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A Study on the Ulam Stability of Impulsive Dynamic Equations on Time Scales

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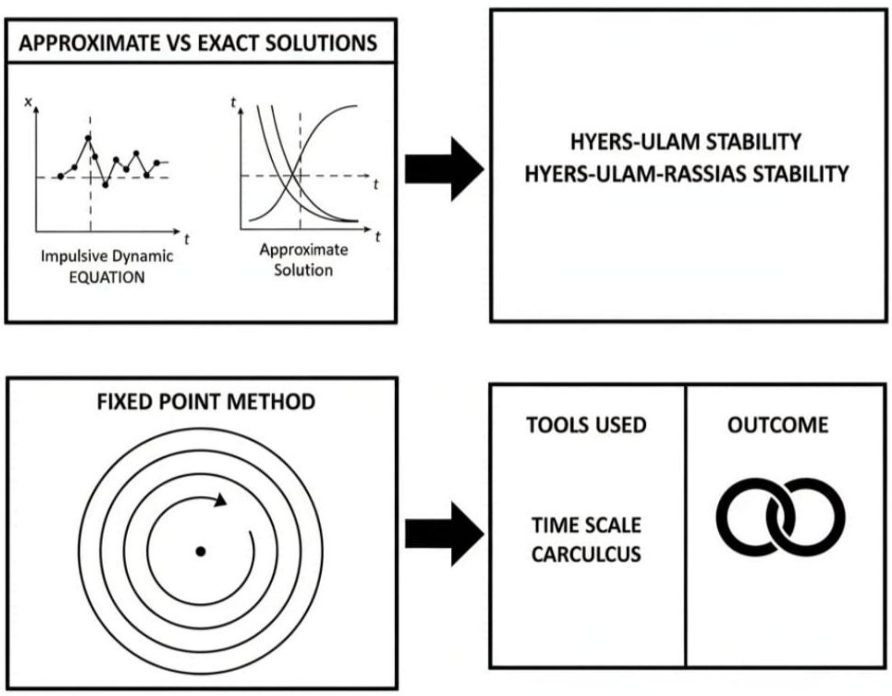
ABSTRACT

This paper investigates the Hyers-Ulam and Hyers-Ulam-Rassias stability of first-order nonlinear impulsive dynamic equations defined on finite time scale intervals. Stability in the sense of Ulam addresses the behavior of approximate solutions and their closeness to exact ones, which is key to the qualitative examination of dynamic systems. The aim is to establish sufficient conditions ensuring such stability properties within the time scale framework that unifies discrete and continuous cases. To achieve this, we utilize tools from time scale calculus combined with an extended integral inequality technique to effectively handle impulsive effects. The analysis is carried out using a fixed-point approach based on the contraction mapping principle, which guarantees both existence and uniqueness of solutions. Explicit stability constants related to Hyers-Ulam and Hyers-Ulam-Rassias stability are derived. To validate the theoretical outcomes, an illustrative example is included. This study contributes to extending stability theory for nonlinear impulsive dynamic equations on time scales, offering a unified perspective for both continuous and discrete models.

Graphical abstract



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1. Introduction

Within the framework of dynamic equations, solution stability is broadly acknowledged as one of the most significant and fascinating qualitative attributes. Among the various types of stability, Ulam stability has garnered notable attention owing to its theoretical significance and broad range of applications. Both differential and difference equations are covered by numerous stability theories see, ([1-4]) and ([5, 6]). Ulam stability refers to the principle that an exact solution may exist near any approximate solution of a given equation. This concept is especially significant in scenarios where determining an exact solution is difficult or impossible through direct methods. It provides a theoretical foundation for analyzing approximate solutions and has been widely applied across different fields of mathematical analysis, especially in relation to functional and differential equations. The concept of this form of stability in functional equations was initially introduced by Ulam [7] and was investigated after one year by D.H.Hyers [8]. Since that time, many researchers have actively explored the Ulam stability of different types of differential and integral equations [9-15]. I.A. Rus [16] introduced four categories of Ulam stability for

