehat{\overline{X}}}_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}}{\overline{X}_{(st)}}\right)\left(\frac{\overline{R}x_{(st)}}{\widehat{\widehat{\overline{R}}}x_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}}+\frac{\widehat{\widehat{\overline{R}}}x_{(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}}{\overline{R}x_{(st)}}\right).\end{aligned} \tag{4.8}$$

Simplifying equation (4.8) we get(4.9)

$$\begin{aligned}\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}-\overline{Y}_{st} &= \frac{k_{10}\overline{Y}_{(st)}(\varepsilon_{0(st)}-1)-k_{11}\overline{X}_{(st)}\varepsilon_{1(st)}-k_{12}\overline{R}_{x(st)}\varepsilon_{2(st)}}{4}\\ &\times\left(\frac{1}{\varepsilon_{1(st)}+1}+\varepsilon_{1(st)}+1\right)\left(\frac{1}{\varepsilon_{2(st)}+1}+\varepsilon_{2(st)}+1\right).\end{aligned} \tag{4.9}$$

Expanding the right-hand side of (4.9), we get (4.10)

$$\begin{aligned}\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}ii\text{-}of\ non\text{-}r)} &= \frac{k_{10}\overline{Y}_{(st)}(\varepsilon_{0(st)}-1)-k_{11}\overline{X}_{(st)}\varepsilon_{1(st)}-k_{12}\overline{R}_{x(st)}\varepsilon_{2(st)}}{4}\\ &\times\left[\varepsilon_{1(st)H}+1+(1-\varepsilon_{1(st)H}+\varepsilon_{1(st)H}^2+\ldots)\right]\\ &\times\left[\varepsilon_{2(st)H}+1+(1-\varepsilon_{2(st)H}+\varepsilon_{2(st)H}^2+\ldots)\right].\end{aligned} \tag{4.10}$$

In Eq. (4.10) we will neglect the terms of $\varepsilon_{(st)}$'s, power having greater than two, we get (4.11)

$$\begin{aligned}\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}-\overline{Y}_{st} &= \overline{Y}_{(st)}(k_{10}-1)\\ &+k_{10}\overline{Y}_{(st)}\varepsilon_{0(st)H}-k_{11}\overline{X}_{(st)}\varepsilon_{1(st)H}-k_{12}\overline{R}_{x(st)}\varepsilon_{2(st)H}\\ &+\tfrac{1}{2}k_{10}\overline{Y}_{(st)}\varepsilon_{1(st)H}^2+\tfrac{1}{2}k_{10}\overline{Y}_{(st)}\varepsilon_{2(st)H}^2.\end{aligned} \tag{4.11}$$

To get the bias of the proposed estimator, we need to take expectation on both the sides of (4.11), hence we will get the bias of the proposed estimator up to the first order of approximation

$$E\left(\widehat{\widehat{\overline{Y}}}_{Prop(st)H(s\text{-}ii\text{-}of\ non\text{-}r)}-\overline{Y}_{st}\right) = \overline{Y}_{(st)}(k_{10}-1)+k_{10}\overline{Y}_{(st)}E(\varepsilon_{0(st)})-k_{11}\overline{X}_{(st)}E(\varepsilon_{1(st)})-k_{12}\overline{R}_{x(st)}E(\varepsilon_{2(st)})+\tfrac{1}{2}k_{10}\overline{Y}_{(st)}E\left(\varepsilon_{1(st)}^2\right)+\tfrac{1}{2}k_{10}\overline{Y}_{(st)}E$$

Using (4.12), the bias and MSE of $\widehat{\overline{Y}}_{Prop(st)H}$, are given by:

$$B(\widehat{\overline{Y}}_{Prop(st)H(s-ii-of\ non-r)}) = \overline{Y}_{st}[(k_{10}-1)+\frac{1}{2}k_{10}\mathcal{B}_{020}+\frac{1}{2}k_{10}\mathcal{B}_{002}]$$

, and MSE is given by (4.13)

$$\begin{aligned} MSE\left(\widehat{\overline{Y}}_{Prop(st)H(s-ii-of\ non-r)}\right) &= \overline{Y}_{st}^{2}\Big[(k_{10}-1)^2+k_{10}(\mathcal{B}_{020}+\mathcal{B}_{002})+k_{10}(\mathcal{B}_{020}+\mathcal{B}_{002}+\mathcal{B}_{200}) \\ &\quad -2k_{10}\overline{Y}_{st}\overline{X}_{st}\overline{R}x_{(st)}\{k_{11}\mathcal{B}_{110}+k_{12}\mathcal{B}_{101}\}+2\overline{X}_{st}\overline{R}x_{(st)}k_{11}k_{12}\mathcal{B}_{011} \\ &\quad +\overline{X}_{st}^{2}k_{11}^{2}\mathcal{B}_{020}+k_{12}^{2}\overline{R}x_{(st)}^{2}\mathcal{B}_{002}\Big]. \end{aligned} \tag{4.13}$$

The values of $k_{10},\ k_{11}$ and k_{12} by diminishing (4.13): Constants $k_{10},\ k_{11}$ and k_{12} are obtained by contracting expression (4.13). That is, with a successive decrease of the terms in Eq. (4.13) the explicit $k_{10},\ k_{11}$ and k_{12} are establish.

$$k_{10(\text{opt})}=\frac{\beta[2+\mathcal{B}_{020}+\mathcal{B}_{002}]}{2[\beta+\mathcal{B}_{200}\alpha+\beta(\mathcal{B}_{020}+\mathcal{B}_{002})]},$$

$$k_{11(\text{opt})}=\frac{\sqrt{\mathcal{B}_{200}}\ \overline{Y}_{(st)}\left(\frac{\mathcal{B}_{110}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{020}}}-\frac{\mathcal{B}_{011}}{\sqrt{\mathcal{B}_{020}}\ \sqrt{\mathcal{B}_{002}}}\frac{\mathcal{B}_{101}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{002}}}\right)[2+\mathcal{B}_{020}+\mathcal{B}_{002}]}{2\overline{X}_{st}\sqrt{\mathcal{B}_{020}}\,(\beta+\mathcal{B}_{200}\alpha+\beta[\mathcal{B}_{020}+\mathcal{B}_{002}])},$$

$$k_{12(\text{opt})}=\frac{\sqrt{\mathcal{B}_{200}}\ \overline{Y}_{(st)}\left(\frac{\mathcal{B}_{101}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{002}}}-\frac{\mathcal{B}_{011}}{\sqrt{\mathcal{B}_{020}}\ \sqrt{\mathcal{B}_{002}}}\frac{\mathcal{B}_{110}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{020}}}\right)[2+\mathcal{B}_{020}+\mathcal{B}_{002}]}{2\overline{R}x_{(st)}\sqrt{\mathcal{B}_{020}}\,(\beta+\mathcal{B}_{200}\alpha+\beta[\mathcal{B}_{020}+\mathcal{B}_{002}])},$$

The minimum MSE of $\widehat{\overline{Y}}_{Prop(st)H(s-ii-of\ non-r)}$ at the values of $w_5,\ w_6$ and w_7 is given by (4.14)

$$MSE_{\min}\left(\widehat{\overline{Y}}_{Prop(st)H(s-ii-of\ non-r)}\right)\cong\frac{\overline{Y}_{(st)}^{2}\left[4\alpha\mathcal{B}_{200}-\beta(\mathcal{B}_{020}+\mathcal{B}_{002})^2\right]}{4[\beta+\mathcal{B}_{200}\alpha+\beta(\mathcal{B}_{020}+\mathcal{B}_{002})]} \tag{4.14}$$

