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Representation and solution of two fifth-order nonlinear difference equations

Raafat Abo-Zeid *

The high institute for Engineering & Technology, Al-Obour, Cairo, Egypt

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ABSTRACT

Difference equations appear as an approximation to differential equations (in numerical analysis). These equations appear in nature, in modeling many situations in biology and economics as well as in ecology and engineering. Solvable difference equations and systems of difference equations occur in many areas of science and mathematics. In this paper, we represent and study the well-defined solutions of the difference equation:

$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{\omega_n + \omega_{n-4}}, n \in \mathbb{N}_0,$$

where the initial values $\omega_0, \omega_{-1}, \omega_{-2}, \omega_{-3}$ and ω_{-4} are real numbers.

We give a representation to the above-mentioned equation using a sequence $\{\sigma_n\}_{n=0}^{\infty}$ that satisfies the linear second-order difference equation:

$$\sigma_{n+2} - \sigma_{n+1} - \sigma_n = 0, n \in \mathbb{N}_0,$$

with $\sigma_0 = 0$,$\sigma = 1$, and give the solution of the difference equation

$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{-\omega_n + \omega_{n-4}}, n \in \mathbb{N}_0,$$

where the initial values $\omega_0, \omega_{-1}, \omega_{-2}, \omega_{-3}$ and ω_{-4} are real numbers.

The important result in this paper is that every well-defined solution $\{\omega_n\}_{n=-4}^{\infty}$ to the first above-mentioned equation converges to zero and every well-defined solution $\{\omega_n\}_{n=-4}^{\infty}$ to the second above-mentioned equation is periodic. A very important tool in studying difference equations is the forbidden set. We introduce here the forbidden set for the above-mentioned equations. We give some examples to show the theoretical results.

1. Introduction

Difference equations became one of the fundamental topics in mathematics. It is used to understand and deal with the behavior of discrete models. Difference equations are used as an approximation to the differential equations. Although a difference equation may has a simple form, it may has a complicated behavior [1]. Hilal et al. [2], studied the two difference equations:

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{ax_n+bx_{n-3}},n\in\mathbb{N}_0,$$

and

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{-ax_n+bx_{n-3}},n\in\mathbb{N}_0,$$

In this paper, we study the well-defined solutions of the two difference equations

$$\omega_{n+1}=\frac{\omega_{n-3}\omega_{n-4}}{\omega_n+\omega_{n-4}},\qquad n\in\mathbb{N}_0, \tag{1}$$

and

$$\omega_{n+1}=\frac{\omega_{n-3}\omega_{n-4}}{-\omega_n+\omega_{n-4}},n\in\mathbb{N}_0, \tag{2}$$

with real initial values $\omega_0,\omega_{-1},\omega_{-2},\omega_{-3},\omega_{-4}$.

For more on difference equations (see [3]-[20]) and the references therein.

2. The difference equation (1)

In this section we introduce a representation for Eq. (1) and prove the main result in this paper.

2.1 Representation and the behavior of Eq. (1)

Theorem 2.1.1.

Assume that $\{\omega_n\}_{n=-4}^{\infty}$ is a well-defined solution for Eq. (1). The solution $\{\omega_n\}_{n=-4}^{\infty}$ can be represented as

$$\begin{cases}\omega_{4n+1}=\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\\ \omega_{4n+2}=\omega_{-2}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\\ \omega_{4n+3}=\omega_{-1}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}},\\ \omega_{4n+4}=\omega_{0}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}\end{cases}\quad n\in\mathbb{N}_0, \tag{3}$$

where $\{\sigma_n\}_{n=0}^{\infty}$ is a solution of the difference equation

$$\sigma_{n+2}-\sigma_{n+1}-\sigma_n=0,n\in\mathbb{N}_0,$$

with $\sigma_0=0$ and $\sigma=1$.

Proof.

Let $\{\omega_n\}_{n=-4}^{\infty}$ be a well-defined solution for Eq. (1). We prove by induction on $n\in\mathbb{N}_0$.