$$\zeta'(\mathfrak{a}) = \mathfrak{h}(\mathfrak{a}, \zeta(\mathfrak{a})), \quad \textbf{(1.1)}$$

on both finite and infinite intervals.
He later extended this classification by identifying four types of Ulam stability for the more general equation

$$\zeta'(\mathfrak{a}) = \omega(\zeta(\mathfrak{a})) + \mathfrak{h}(\mathfrak{a}, \zeta(\mathfrak{a})), \quad \textbf{(1.2)}$$

in Banach spaces (see [17]).
In [19], Y. Shen analyzed Ulam stability for the equation

$$\zeta^{\Delta}(\mathfrak{a}) = \omega(\mathfrak{a})\zeta(\mathfrak{a}) + \mathfrak{h}(\mathfrak{a}), \quad \textbf{(1.3)}$$

and its adjoint

$$\zeta^{\Delta}(\mathfrak{a}) = -\omega(\mathfrak{a})\zeta^{\sigma}(\mathfrak{a}) + \mathfrak{h}(\mathfrak{a}), \quad \textbf{(1.4)}$$

over a finite interval.
In 2021, M. A. Alghamdi et al. [20, 21] employed dynamic inequalities to derive results concerning the Hyers-Ulam and Hyers-Ulam-Rassias stability of

$$\zeta^{\Delta}(\mathfrak{a}) + \omega(\mathfrak{a})\zeta(\mathfrak{a}) = \mathfrak{h}(\mathfrak{a}), \quad \textbf{(1.5)}$$

and

$$\zeta^{\Delta}(\mathfrak{a}) = \omega(\mathfrak{a})\zeta(\mathfrak{a}) + \mathfrak{h}(\mathfrak{a}, \zeta(\mathfrak{a}),\ \mathfrak{z}(\zeta(\mathfrak{a}))) + g(\mathfrak{a}). \quad \textbf{(1.6)}$$

In 2022, Martin Bohner and Sanket Tikare [22] examined the Ulam–Hyers–Rassias stability of

$$\zeta^{\Delta}(\mathfrak{a}) = \omega(\mathfrak{a})\zeta^{\sigma}(\mathfrak{a}) + \mathfrak{h}\big(\mathfrak{a}, \zeta(\mathfrak{a})\big). \quad \textbf{(1.7)}$$

Since the late 1990s, significant advancements have been achieved in the analysis of differential equations and differential inclusions that incorporate impulsive effects. These types of equations are widely employed to represent dynamic systems that undergo abrupt and discontinuous changes during their evolution. The theory surrounding such equations has seen significant advancement, with numerous key books and research articles available (see [24-31]). Recent years have witnessed considerable development in the analysis and utilization of impulsive dynamic equations (see [32, 34-37]). An important aspect in the analysis of such dynamic systems is the study of Ulam stability, which provides a foundational framework for understanding the behavior of approximate solutions in relation to exact ones. Ulam-type stability is essential for evaluating the robustness of mathematical models under small perturbations, especially in impulsive systems where discontinuities are intrinsic.

In [40], authors introduced the Ulam stability for the following equations:

$$\begin{aligned} &\zeta^{\Delta}(\mathfrak{a}) + \omega(\mathfrak{a})\zeta^{\sigma}(\mathfrak{a}) = \mathfrak{h}\big(\mathfrak{a}, \zeta(\mathfrak{a})\big),\\ &\zeta(\mathfrak{a}_i^+) - \zeta(\mathfrak{a}_i^-) = \mathrm{I}_i\big(\zeta(\mathfrak{a}_i^-)\big),\\ &\zeta(\mathfrak{a}_0) = \mathrm{A} \in \mathbb{R}. \end{aligned} \quad \textbf{(1.8)}$$