$\alpha=1-\frac{\mathcal{B}_{011}^{2}}{\mathcal{B}_{020}\mathcal{B}_{002}}-\frac{\mathcal{B}_{101}^{2}}{\mathcal{B}_{200}\mathcal{B}_{002}}-\frac{\mathcal{B}_{110}^{2}}{\mathcal{B}_{200}\mathcal{B}_{020}}+2\frac{\mathcal{B}_{101}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{002}}}\times\frac{\mathcal{B}_{011}}{\sqrt{\mathcal{B}_{020}}\ \sqrt{\mathcal{B}_{002}}}\times\frac{\mathcal{B}_{110}}{\sqrt{\mathcal{B}_{200}}\ \sqrt{\mathcal{B}_{020}}}$ and $\beta=1-\frac{\mathcal{B}_{011}^{2}}{\mathcal{B}_{020}\mathcal{B}_{002}}$

5. Numerical Study

One real-life data set has been used in this section to compare the performance of the proposed estimators with their existing counterparts. The idea behind using real data is to obtain a more realistic estimation of the effectiveness and consistency of the proposed approaches outside the theoretical context. The findings of such a set are reported in Tables 2 through 9, in which the (MSE) and the (PRE) of the population means are reported. The values of MSE are used to evaluate the accuracy of the estimators. The smaller the values, the better the performance of the estimators. In contrast, the values of PRE provide a relative measure of the efficiency of the suggested estimators compared to conventional estimators. These combined measures can be used to comprehensively evaluate the benefits presented by the enhanced class of estimators in non-response conditions.

Population-I: [Source: Koyuncu and Kadilar [16]]

We used an education dataset from Turkey in 2007, which includes records of 923 districts and is published as an open data resource. For each of the six major regions (Marmara, Aegean, Mediterranean, Central Anatolia, Black Sea and East & South-east Anatolia), the districts were grouped to address regional heterogeneity. This stratification simulates both geographical and socio-economic variation in the data, making it suitable for checking the performance of the proposed estimators under a stratified random sampling design.

The study outcome (Y) was the number of educators, a key determinant of educational resources within each district. Two additional variables were used as covariates, with the first auxiliary variable (X). To be more specific, representing the number of pupils would show a relatively high correlation with the study variable, as it is related to teacher demand. Secondly, the second auxiliary variable (Z) represented the total number of courses in primary and secondary schools at each district level, involving an extra layer of educational structure and capacity information. Together, these secondary variables account for graduations and measure the size (number of students enrolled) and structure of the educational system (courses offered) in terms of their effect on precision when used in conjunction with their approach.

Descriptive statistics on the dataset showed that a district had an average of 340 teachers, 2,650 pupils, and 120 courses across all school levels. There was regional variation: for example, the Marmara region had the highest average (around 520 teachers and 3,800 pupils per district), followed by eastern and southern Anatolia, which had around 210 teachers and 1,700 pupils per district. The number of courses per district varied from about 85 in less developed districts to more than 160 in urbanised districts. This stratification heterogeneity emphasises the relevance of the estimation problem in stratified sampling and why generalising dual auxiliaries applicable to such heterogeneous populations is methodologically essential.

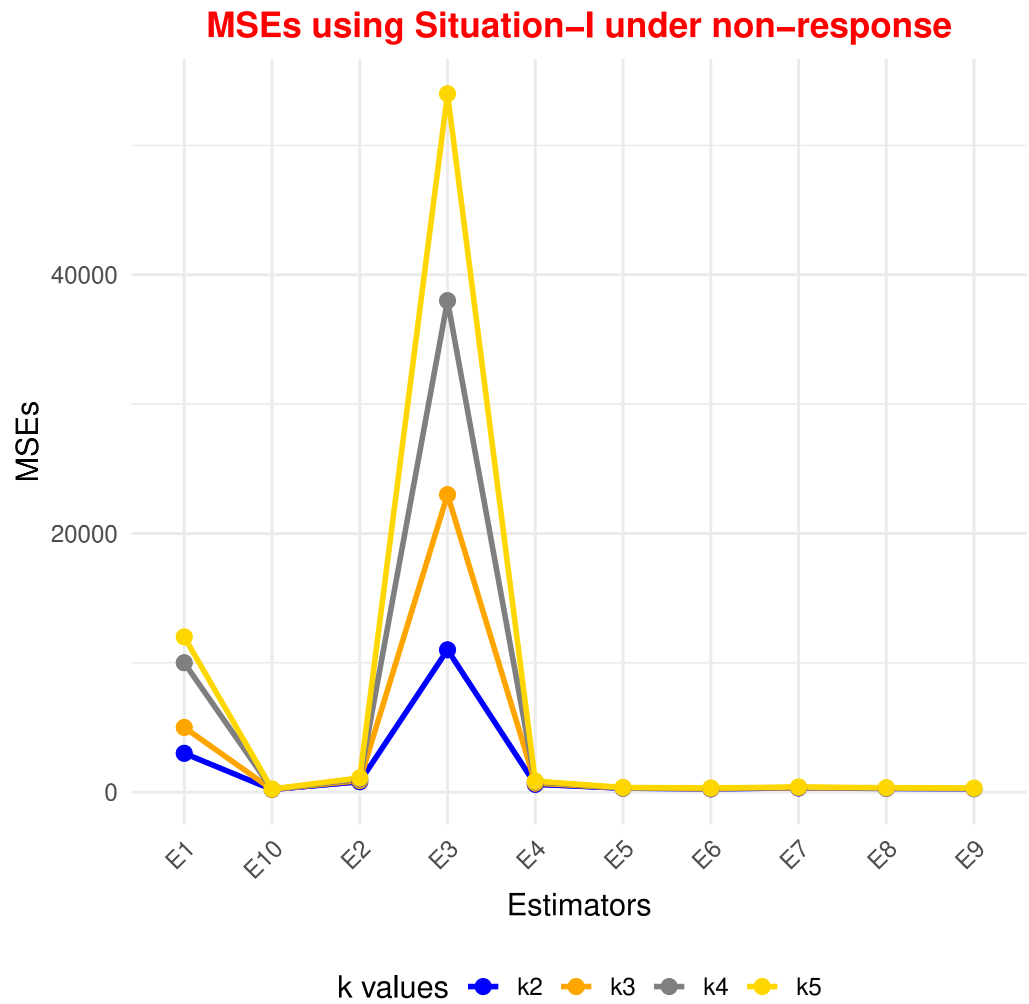

Figure 1. Showing MSEs using Situation-I under non-response using real data set.

From Figures 1, 2, 3, and 4, we conclude that the plot of MSEs and PREs shows that the proposed estimator is efficient then existing estimators, using real-life date set.