For $n=0$, using formula (3), we get

$$\omega_1=\omega_{-3}\frac{\omega_0\sigma_0+\omega_{-4}\sigma_1}{\omega_0\sigma_1+\omega_{-4}\sigma_2}=\frac{\omega_{-3}\omega_{-4}}{\omega_0+\omega_{-4}},$$

$$\omega_2=\omega_{-2}\frac{\omega_0\sigma_1+\omega_{-4}\sigma_2}{\omega_0\sigma_2+\omega_{-4}\sigma_3}=\omega_{-2}\frac{\omega_0+\omega_{-4}}{\omega_0+2\omega_{-4}},$$

$$\omega_3=\omega_{-1}\frac{\omega_0\sigma_2+\omega_{-4}\sigma_3}{\omega_0\sigma_3+\omega_{-4}\sigma_4}=\omega_{-1}\frac{\omega_0+2\omega_{-4}}{2\omega_0+3\omega_{-4}},$$

and

$$\omega_4=\omega_0\frac{\omega_0\sigma_3+\omega_{-4}\sigma_4}{\omega_0\sigma_4+\omega_{-4}\sigma_5}=\omega_0\frac{2\omega_0+3\omega_{-4}}{3\omega_0+5\omega_{-4}}.$$

Now, assume that formula (3) is true for a certain $n\in\mathbb{N}$. Then

$$\omega_{4(n+1)+1}=\frac{\omega_{4n+1}\omega_{4n}}{\omega_{4n+4}+\omega_{4n}}$$

$$=\frac{\omega_{-3}\omega_0\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times\prod_{t=0}^{n-1}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}}{\omega_0\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}+\omega_0\prod_{t=0}^{n-1}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}}$$

$$=\frac{\omega_{-3}\omega_0\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times\prod_{t=0}^{n-1}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}}{\omega_0\prod_{t=0}^{n-1}\frac{\omega_0\sigma_{4t+3}+\omega_{-4}\sigma_{4t+4}}{\omega_0\sigma_{4t+4}+\omega_{-4}\sigma_{4t+5}}(\frac{\omega_0\sigma_{4n+3}+\omega_{-4}\sigma_{4n+4}}{\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5}}+1)}$$

$$=\frac{\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}}{(\frac{\omega_0\sigma_{4n+3}+\omega_{-4}\sigma_{4n+4}}{\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5}}+1)}$$

$$=\frac{\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times(\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5})}{(\omega_0\sigma_{4n+3}+\omega_{-4}\sigma_{4n+4}+\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5})}$$

$$=\frac{\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times(\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5})}{\omega_0(\sigma_{4n+3}+\sigma_{4n+4})+\omega_{-4}(\sigma_{4n+4}+\sigma_{4n+5})}$$

$$=\frac{\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times(\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5})}{\omega_0\sigma_{4n+5}+\omega_{-4}\sigma_{4n+6}}$$

$$=\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}\times\frac{\omega_0\sigma_{4n+4}+\omega_{-4}\sigma_{4n+5}}{\omega_0\sigma_{4n+5}+\omega_{-4}\sigma_{4n+6}}$$

$$=\omega_{-3}\prod_{t=0}^{n+1}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}.$$

Also,

$$\omega_{4(n+1)+2}=\frac{\omega_{4n+2}\omega_{4n+1}}{\omega_{4n+5}+\omega_{4n+1}}$$

$$=\frac{\omega_{-2}\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\times\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}}{\omega_{-3}\prod_{t=0}^{n+1}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}+\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}}$$

$$=\frac{\omega_{-2}\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\times\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}}{\omega_{-3}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t}+\omega_{-4}\sigma_{4t+1}}{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}(\frac{\omega_0\sigma_{4(n+1)}+\omega_{-4}\sigma_{4(n+1)+1}}{\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2}}+1)}$$

$$=\frac{\omega_{-2}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}}{(\frac{\omega_0\sigma_{4(n+1)}+\omega_{-4}\sigma_{4(n+1)+1}}{\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2}}+1)}$$

$$=\frac{\omega_{-2}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\times(\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2})}{(\omega_0\sigma_{4(n+1)}+\omega_{-4}\sigma_{4(n+1)+1}+\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2})}$$

$$=\frac{\omega_{-2}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\times(\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2})}{\omega_0(\sigma_{4(n+1)}+\sigma_{4(n+1)+1})+\omega_{-4}(\sigma_{4(n+1)+1}+\sigma_{4(n+1)+2})}$$

$$=\frac{\omega_{-2}\prod_{t=0}^{n}\frac{\omega_0\sigma_{4t+1}+\omega_{-4}\sigma_{4t+2}}{\omega_0\sigma_{4t+2}+\omega_{-4}\sigma_{4t+3}}\times(\omega_0\sigma_{4(n+1)+1}+\omega_{-4}\sigma_{4(n+1)+2})}{\omega_0\sigma_{4(n+1)+2}+\omega_{-4}\sigma_{4(n+1)+3}}$$

$$= \omega_{-2} \prod_{t=0}^{n} \frac{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}}{\omega_0 \sigma_{4t+2} + \omega_{-4}\sigma_{4t+3}} \times \frac{\omega_0 \sigma_{4(n+1)+1} + \omega_{-4}\sigma_{4(n+1)+2}}{\omega_0 \sigma_{4(n+1)+2} + \omega_{-4}\sigma_{4(n+1)+3}}$$

$$= \omega_{-2} \prod_{t=0}^{n+1} \frac{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}}{\omega_0 \sigma_{4t+2} + \omega_{-4}\sigma_{4t+3}}.$$