In [44], the authors investigated stability properties, including existence, uniqueness, and various forms of Ulam-type stability for the following equations:

$$\begin{aligned} &\zeta^{\Delta}(\mathfrak{a}) + \omega(\mathfrak{a})\zeta^{\sigma}(\mathfrak{a}) = \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} \mathfrak{h}\big(\mathfrak{s}, \zeta(\mathfrak{s})\big)\Delta\mathfrak{s} + \int_{a}^{b} \mathfrak{h}\big(\mathfrak{s}, \zeta(\mathfrak{s})\big)\Delta\mathfrak{s},\\ &\zeta(\mathfrak{a}_i^+) - \zeta(\mathfrak{a}_i^-) = \mathrm{I}_i\big(\zeta(\mathfrak{a}_i^-)\big),\ \zeta(\mathfrak{a}_0) = \mathrm{A}. \end{aligned} \quad \textbf{(1.9)}$$

Motivated by these considerations, the present work aims to examine the Hyers-Ulam and Hyers-Ulam-Rassias stability of first-order nonlinear impulsive dynamic equations defined on a time scale $\mathbb{T}$, described by the following system:

$$\begin{aligned} &\zeta^{\Delta}(\mathfrak{a}) - \omega(\mathfrak{a})\zeta(\mathfrak{a}) = \mathfrak{h}\big(\mathfrak{a}, \zeta(\mathfrak{a})\big), \quad \mathfrak{a} \in \mathbb{S}^k \backslash \{\mathfrak{a}_i\}, i \in \mathcal{N} = \\ &\{1,2,\ldots\ldots,m\} \subset \mathbb{N}, \zeta(\mathfrak{a}_i^+) - \zeta(\mathfrak{a}_i^-) = \mathrm{I}_i\big(\zeta(\mathfrak{a}_i^-)\big),\\ &\zeta(\mathfrak{a}_0) = \mathrm{A} \in \mathbb{R}, \end{aligned} \quad \textbf{(1.10)}$$

where $\mathbb{S} := [\mathfrak{a}_0, \mathrm{T}]_{\mathbb{T}}$, $0 \leq \mathfrak{a}_0 < \mathrm{T} < \infty$, and $\zeta: \mathbb{S} \to \mathbb{R}$ is the unknown function. The coefficient $\omega: \mathbb{T} \to \mathbb{R}$ is rd-continuous and positively regressive, and $\mathfrak{h}: \mathbb{S} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous in the first variable and continuous in its second. The points $\{\mathfrak{a}_i\}_{i \in \mathcal{N}} \subset \mathbb{S}$, with $\mathfrak{a}_0 < \mathfrak{a}_i < \mathfrak{a}_{i+1} < \mathrm{T}$, indicates known impulse moments. The limits $\zeta(\mathfrak{a}_i^+) = \lim_{b \to 0^+} \zeta(\mathfrak{a}_i + b)$ and $\zeta(\mathfrak{a}_i^-) = \lim_{b \to 0^-} \zeta(\mathfrak{a}_i - b)$ denote the right and left limits of ζ at $\mathfrak{a}_i$, with $\zeta(\mathfrak{a}_i^+) = \zeta(\mathfrak{a}_i)$ if $\mathfrak{a}_i$ is right-scattered and $\zeta(\mathfrak{a}_i^-) = \zeta(\mathfrak{a}_i)$ if $\mathfrak{a}_i$ is left-scattered. Here, ζ^{Δ} denotes the delta derivative, and $\mathrm{I}_i: \mathbb{R} \to \mathbb{R}$ characterizes the discontinuity of ζ at $\mathfrak{a}_i$.

The structure of the paper is as follows: Section 2 outlines the fundamental definitions and key concepts needed for the analysis. Section 3 provides auxiliary results that will be utilized in deriving the main findings. Section 4 focuses on examining the stability properties of (1.1) over finite intervals of the time scale. Lastly, Section 5 concludes with an illustrative example demonstrating the theoretical results.

2. Preliminaries

This section recalls key results from time scale calculus that will be used throughout this work (see [41, 42]).

Definition 2.1 A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. For any $\mathfrak{q} \in \mathbb{T}$, the following are defined:

- The forward jump operator: $\sigma(\mathfrak{q}) = \inf\{z \in \mathbb{T}: z > \mathfrak{q}\}$;
- The backward jump operator: $\rho(\mathfrak{q}) = \sup\{z \in \mathbb{T}: z < \mathfrak{q}\}$;
- The graininess function: $\mu(\mathfrak{q}) = \sigma(\mathfrak{q}) - \mathfrak{q}$.