Table 1. Population and Sample Characteristics under Stratified Sampling with Non-Response

Parameter	**Value**
N	923
N_{non}	231
$N_1, N_2, N_3, N_4, N_5, N_6$	127, 117, 103, 170, 205, 201
n	180
n_{non}	45
$n_1, n_2, n_3, n_4, n_5, n_6$	31, 21, 29, 38, 22, 39
$N_{1(2)}, N_{2(2)}, N_{3(2)}, N_{4(2)}, N_{5(2)}, N_{6(2)}$	32, 30, 26, 43, 52, 51
$n_{1(2)}, n_{2(2)}, n_{3(2)}, n_{4(2)}, n_{5(2)}, n_{6(2)}$	8, 5, 7, 10, 6, 10
λ	0.004472132
λ_{11}-λ_{16}	0.0244, 0.0390, 0.0248, 0.0204, 0.0406, 0.0207
When $k = 2$	$\lambda_{non} = 0.001388889$
λ_{21}-λ_{26}	0.0081, 0.0119, 0.0086, 0.0066, 0.0227, 0.0114
When $k = 3$	$\lambda_{non} = 0.002777778$
λ_{21}-λ_{26}	0.0161, 0.0238, 0.0172, 0.0132, 0.0227, 0.0128
When $k = 4$	$\lambda_{non} = 0.004166667$
λ_{21}-λ_{26}	0.0242, 0.0357, 0.0259, 0.0197, 0.0341, 0.0192
When $k = 5$	$\lambda_{non} = 0.005555556$
λ_{21}-λ_{26}	0.0323, 0.0476, 0.0345, 0.0263, 0.0455, 0.0256
$\bar{Y}$, $\bar{Y}_{non}$	436.4345, 528.5801
$\bar{Y}_{11}$-$\bar{Y}_{16}$	703.74, 413.00, 573.17, 424.66, 267.03, 393.84
$\bar{Y}_{21}$–$\bar{Y}_{26}$	501.22, 373.72, 403.46, 256.70, 311.43, 428.96
$\bar{X}$, $\bar{X}_{non}$	11440.5, 14557.35
$\bar{X}_{11}$-$\bar{X}_{16}$	20804.59, 9211.80, 14309.30, 9478.85, 5569.95, 12997.59
$\bar{X}_{21}$-$\bar{X}_{26}$	14314.31, 8296.86, 8948.50, 5312.70, 6175.41, 16897.64
$\bar{R}_x$, $\bar{R}_{x,non}$	462, 116
$\bar{R}_{x,11}$-$\bar{R}_{x,16}$	64, 59, 52, 86, 103, 101
$\sigma^2_{y_{11}}$-$\sigma^2_{y_{16}}$	781163.9, 415924.8, 1068054, 657047.8, 162936.9, 506549
$\sigma^2_{x_{11}}$-$\sigma^2_{x_{16}}$	9294420, 2304560, 7589858, 33192950, 7221220, 5333395
$\sigma^2_{y_{21}}$-$\sigma^2_{y_{26}}$	418457, 160450.1, 151741.3, 167191, 165085.8, 431511.1
$\sigma^2_{x_{21}}$-$\sigma^2_{x_{26}}$	440651, 880443, 772197, 698836, 632839, 644660
$C_{y_{11}}$-$C_{y_{16}}$	1.2559, 1.5616, 1.8031, 1.9088, 1.5116, 1.8071
$C_{x_{11}}$-$C_{x_{16}}$	1.4654, 1.6480, 1.9253, 1.9221, 1.5256, 1.7768
$C_{y_{21}}$-$C_{y_{26}}$	1.2906, 1.0718, 0.9655, 1.5929, 1.3046, 1.5314
$C_{x_{21}}$-$C_{x_{26}}$	1.4665, 1.1309, 0.9820, 1.5735, 1.2882, 1.5026
$\varrho_{y_{11},x_{11}}$-$\varrho_{y_{16},x_{16}}$	0.9366, 0.9957, 0.9938, 0.9835, 0.9893, 0.9652
$\varrho_{y_{11},rx_{11}}$–$\varrho_{y_{16},rx_{16}}$	0.8239, 0.6584, 0.6337, 0.6360, 0.6595, 0.5863
$\varrho_{x_{11},rx_{11}}$–$\varrho_{x_{16},rx_{16}}$	0.7834, 0.6517, 0.6237, 0.6442, 0.6655, 0.6162
$\varrho_{y_{21},x_{21}}$–$\varrho_{y_{26},x_{26}}$	0.9623, 0.9927, 0.9911, 0.9972, 0.9926, 0.9867
$S_{y_{11},x_{11}}$–$S_{y_{16},x_{16}}$	25237154, 9747943, 28294397, 14523886, 3393592, 15864574
$S_{y_{21},x_{21}}$–$S_{y_{26},x_{26}}$	13066923, 3731279, 3392565, 3408435, 3208166, 16456424
$\beta_{x_{11}}$-$\beta_{x_{16}}$	7.367, 20.709, 17.648, 12.830, 24.386, 25.513
$\beta_{x_{21}}$-$\beta_{x_{26}}$	7.371, 6.744, 5.416, 10.632, 8.869, 15.888

Table 2. Population–I, Situation–I, when $k = 2, 3$

$k = 2$			$k = 3$		
Estimators	**MSE**	**PRE**	**Estimators**	**MSE**	**PRE**
$\widehat{\overline{\overline{Y}}}_{SRS}$	2601.169	100	$\widehat{\overline{\overline{Y}}}_{SRS}$	2973.071	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	270.9321	960.0813	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	325.4459	913.5378
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	10882.59	23.9021	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	12559.89	23.67116
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	235.9403	1102.469	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	276.7634	1074.229
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	235.6484	1103.835	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	276.3618	1075.789
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	692.2564	375.7522	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	781.9783	380.1987
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	234.0031	1111.596	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	274.1178	1084.596
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	223.1232	1165.799	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	260.138	1142.882
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	220.3652	1180.390	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	255.5571	1163.368
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	212.3298	1225.061	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	240.0983	1238.273

Table 3. Population–I, Situation–I, when $k = 4, 5$

$k = 4$			$k = 5$		
Estimators	**MSE**	**PRE**	**Estimators**	**MSE**	**PRE**
$\widehat{\overline{\overline{Y}}}_{SRS}$	3344.973	100	$\widehat{\overline{\overline{Y}}}_{SRS}$	3716.876	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	379.9596	880.3497	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	434.4734	855.4898
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	14237.18	23.4946	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	15914.47	23.3553
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	317.0643	1054.983	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	357.0166	1041.093
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	316.5374	1056.739	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	356.3487	1043.045
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	871.7003	383.7298	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	961.4222	386.6018
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	313.6048	1066.621	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	352.6379	1054.021
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	296.5864	1127.824	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	332.6087	1117.492
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	289.3737	1155.935	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	321.9891	1154.348
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	267.2104	1251.812	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	293.6669	1265.678

Table 4. Population –I, Situation –II, when k=2, 3

k=2			k=3		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{Y}}_{SRS}$	2601.169	100	$\widehat{\overline{Y}}_{SRS}$	2973.071	100
$\widehat{\overline{Y}}_{R(st)H}$	588.3207	442.1345	$\widehat{\overline{Y}}_{R(st)H}$	960.2231	309.6229
$\widehat{\overline{Y}}_{P(st)H}$	9577.201	27.1600	$\widehat{\overline{Y}}_{P(st)H}$	9949.103	29.8828
$\widehat{\overline{Y}}_{reg(st)H}$	566.1856	459.4198	$\widehat{\overline{Y}}_{reg(st)H}$	938.0880	316.9288
$\widehat{\overline{Y}}_{R,D(st)H}$	564.5076	460.7854	$\widehat{\overline{Y}}_{R,D(st)H}$	933.4906	318.4897
$\widehat{\overline{Y}}_{S(st)H}$	974.4368	266.9407	$\widehat{\overline{Y}}_{S(st)H}$	1346.339	220.8263
$\widehat{\overline{Y}}_{G,K(st)H}$	562.1658	462.7049	$\widehat{\overline{Y}}_{G,K(st)H}$	929.9481	319.7029
$\widehat{\overline{Y}}_{A,H(st)H}$	554.5192	469.0854	$\widehat{\overline{Y}}_{A,H(st)H}$	922.3313	322.3431
$\widehat{\overline{Y}}_{H,S(st)H}$	549.0983	473.7164	$\widehat{\overline{Y}}_{H,S(st)H}$	913.2905	325.5340
$\widehat{\overline{Y}}_{Prop(st)H}$	215.9077	1204.759	$\widehat{\overline{Y}}_{Prop(st)H}$	247.9003	1199.301