Similarly, we can prove formula (3) for ω_{4n+3} and ω_{4n+4}.
This completes the proof.

The main result in this section is the following:

Theorem 2.1.2.
Every well-defined solution $\{\omega_n\}_{n=-4}^{\infty}$ of Eq. (1) converges to zero.
Proof.
Let $\{\omega_n\}_{n=-4}^{\infty}$ be a well-defined solution of Eq. (1). We show that $\omega_{4n+1} \to 0$, as $n \to \infty$. We have first that

$$L(t) := \frac{\omega_0 \sigma_{4t} + \omega_{-4}\sigma_{4t+1}}{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}} = \frac{\sigma_{4t+1}\left(\omega_0 \frac{\sigma_{4t}}{\sigma_{4t+1}} + \omega_{-4}\right)}{\sigma_{4t+2}\left(\omega_0 \frac{\sigma_{4t+1}}{\sigma_{4t+2}} + \omega_{-4}\right)}$$

$$\to \frac{1}{\alpha} \frac{(\omega_0 \frac{1}{\alpha} + \omega_{-4})}{(\omega_0 \frac{1}{\alpha} + \omega_{-4})} = \frac{1}{\alpha}, \quad \text{as } t \to \infty,$$

where α is the Golden number ($\alpha = \frac{1+\sqrt{5}}{2}$).
Then for $1 - \frac{1}{\alpha} > \epsilon > 0$, then there exists $t_0 \in \mathbb{N}$ such that:

$$|\omega_{4n+1}| = |\omega_{-3}| \left| \prod_{t=0}^{n} \frac{\omega_0 \sigma_{4t} + \omega_{-4}\sigma_{4t+1}}{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}} \right|$$

$$= |\omega_{-3}| \left| \prod_{t=0}^{t_0-1} \frac{\omega_0 \sigma_{4t} + \omega_{-4}\sigma_{4t+1}}{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}} \right| \times \left| \prod_{t=t_0}^{n} \frac{\omega_0 \sigma_{4t} + \omega_{-4}\sigma_{4t+1}}{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}} \right|$$

$$< |\omega_{-3}| \left| \prod_{t=0}^{t-1} \frac{\omega_0 \sigma_{4t} + \omega_{-4}\sigma_{4t+1}}{\omega_0 \sigma_{4t+1} + \omega_{-4}\sigma_{4t+2}} \right| \left(\frac{1}{\alpha} + \epsilon\right)^{n-t_0+1}$$

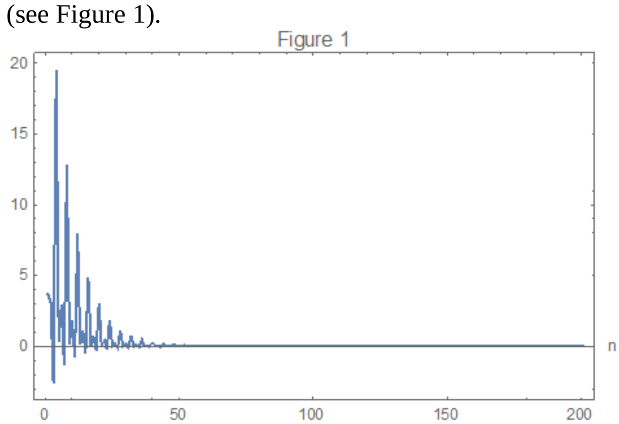
as $n \to \infty$, $\omega_{4n+1} \to 0$.
Similarly, we can show that, as $n \to \infty$, $\omega_{4n+2} \to 0$, $\omega_{4n+3} \to 0$ and $\omega_{4n+4} \to 0$.
Therefore, the solution $\{\omega_n\}_{n=-4}^{\infty}$ of Eq. (1) converges to zero.
This completes the proof.