Definition 2.2 A point $\mathfrak{q} \in \mathbb{T}$ with $\inf\mathbb{T} < \mathfrak{q} < \sup\mathbb{T}$ is classified as follows:

- Right-scattered if $\sigma(\mathfrak{q}) > \mathfrak{q}$;
- Left-scattered if $\rho(\mathfrak{q}) < \mathfrak{q}$;
- Right-dense if $\sigma(\mathfrak{q}) = \mathfrak{q}$;
- Left-dense if $\rho(\mathfrak{q}) = \mathfrak{q}$.

Definition 2.3 A function $Q: \mathbb{T} \to \mathbb{R}$ is rd-continuous, written as $Q \in C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at right-dense points and has left limits at left-dense points of $\mathbb{T}$.

Definition 2.4 For a time scale $\mathbb{T}$, define

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash\{B\}, & \text{if } \mathbb{T} \text{ has a left} - \text{scattered maximum } B, \\ \mathbb{T}, & \text{Otherwise.} \end{cases}$$

Definition 2.5 Assume $Q: \mathbb{T} \to \mathbb{R}$ and $\varsigma \in \mathbb{T}^{\kappa}$. The delta derivative $Q^{\Delta}(\varsigma)$ is defined (if it exists) as the number satisfying: $\forall\delta > 0$, $\exists$ $\Lambda = (\varsigma - \varrho, \varsigma + \varrho) \cap \mathbb{T}$, $\varrho > 0$ a neighborhood, such that

$$\left|Q\big(\sigma(\varsigma)\big) - Q(z) - Q^{\Delta}(\varsigma)[\sigma(\varsigma) - z]\right| \leq \kappa|\sigma(\varsigma) - z| \quad \forall z \in \Lambda.$$

We say Q is delta differentiable on $\mathbb{T}^{\kappa}$ if $Q^{\Delta}(\varsigma)$ exists for all $\varsigma \in \mathbb{T}^{\kappa}$.

Definition 2.6 A function $\omega: \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(\varsigma)\omega(\varsigma) \neq 0 \quad \forall\varsigma \in \mathbb{T}^{\kappa}$.

Denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ the set of all rd-continuous regressive functions and by $\mathcal{R}^+$ the subset of functions that are both positively regressive and rd-continuous.

Definition 2.7 For $\omega \in \mathcal{R}$, the exponential function $e_{\omega}(\varsigma, z)$ on $\mathbb{T}$ is given as

$$e_{\omega}(\varsigma, z) := \begin{cases} \exp\left(\int_z^{\varsigma} \frac{\log|1 + \mu(\iota)\omega(\iota)|}{\mu(\iota)} \Delta\iota\right), & \text{if} \mu(\iota) \neq 0, \\ \exp\left(\int_z^{\varsigma} \omega(\iota)\, \Delta\iota\right), & \text{if} \mu(\iota) = 0. \end{cases}$$

For $\omega, \lambda \in \mathcal{R}$, define the operations:

$$\omega \oplus \lambda = \omega + \lambda + \mu\omega\lambda, \quad \ominus\, \omega = \frac{-\omega}{1 + \mu\omega},$$
$$\omega \ominus \lambda = \omega \oplus (\ominus\, \lambda).$$

Theorem 2.1 Let $\omega \in \mathcal{R}$ and $\varsigma, \iota, z \in \mathbb{T}$. Then

i. $e_{\omega}(\varsigma, z) = \frac{1}{e_{\omega}(z,\varsigma)} = e_{\ominus\omega}(z, \varsigma)$;

ii. $e_{\omega}(\varsigma, z)e_{\omega}(z, \iota) = e_{\omega}(\varsigma, \iota)$;

iii. $e_{\omega}(\sigma(\varsigma), z) = (1 + \mu(\varsigma)\omega(\varsigma))e_{\omega}(\varsigma, z)$;

iv. $e_{\omega}(\varsigma, \sigma(z)) = \frac{e_{\omega}(\varsigma,z)}{1+\mu(z)\omega(z)}$;

v. $(e_{\omega}(., z))^{\Delta} = \omega e_{\omega}(., z)$;

vi. $(e_{\omega}(\varsigma, .))^{\Delta} = (\ominus\, \omega)e_{\omega}(\varsigma, .)$.