Table 5. Population –I, Situation –II, when k=4, 5

k=4			k=5		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{Y}}_{SRS}$	3344.973	100	$\widehat{\overline{Y}}_{SRS}$	3716.876	100
$\widehat{\overline{Y}}_{R(st)H}$	1332.126	251.1005	$\widehat{\overline{Y}}_{R(st)H}$	1704.028	218.1229
$\widehat{\overline{Y}}_{P(st)H}$	10321.01	32.4094	$\widehat{\overline{Y}}_{P(st)H}$	10692.91	34.7602
$\widehat{\overline{Y}}_{reg(st)H}$	1309.990	255.3434	$\widehat{\overline{Y}}_{reg(st)H}$	1681.893	220.9936
$\widehat{\overline{Y}}_{R,D(st)H}$	1301.043	257.0995	$\widehat{\overline{Y}}_{R,D(st)H}$	1667.172	222.9450
$\widehat{\overline{Y}}_{S(st)H}$	1718.242	194.6742	$\widehat{\overline{Y}}_{S(st)H}$	2090.144	177.8287
$\widehat{\overline{Y}}_{G,K(st)H}$	1296.304	258.0393	$\widehat{\overline{Y}}_{G,K(st)H}$	1661.242	223.7408
$\widehat{\overline{Y}}_{A,H(st)H}$	1288.717	259.5585	$\widehat{\overline{Y}}_{A,H(st)H}$	1653.684	224.7634
$\widehat{\overline{Y}}_{H,S(st)H}$	1276.070	262.1308	$\widehat{\overline{Y}}_{H,S(st)H}$	1637.446	226.9923
$\widehat{\overline{Y}}_{Prop(st)H}$	279.8821	1195.136	$\widehat{\overline{Y}}_{Prop(st)H}$	311.8532	1191.867

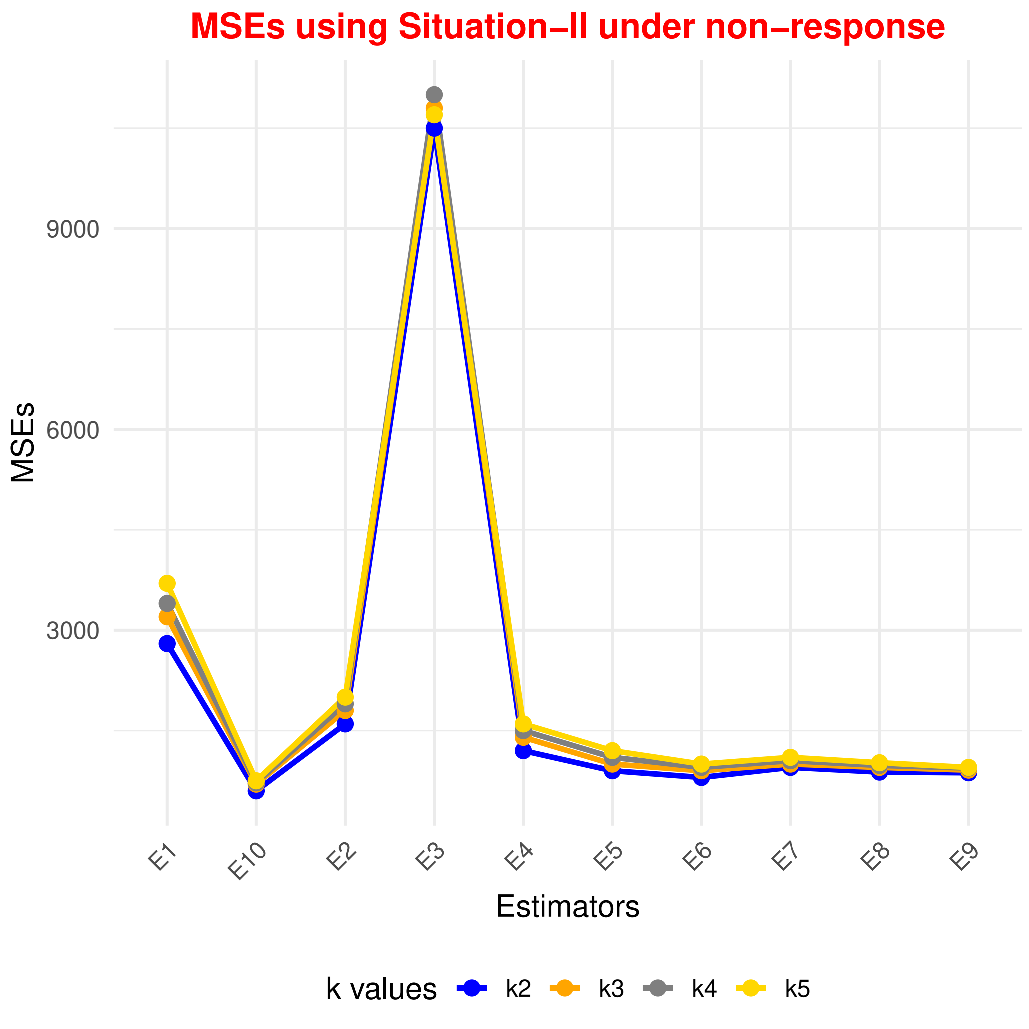

Figure 2. Showing MSEs using Situation-II under non-response using real data set.