A very important tool in studying difference equations is the forbidden set. It is useful to state that for Eq. (1). The forbidden set for Eq. (1) is:

$$\Omega = \cup_{n=1}^{\infty} \left\{ (\omega_0, \omega_{-1}, \omega_{-2}, \omega_{-3}, \omega_{-4}) \in \mathbb{R}^5 : \omega_{-4} = -\omega_0 \frac{\sigma_{n-1}}{\sigma_n} \right\} \cup \cup_{l=-4}^{-1} \{ (\omega_0, \omega_{-1}, \omega_{-2}, \omega_{-3}, \omega_{-4}) \in \mathbb{R}^5 : \omega_l = 0 \}.$$

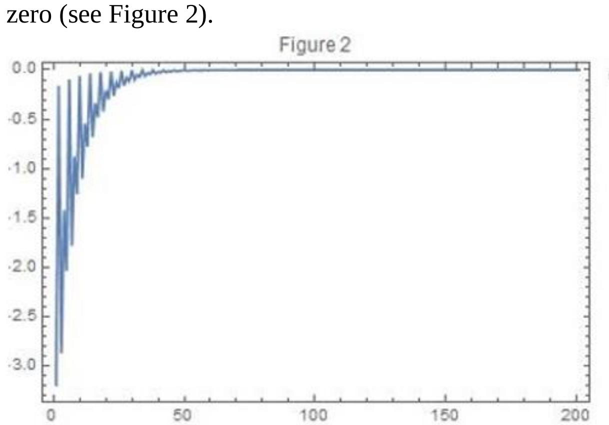
2.2 Illustrative examples:

Example 2.2.1.
If $\{\omega_n\}_{n=-4}^{\infty}$ is a well-define solution of Eq. (1) such that $\omega_{-4} = -2.5$, $\omega_{-3} = -1.2$, $\omega_{-2} = 2.8$, $\omega_{-1} = 1.5$ and $\omega_0 = 3.7$, then the solution $\{\omega_n\}_{n=-4}^{\infty}$ converges to zero (see Figure 1).



$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{\omega_n + \omega_{n-4}}$$

Example 2.2.2.
If $\{\omega_n\}_{n=-4}^{\infty}$ is a well-define solution of Eq. (1) such that $\omega_{-4} = -1.5$, $\omega_{-3} = -0.5$, $\omega_{-2} = -3.8$, $\omega_{-1} = -2.5$ and $\omega_0 = -3.2$, then the solution $\{\omega_n\}_{n=-4}^{\infty}$ converges to zero (see Figure 2).



$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{\omega_n + \omega_{n-4}}$$

3. The difference equation (2)

In this section we introduce a representation for Eq. (2) and prove the main result in this paper.

3.1 Representation and the behavior of Eq. (2)

Theorem 3.1.

Assume that $\{\omega_n\}_{n=-4}^{\infty}$ is a well-defined solution for Eq. (2). The solution $\{\omega_n\}_{n=-4}^{\infty}$ can be represented as

$$\begin{cases} \omega_{4n+1} = \omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)} \\ \omega_{4n+2} = \omega_{-2}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)} \\ \omega_{4n+3} = \omega_{-1}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)}{\omega_{-4}\sin\frac{\pi}{3}(4j+4)-\omega_0\sin\frac{\pi}{3}(4j+3)}, \\ \omega_{4n+4} = \omega_0\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+4)-\omega_0\sin\frac{\pi}{3}(4j+3)}{\omega_{-4}\sin\frac{\pi}{3}(4j+5)-\omega_0\sin\frac{\pi}{3}(4j+4)} \end{cases} n \in \mathbb{N}_0. \quad (4)$$

Proof.

Let $\{\omega_n\}_{n=-4}^{\infty}$ be a well-defined solution for Eq. (2). We prove by induction on $n \in \mathbb{N}_0$.

For $n = 0$, using formula (4), we get the result.

Now, assume that formula (4) is true for a certain $n \in \mathbb{N}$. Then

$$\omega_{4(n+1)+1} = \frac{\omega_{4n+1}\omega_{4n}}{-\omega_{4n+4}+\omega_{4n}}$$

$$= \omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}\times$$

$$\times\prod_{t=0}^{n-1}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+4)-\omega_0\sin\frac{\pi}{3}(4j+3)}{\omega_{-4}\sin\frac{\pi}{3}(4j+5)-\omega_0\sin\frac{\pi}{3}(4j+4)}$$

$$/(-\omega_0\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+4)-\omega_0\sin\frac{\pi}{3}(4j+3)}{\omega_{-4}\sin\frac{\pi}{3}(4j+5)-\omega_0\sin\frac{\pi}{3}(4j+4)}$$