Now, let $\mathcal{C}(\mathbb{S}, \mathbb{R})$ be the Banach space of all continuous functions $\zeta: \mathbb{S} \to \mathbb{R}$ with
$\| \zeta \| := \sup_{\mathfrak{a}\in\mathbb{S}}|\zeta(\mathfrak{a})|$.

For each $i \in \mathcal{N}$, let

$J_{\circ} := [\mathfrak{a}_0, \mathfrak{a}_1]$ and $J_i := (\mathfrak{a}_i, \mathfrak{a}_{i+1}]$.

Define the following sets:

$\mathcal{PC}(\mathbb{S}, \mathbb{R}) := \{\zeta: \zeta \in \mathcal{C}(J_i, \mathbb{R})$ and $\zeta(\mathfrak{a}_i^+), \zeta(\mathfrak{a}_i^-)$ exist with $\zeta(\mathfrak{a}_i^-) = \zeta(\mathfrak{a}_i), i \in \mathcal{N}\}$,
And
$\mathcal{PC}^1(\mathbb{S}, \mathbb{R}) := \{\zeta \in \mathcal{PC}(\mathbb{S}, \mathbb{R}): \zeta^{\Delta} \in \mathcal{PC}(\mathbb{S}, \mathbb{R})\}$.

Clearly, both $\mathcal{PC}$ and $\mathcal{PC}^1$ constitute a Banach space with norms:
$\| \zeta \|_{\mathcal{PC}} := \max_{i\in \mathcal{N}}\{\| \zeta_i \|\}$, where $\| \zeta \|_i = \sup_{\mathfrak{a}\in J_i}|\zeta(\mathfrak{a})|$
and $\| \zeta \|_{\mathcal{PC}^1} := \max\{\| \zeta \|_{\mathcal{PC}}, \| \zeta^{\Delta} \|_{\mathcal{PC}}\}$.

Definition 2.8 A function $\zeta \in \mathcal{PC}^1$ is considered a solution of (1.10), if it satisfies:

$$\zeta^{\Delta}(\mathfrak{a}) - \omega(\mathfrak{a})\zeta(\mathfrak{a}) = \mathfrak{h}(\mathfrak{a}, \zeta(\mathfrak{a})), \ \mathfrak{a} \in \mathbb{S}^k\backslash\{\mathfrak{a}_i\}, i \in \mathcal{N}, \quad \textbf{(2.1)}$$

$$\zeta(\mathfrak{a}_i^+) - \zeta(\mathfrak{a}_i^-) = I_i(\zeta(\mathfrak{a}_i^-)), \quad \zeta(\mathfrak{a}_{\circ}) = A. \quad \textbf{(2.2)}$$

Definition 2.9 Equation (1.10) has Hyers-Ulam stability (HUS) if $\exists$ $K_{\mathfrak{h},\mathcal{N}} > 0$ such that for any $\delta > 0$, every $\xi \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ satisfying

$$|\xi^{\Delta}(\mathfrak{a}) - \omega(\mathfrak{a})\xi(\mathfrak{a}) - \mathfrak{h}(\mathfrak{a}, \xi(\mathfrak{a}))| \leq \delta, \quad \mathfrak{a} \in \mathbb{S}^k\backslash\{\mathfrak{a}_i\}, \quad \textbf{(2.3)}$$

$$|\xi(\mathfrak{a}_i^+) - \xi(\mathfrak{a}_i^-) - I_i(\xi(\mathfrak{a}_i^-)| \leq \delta, \quad i \in \mathcal{N}, \quad \textbf{(2.4)}$$

has a solution $\zeta \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ to (1.10) such that

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq K_{\mathfrak{h},\mathcal{N}}\delta, \quad \mathfrak{a} \in \mathbb{S}. \quad \textbf{(2.5)}$$

Here, $K_{\mathfrak{h},\mathcal{N}}$ is referred to as the HUS constant.