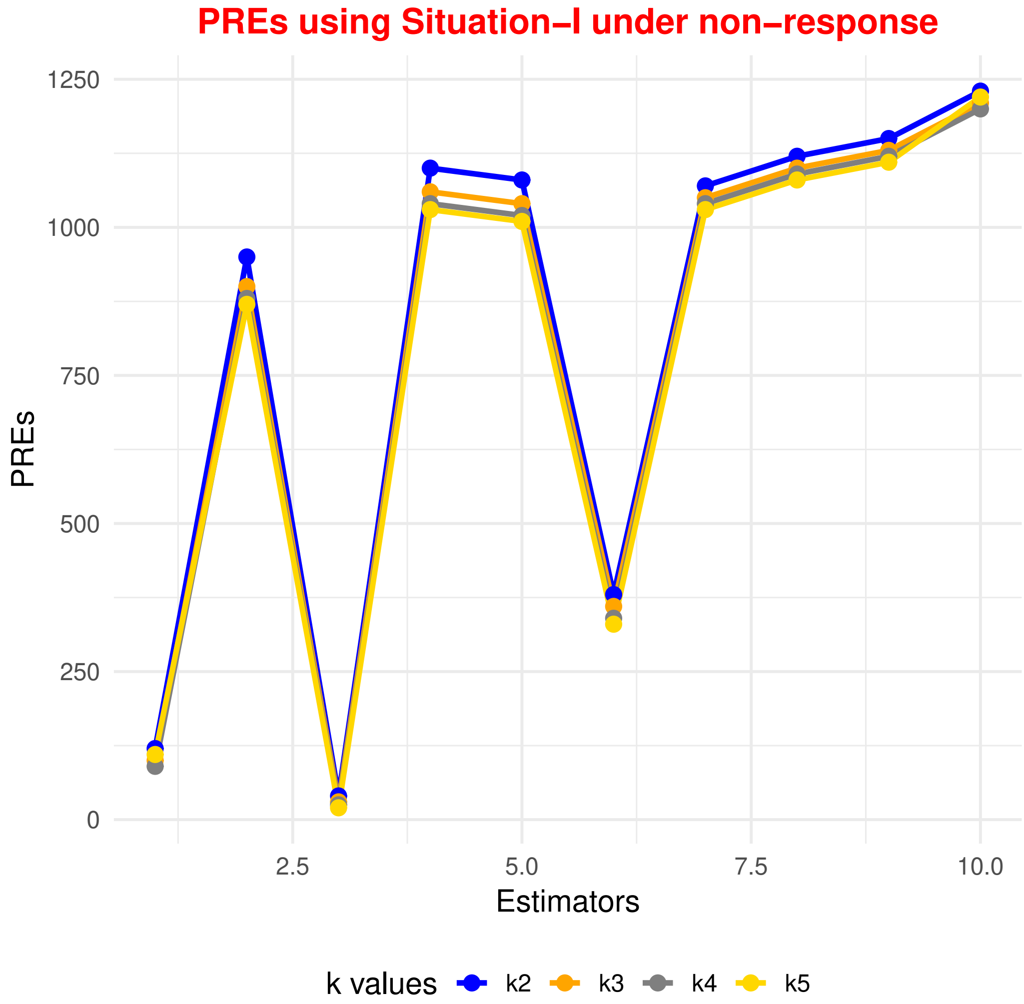

Figure 3. Showing PREs using Situation-I under non-response using real data set.

6. Simulation study

The purpose of the simulation study is to evaluate how different non-response (NR) rates affect stratified sample estimators over a 10,0000-iteration period. There are six different groupings, or strata, in the study. The simulation chooses a population size of 1000–2000 people for each stratum at random in each iteration. Each group's initial sample numbers range from 200 to 400 people. In the simulation model, the sample size with and without non-response (NR) could vary between 500 and 1000 with a fixed NR rate at 25%, based on realistic survey scenarios. Correlations between variables were created for each stratum, and artificial means and standard deviations were generated for the three variables studied: the variable under study (Y), the auxiliary variable (X) and the rank of X. In particular, the mean parameters for Y were drawn from a normal distribution between 400 and 800, X from a normal distribution of range (5000; 10000), and the rank of X was sampled from a normal distribution within the interval [300; 600]. Finally, overall population-level variables were calculated as weighted sums using the sizes of the strata as weights. These weighted values described the aggregated means and covariances that accurately represented the stratified population. Such a well-defined setup guarantees that the simulation study can thoroughly examine how non-response affects the efficiency and consistency of our estimators under

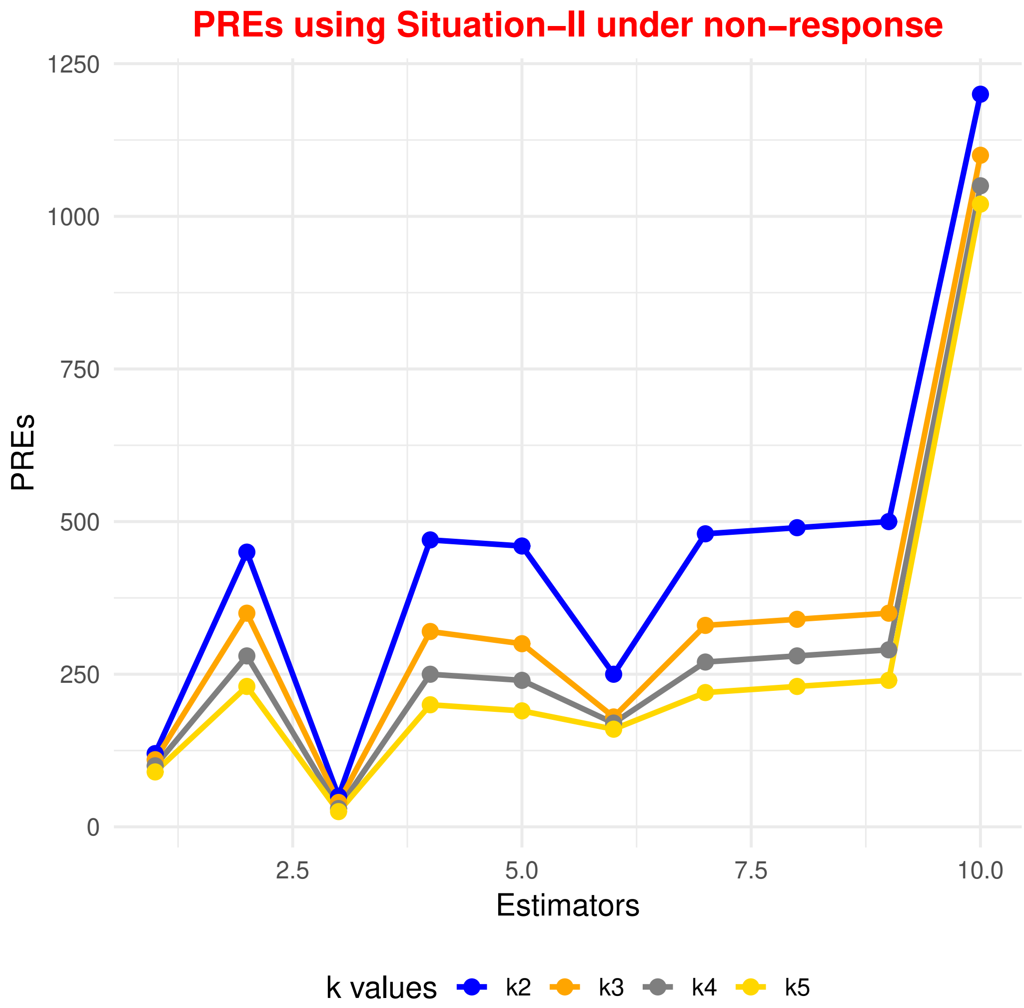

Figure 4. Showing PREs using Situation-I under non-response using real data set.

stratified random sampling.

Simulation results are presented in tables 6–9, where we represent the Mean Squared Error and Percentage Relative Efficiency of the estimator under different non-response. Two distinct scenarios were considered, the first being only non-response on the study variable (Y) and the second comprising non-response on both the study variable and any covariates (X), or rank of X. Comparing these examples, we demonstrate the effect of non-response on auxiliary information in terms of the estimator's efficiency. In brief, the results verify that our proposed estimator has a lower MSE and higher PRE than the co-specific estimators, indicating it is more robust and efficient even under multi-variable non-response.

From Figures 5, 6, 7, and 8, one can conclude that the plot of MSEs and PREs shows that the proposed estimator is efficient then existing estimators, using simulation study.