$$+\omega_0\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+4)-\omega_0\sin\frac{\pi}{3}(4j+3)}{\omega_{-4}\sin\frac{\pi}{3}(4j+5)-\omega_0\sin\frac{\pi}{3}(4j+4)})$$

$$= \omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}$$

$$/(-\frac{\omega_{-4}\sin\frac{\pi}{3}(4n+4)-\omega_0\sin\frac{\pi}{3}(4n+3)}{\omega_{-4}\sin\frac{\pi}{3}(4n+5)-\omega_0\sin\frac{\pi}{3}(4n+4)}+1)$$

$$= \omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}$$

$$/(\frac{\omega_{-4}\sin\frac{\pi}{3}(4n+6)-\omega_0\sin\frac{\pi}{3}(4n+5)}{\omega_{-4}\sin\frac{\pi}{3}(4n+5)-\omega_0\sin\frac{\pi}{3}(4n+4)})$$

$$= \omega_{-3}\prod_{t=0}^{n+1}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}.$$

Also,

$$\omega_{4(n+1)+2} = \frac{\omega_{4n+2}\omega_{4n+1}}{-\omega_{4n+5}+\omega_{4n+1}}$$

$$= \omega_{-3}\omega_{-2}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)}$$

$$\times\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}$$

$$/(-\omega_{-3}\prod_{t=0}^{n+1}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}$$

$$+\omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)})$$

$$= \omega_{-2}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)}$$

$$/(-\frac{\omega_{-4}\sin\frac{\pi}{3}(4n+5)-\omega_0\sin\frac{\pi}{3}(4n+4)}{\omega_{-4}\sin\frac{\pi}{3}(4n+6)-\omega_0\sin\frac{\pi}{3}(4n+5)}+1)$$

$$= \omega_{-2}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)}$$

$$/(-\frac{\omega_{-4}\sin\frac{\pi}{3}(4n+7)-\omega_0\sin\frac{\pi}{3}(4n+6)}{\omega_{-4}\sin\frac{\pi}{3}(4n+6)-\omega_0\sin\frac{\pi}{3}(4n+5)})$$

$$= \omega_{-2}\prod_{t=0}^{n+1}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\omega_0\sin\frac{\pi}{3}(4j+1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+3)-\omega_0\sin\frac{\pi}{3}(4j+2)}.$$

Similarly, we can prove formula (4) for ω_{4n+3} and ω_{4n+4}.

This completes the proof.

We give the following lemma without proof, as it is a direct calculations.

Lemma 3.2.

The following statement is true for all $j \in \mathbb{N}_0$:

$$\prod_{t=0}^{2}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+t)-\omega_0\sin\frac{\pi}{3}(4j+t-1)}{\omega_{-4}\sin\frac{\pi}{3}(4j+t+1)-\omega_0\sin\frac{\pi}{3}(4j+t)} = -1, t = 1,2,3,4.$$

Theorem 3.3.

Assume that $\{\omega_n\}_{n=-4}^{\infty}$ is a well-defined solution for Eq. (2). The solution $\{\omega_n\}_{n=-4}^{\infty}$ is periodic with period 24.

roof.

Let $\{\omega_n\}_{n=-4}^{\infty}$ be a well-defined solution for Eq. (2).

Then

$$
\begin{aligned}
&\omega_{4(n+6)+1} \\
&= \omega_{-3}\prod_{t=0}^{n+6}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&= \omega_{-3}\prod_{t=0}^{2}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&\times\prod_{t=3}^{5}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&\times\prod_{t=6}^{n+6}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&= \omega_{-3}(-1)\prod_{t=0}^{2}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+13)-\ \omega_0\sin\frac{\pi}{3}(4j+12)}{\omega_{-4}\sin\frac{\pi}{3}(4j+14)-\ \omega_0\sin\frac{\pi}{3}(4j+13)} \\
&\times\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+25)-\ \omega_0\sin\frac{\pi}{3}(4j+24)}{\omega_{-4}\sin\frac{\pi}{3}(4j+26)-\ \omega_0\sin\frac{\pi}{3}(4j+25)} \\
&= \omega_{-3}(-1)\prod_{t=0}^{2}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&\times\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&= \omega_{-3}\prod_{t=0}^{n}\frac{\omega_{-4}\sin\frac{\pi}{3}(4j+1)-\ \omega_0\sin\frac{\pi}{3}(4j)}{\omega_{-4}\sin\frac{\pi}{3}(4j+2)-\ \omega_0\sin\frac{\pi}{3}(4j+1)} \\
&\qquad= \omega_{4n+1}.
\end{aligned}
$$