Definition 2.10 Equation (1.10) has generalized Hyers-Ulam stability if $\exists$ $\Xi_{\mathfrak{h},\mathcal{N}} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\Xi_{\mathfrak{h}}(0) = 0$, such that for any $\delta > 0$, every $\xi \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ satisfying

$$|\xi^{\Delta}(\mathfrak{a}) - \omega(\mathfrak{a})\xi(\mathfrak{a}) - \mathfrak{h}(\mathfrak{a}, \xi(\mathfrak{a}))| \leq \delta, \quad \mathfrak{a} \in \mathbb{S}^k\backslash\{\mathfrak{a}_i\}, \quad \textbf{(2.6)}$$

$$|\xi(\mathfrak{a}_i^+) - \xi(\mathfrak{a}_i^-) - I_i(\xi(\mathfrak{a}_i^-)| \leq \delta, \quad i \in \mathcal{N}, \quad \textbf{(2.7)}$$

has a solution $\zeta \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ of (1.10) such that

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq \Xi_{\mathfrak{h},\mathcal{N}}(\delta), \quad \mathfrak{a} \in \mathbb{S}. \quad \textbf{(2.8)}$$

Definition 2.11 Equation (1.10) has Hyers-Ulam-Rassias stability (HURS) with respect to (Ω,Υ) if $\exists\ K_{\mathfrak{h},\mathcal{N},\Omega}>0$ such that for any nondecreasing $\Omega\in\mathcal{PC}^1(\mathbb{S},\mathbb{R}^+)$, $\delta>0$, and $\Upsilon\geq 0$, every $\xi\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ satisfying

$$\left|\xi^\Delta(\mathfrak{a})-\omega(\mathfrak{a})\xi(\mathfrak{a})-\mathfrak{h}(\mathfrak{a},\xi(\mathfrak{a}))\right|\leq\delta\Omega(\mathfrak{a}),\quad \mathfrak{a}\in\mathbb{S}^k\backslash\{\mathfrak{a}_i\},\tag{2.9}$$

$$|\xi(\mathfrak{a}_i^+)-\xi(\mathfrak{a}_i^-)-I_i(\xi(\mathfrak{a}_i^-)|\leq\delta\Upsilon,\quad i\in\mathcal{N},\tag{2.10}$$

admits a solution $\zeta\in\mathcal{PC}^1(\mathbb{J},\mathbb{R})$ of (1.10) such that

$$|\xi(\mathfrak{a})-\zeta(\mathfrak{a})|\leq K_{\mathfrak{h},\mathcal{N},\Omega}\,\delta(\Omega(\mathfrak{a})+\Upsilon),\quad \mathfrak{a}\in\mathbb{S}.\tag{2.11}$$

Here, $K_{\mathfrak{h},\mathcal{N},\Omega}$ is referred to as the HURS constant.

Definition 2.12 Equation (1.10) has generalized Hyers-Ulam-Rassias stability (GHURS) with respect to (Ω,Υ), if $\exists\ K_{\mathfrak{h},\mathcal{N},\Omega}>0$ such that for any nondecreasing $\Omega\in\mathcal{PC}^1(\mathbb{S},\mathbb{R}^+)$, and $\Upsilon\geq 0$, every $\xi\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ satisfying

$$\left|\xi^\Delta(\mathfrak{a})-\omega(\mathfrak{a})\xi(\mathfrak{a})-\mathfrak{h}(\mathfrak{a},\xi(\mathfrak{a}))\right|\leq\Omega(\mathfrak{a}),\quad \mathfrak{a}\in\mathbb{S}^k\backslash\{\mathfrak{a}_i\},\tag{2.12}$$

$$|\xi(\mathfrak{a}_i^+)-\xi(\mathfrak{a}_i^-)-I_i(\xi(\mathfrak{a}_i^-)|\leq\Upsilon,\quad i\in\mathcal{N},\tag{2.13}$$

admits a solution $\zeta\in\mathcal{PC}^1(\mathbb{J},\mathbb{R})$ of (1.1) such that

$$|\xi(\mathfrak{a})-\zeta(\mathfrak{a})|\leq K_{\mathfrak{h},\mathcal{N},\Omega}(\Omega(\mathfrak{a})+\Upsilon),\quad \mathfrak{a}\in\mathbb{S}.\tag{2.14}$$