7. Discussion

The performance of the proposed estimators for stratified random sampling with non-response was investigated by simulation studies and applied to some real data as described above. We studied two populations, Scenario-I and Scenario-II, and

Table 6. Simulation, Situation –I, when k=2, 3

k=2			k=3		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{\overline{Y}}}_{SRS}$	2744.233	100	$\widehat{\overline{\overline{Y}}}_{SRS}$	3136.59	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	285.8334	1012.89	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	343.3454	963.78
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	11481.13	25.22	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	13250.68	24.97
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	248.917	1163.1	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	291.9854	1133.31
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	248.6091	1164.55	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	291.5617	1134.96
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	730.3305	396.42	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	824.9871	401.11
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	246.8733	1172.73	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	289.1943	1144.25
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	235.395	1229.92	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	274.4456	1205.74
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	232.4853	1245.31	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	269.6127	1227.35
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	224.0079	1292.44	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	253.3037	1306.38

Table 7. Simulation, Situation –I, when k=4, 5

k=4			k=5		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{\overline{Y}}}_{SRS}$	2532.927	100	$\widehat{\overline{\overline{Y}}}_{SRS}$	2895.073	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	263.8242	934.89	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	316.9078	889.57
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	10597.09	23.28	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	12230.38	23.05
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	229.7504	1073.55	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	269.5025	1046.05
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	229.4662	1074.88	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	269.1114	1047.57
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	674.0951	365.89	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	761.4631	370.22
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	227.864	1082.43	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	266.9263	1056.14
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	217.2696	1135.21	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	253.3133	1112.9
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	214.5839	1149.42	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	248.8526	1132.85
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	206.7593	1192.92	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	233.7993	1205.79

Table 8. Simulation, Situation –II, when k=2, 3

k=2			k=3		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{Y}}_{SRS}$	2654.493	100	$\widehat{\overline{Y}}_{SRS}$	3034.019	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	600.3813	451.2	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	979.9077	315.97
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	9773.534	27.72	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	10153.06	30.5
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	577.7924	468.84	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	957.3188	323.43
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	576.08	470.23	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	952.6272	325.02
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	994.4128	272.41	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	1373.939	225.35
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	573.6902	472.19	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	949.012	326.26
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	565.8868	478.7	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	941.2391	328.95
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	560.3548	483.43	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	932.013	332.21
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	220.3338	1229.46	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	252.9823	1223.89

Table 9. Simulation, Situation –II, when k=4, 5

k=4			k=5		
Estimators	MSE	PRE	Estimators	MSE	PRE
$\widehat{\overline{Y}}_{SRS}$	2734.152	100	$\widehat{\overline{Y}}_{SRS}$	3125.067	100
$\widehat{\overline{\overline{Y}}}_{R(st)H}$	618.3981	464.74	$\widehat{\overline{\overline{Y}}}_{R(st)H}$	1009.314	325.45
$\widehat{\overline{\overline{Y}}}_{P(st)H}$	10066.83	28.55	$\widehat{\overline{\overline{Y}}}_{P(st)H}$	10457.74	31.41
$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	595.1314	482.91	$\widehat{\overline{\overline{Y}}}_{reg(st)H}$	986.047	333.13
$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	593.3676	484.34	$\widehat{\overline{\overline{Y}}}_{R,D(st)H}$	981.2145	334.77
$\widehat{\overline{\overline{Y}}}_{S(st)H}$	1024.254	280.59	$\widehat{\overline{\overline{Y}}}_{S(st)H}$	1415.17	232.12
$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	590.9061	486.36	$\widehat{\overline{\overline{Y}}}_{G,K(st)H}$	977.4909	336.05
$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	582.8685	493.07	$\widehat{\overline{\overline{Y}}}_{A,H(st)H}$	969.4847	338.82
$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	577.1705	497.93	$\widehat{\overline{\overline{Y}}}_{H,S(st)H}$	959.9817	342.18
$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	226.9458	1266.3	$\widehat{\overline{\overline{Y}}}_{Prop(st)H}$	260.574	1260.61

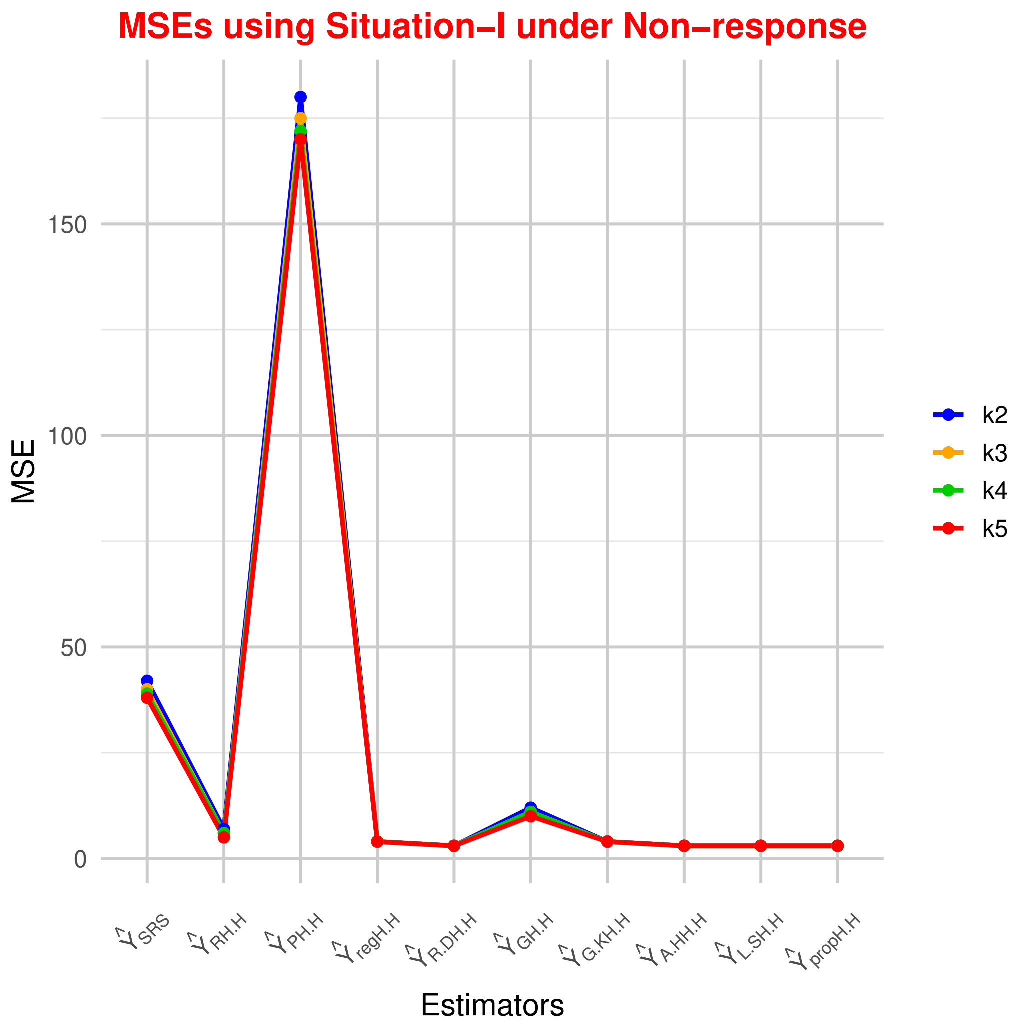

Figure 5. Showing MSEs using Situation-I under non-response using Simulation.

compared estimators in each scenario for a range of threshold values k (i.e., k=2,3,4,5). The performances (Tables 2–9) and the plots of MSE/PRE undoubtedly support that all the proposed estimators perform better compared to their corresponding ones. In particular, the proposed estimators are persistently superior in MSE and PRE to the theory-based regression estimator under the non-response model for stratified sampling. The graphical comparisons (1–8) provide an additional perspective to the tabular results in illustrating how performance varies with k. Varying the threshold from 2 to 5, we observe that MSE values increase, while PRE values decrease slightly as k changes, reflecting that estimator performance depends on the selected k. Crucially, despite these variations, the proposed estimators remain superior to their competitors. These results validate the robustness and effectiveness of our approach, proving that it can be applied to real-world surveys where non-response is inevitable.