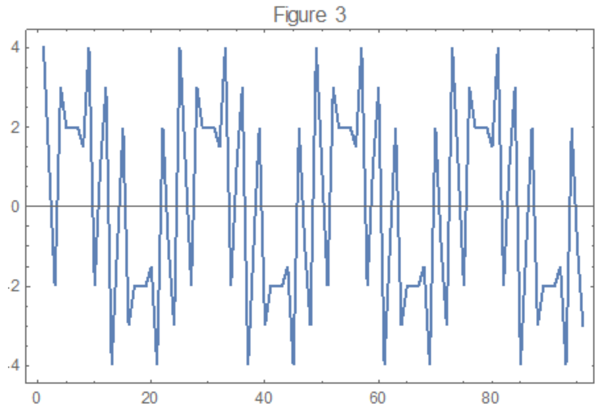
The proof of $\omega_{4n+2}, \omega_{4n+3}, \omega_{4n+4}$is similar and will be omitted.
This completes the proof.

The forbidden set for Eq. (2) is

$$
\Omega = \cup_{n=1}^{\infty}\left\{(\omega_0,\omega_{-1},\omega_{-2},\omega_{-3},\omega_{-4})\in\mathbb{R}^5:\omega_{-4} = \omega_0\frac{\sin\frac{\pi}{3}(n-1)}{\sin\frac{\pi}{3}n}\right\} \cup\cup_{l=-4}^{-1}\{(\omega_0,\omega_{-1},\omega_{-2},\omega_{-3},\omega_{-4})\in\mathbb{R}^5:\omega_l = 0\}.
$$

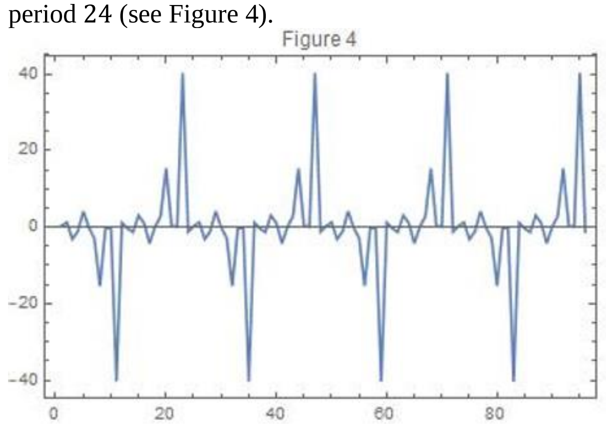
3.2 Illustrative examples:

Example 3.2.1.
If $\{\omega_n\}_{n=-4}^{\infty}$ is a well-define solution of Eq. (2) such that $\omega_{-4} = 4$, $\omega_{-3} = 1$, $\omega_{-2} = -2$, $\omega_{-1} = 3$ and $\omega_0 = 2$, then the solution $\{\omega_n\}_{n=-4}^{\infty}$ is periodic with period 24 (see Figure 3).



$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{-\omega_n+\omega_{n-4}}$$

Example 3.2.2.
If $\{\omega_n\}_{n=-4}^{\infty}$ is a well-define solution of Eq. (2) such that $\omega_{-4} = 0.3$, $\omega_{-3} = 1.3$, $\omega_{-2} = -3.1$, $\omega_{-1} = -1.1$ and $\omega_0 = 4.2$, then the solution $\{\omega_n\}_{n=-4}^{\infty}$ is periodic with period 24 (see Figure 4).



$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{-\omega_n+\omega_{n-4}}$$

Conclusion

In this paper, we studied the well-defined solutions of the two difference equations

$$\omega_{n+1} = \frac{\omega_{n-3}\omega_{n-4}}{\omega_n+\omega_{n-4}}, n = 0,1,\ldots,$$

and

$$\omega_{n+1} = -\frac{\omega_{n-3}\omega_{n-4}}{\omega_n+\omega_{n-4}}, n = 0,1,\ldots,$$

where the initial values $\omega_0, \omega_{-1}, \omega_{-2}, \omega_{-3}, \omega_{-4}$ are nonzero real numbers. The main result in this paper is that, every well-defined solution of the first above-mentioned equation converges to zero and the second one is periodic with period 24. We provided some examples to show the theoretical results.

Ethics approval

Not applicable.

Availability of data and material

Not applicable.

Conflict of interest

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