8. Conclusion

The paper contributes to the theory of mean estimation with stratified random sampling by presenting a new estimator for the population mean of a research variable when non-response is present in both study and auxiliary variables. For

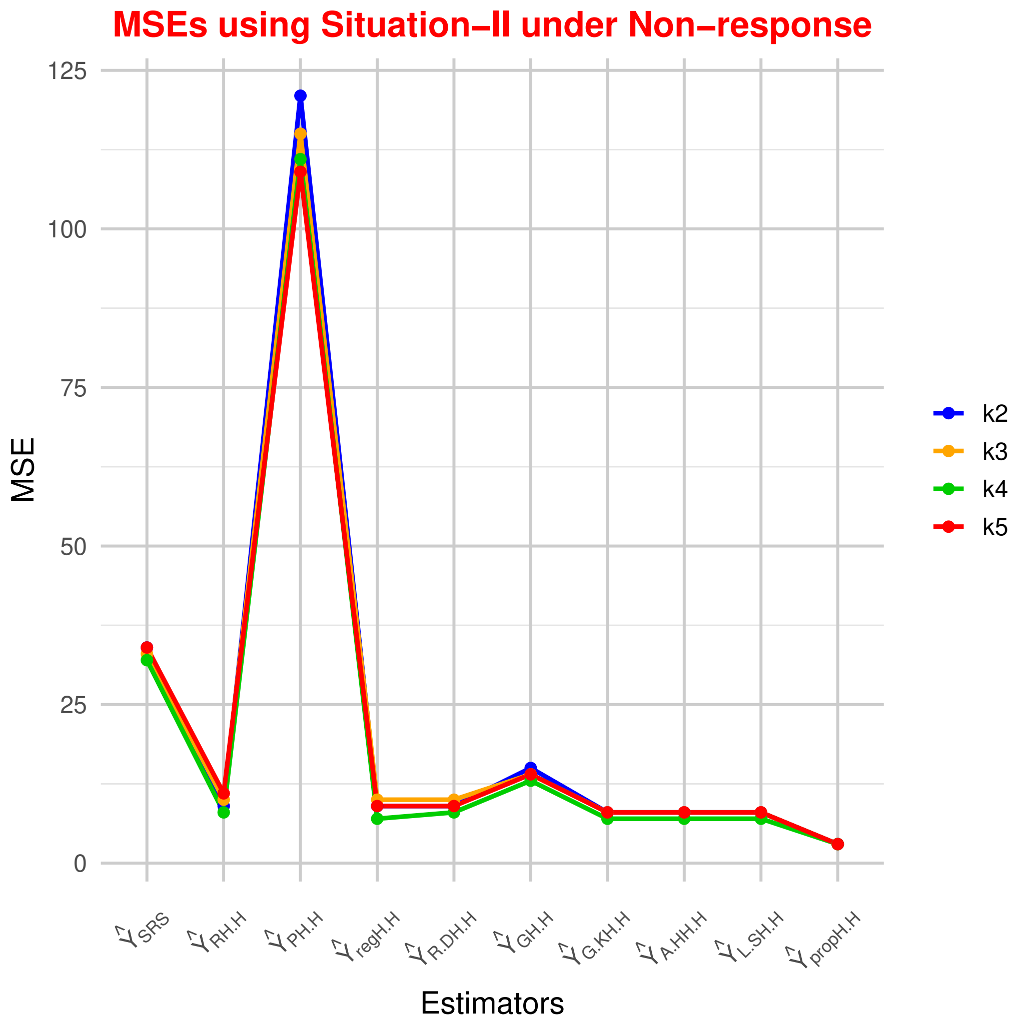

Figure 6. Showing MSEs using Situation-II under non-response using Simulation.

non-response, the proposed estimator utilizes inexpensive upstream abundance information, along with known population parameters of an auxiliary variable, to provide a population mean estimate for a study variable. To enrich its theory, the properties of the suggested estimator (bias, mean squared errors, efficiency and optimality conditions) are then derived to first order approximation.

The results of the Monte Carlo simulation studies, along with those from real data, indicate that the anticipated estimator performs better than the current estimators considered in this study. Specifically, it shows the best numerical results by lowering the (MSE) and increasing the (PRE), hence proving its practical usefulness. The outcomes clearly indicate that the newly created estimator should be preferred over the traditional methods used in non-response situations, as highlighted in the literature reviewed during this study.

Thus, in the case of practitioners and researchers interested in survey sampling, and in situations where non-response can be a significant issue, it is highly advised to adopt the proposed estimator. Their strength and efficiency make it helpful in obtaining more accurate estimates of the population mean in situations involving stratified random sampling designs with non-response. In addition, this development not only enhances the theoretical base of mean estimation but also creates opportunities for further methodological advancements, such as adjusting to more sophisticated sampling designs like

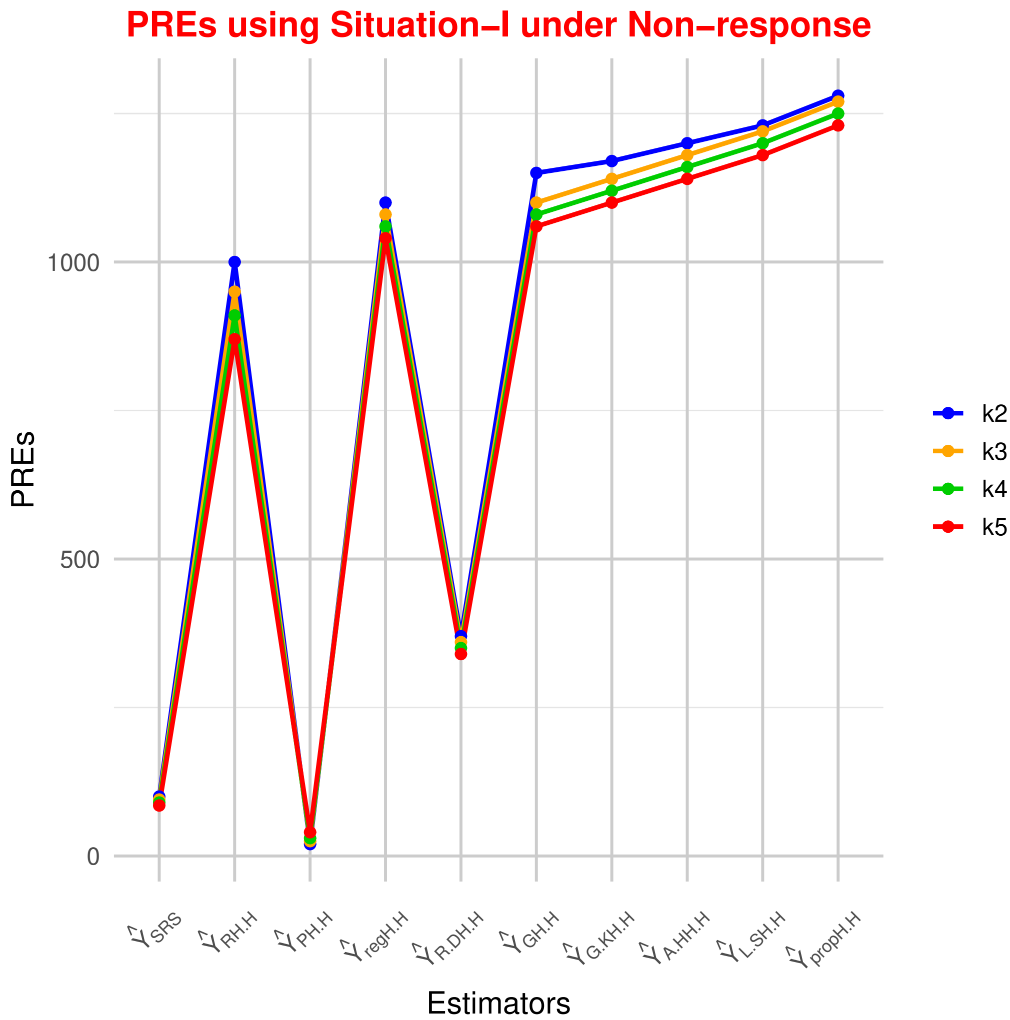

Figure 7. Showing PREs using Situation-I under non-response using Simulation.

simple random sampling (SRS) or cluster sampling, thereby extending its scope to practical survey activities.

9. Future Research Directions

Several necessary extensions could strengthen the theoretical coverage and the practical value of this work. First, as an immediate line of research, it would be interesting to study the effect of measurement error on our proposed estimators. In most large-scale surveys, errors exist in the recording or reporting of variables, and studying robustness under such situations would make the method more practical. Secondly, the introduction of high non-response adaptive estimators would represent a beneficial contribution, as non-response rates in real surveys vary significantly across strata or groups. Although beyond the scope of this work, adaptive methods might even be able to automatically change their estimation strategy and remain efficient under substantial data reduction. Third, the technique could be adapted to more complicated sampling designs, such as probability proportional to size (PPS) or cluster sampling. Such designs are frequently used in ultra-large sample surveys, taking into account homogeneity and economic factors. By including non-response adjustment in those frameworks, the proposed methodology would not only extend its theoretical performance but also contribute

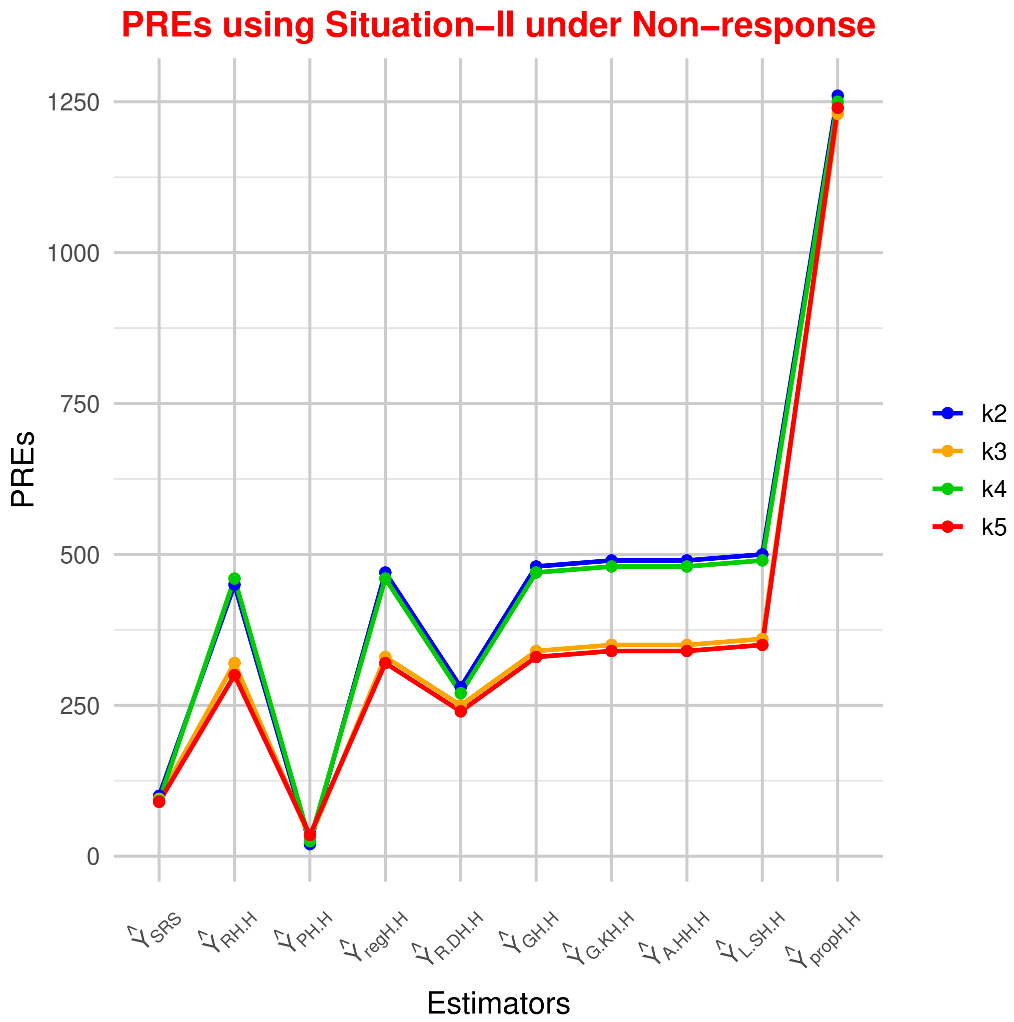

Figure 8. Showing PREs using Situation-II under non-response using Simulation.

to simultaneously handling missing data, measurement error and complex survey design issues typically confronted in practice.

10. Practical Implications of the proposed work

1. **Greater trustworthiness of Survey Results.** One of the significant sources of bias in the sampling of surveys is non-response. By creating estimators that directly account for non-response using stratified random sampling, investigators can obtain population mean estimates that are more valid and closer to the actual population values. This minimizes the chances of biased policy making, especially in national surveys that are carried out on a large scale, like health, education and labour statistics.
2. **Economical Utilization of Auxiliary Information.**
   Stratified sampling is already exact because it clusters the population into similar strata. Adding additional information during the process of dealing with non-response further optimises the estimates. As an illustration, land records or past yield data could be used as auxiliary data to enhance the estimation of the average yield of crops in agricultural

surveys where some farmers fail to respond.

3. **Field Survey Cost-Effectiveness.**
   Non-response follow-ups and callbacks can be costly. When good estimators are available to compensate for non-response, survey organisations need not conduct extensive follow-up, saving on data collection expenses while still achieving accuracy.
4. **Recommendations to Public Policy and Planning.**
   In many areas of life, like in the planning of education or poverty reduction, stable estimates of averages (mean income, mean literacy score, mean nutritional intake, etc.) are essential. Stratified survey results are reliable and allow planners to depend on adjusted estimators when there is incomplete participation in a survey due to non-response.
   To give an example, in a health survey, non-response can be disproportionate in some socio-economic groups; stratified non-response adjustment would be used to make sure that the contribution they make is not under-represented.
5. **Elasticity on the Multiple Non-Response Levels.**
   As efficiency and (MSE) of estimators may differ with the level of non-response, the suggested methods provide statisticians with adaptive instruments to adjust the strategy of estimation based on the real-life situation in the field. This does not imply that survey statisticians cannot retain a high percentage relative efficiency (PRE) in cases where non-response levels differ in strata.
6. **Usages in the Big Data and Administrative Data Sources.**
   In addition to the old-fashioned survey, contemporary applications in big data integration (e.g., tying census data and administrative records) tend to result in incomplete reporting or selective non-response. The techniques that evolved to stratified designs are also applicable to such settings, making them highly relevant to present-day data environments.
7. **Making Survey Statistics More Trustworthy by the public.**
   By having government agencies, NGOs, or international organisations create survey-based statistics, one can reduce non-response bias and build confidence in official statistics. Survey data is also accused of being unrepresentative or biased, but reliable mean estimates minimise these criticisms.

Conflict of Interest: Authors declare no conflict of interest.
Funding: This research received no external funding.

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