Here, $K_{\mathfrak{h},\mathcal{N},\Omega}$ is referred to as the GHURS constant.

Remark 2.1 A function $\xi\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ satisfies (2.3) if and only if there exist $g\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ and $\{g_i\}_{i\in\mathcal{N}}$, both depending on ξ, such that:

$$|g(\mathfrak{a})|\leqslant\delta,\ \forall\mathfrak{a}\in\mathbb{S}\text{ and }|g_i|\leqslant\delta.\tag{2.15}$$

$$\xi^\Delta(\mathfrak{a})-\omega(\mathfrak{a})\xi(\mathfrak{a})=\mathfrak{h}(\mathfrak{a},\xi(\mathfrak{a}))+g(\mathfrak{a}),\ \forall\mathfrak{a}\in\mathbb{S}^k\backslash\{\mathfrak{a}_i\}.\tag{2.16}$$

$$\xi(\mathfrak{a}_i^+)-\xi(\mathfrak{a}_i^-)=I_i(\xi(\mathfrak{a}_i^-)+g_i.\tag{2.17}$$

Remark 2.2 A function $\xi\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ satisfies (2.9) if and only if there exist $w\in\mathcal{PC}^1(\mathbb{S},\mathbb{R})$ and $\{w_i\}_{i\in\mathcal{N}}$, both depending on ξ, such that:

$$|w(\mathfrak{a})|\leqslant\delta\Omega(\mathfrak{a}),\ \forall\mathfrak{a}\in\mathbb{S}\text{ and }|w_i|\leqslant\delta\Psi.\tag{2.18}$$

$$\xi^\Delta(\mathfrak{a})-\omega(\mathfrak{a})\xi(\mathfrak{a})=\mathfrak{h}(\mathfrak{a},\xi(\mathfrak{a}))+w(\mathfrak{a}),\ \mathfrak{a}\in\mathbb{S}^k\backslash\{\mathfrak{a}_i\}.\tag{2.19}$$

$$\xi(\mathfrak{a}_i^+)-\xi(\mathfrak{a}_i^-)=I_i(\xi(\mathfrak{a}_i^-)+w_i.\tag{2.20}$$

The inequalities (2.6) and (2.12) can be treated using similar arguments.

3. Auxiliary Result

In the following lemma, we derive the solution of Equation (1.10) in the absence of impulsive effects.

Lemma 3.1 Let $\mathfrak{a}_\circ\in\mathbb{T}$, $A\in\mathbb{R}$, $\omega\in\mathcal{R}(\mathbb{S},\mathbb{R})$, and $\mathfrak{h}\in C_{rd}(\mathbb{S}\times\mathbb{R},\mathbb{R})$. Then the solution ζ to the initial-value problem

$$\begin{aligned}&\zeta^\Delta(\mathfrak{a})-\omega(\mathfrak{a})\zeta(\mathfrak{a})=\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a})),\quad \mathfrak{a}\in\mathbb{S},\\&\zeta(\mathfrak{a}_\circ)=A.\end{aligned}\tag{3.1}$$

can be written as

$$\zeta(\mathfrak{a})=Ae_\omega(\mathfrak{a},\mathfrak{a}_\circ)+\int_{\mathfrak{a}_\circ}^{\mathfrak{a}}e\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z.\tag{3.2}$$

Proof. Equation (3.1) can be reformulated as

$$\zeta^\Delta(\mathfrak{a})=\omega(\mathfrak{a})[\zeta^\sigma(\mathfrak{a})-\mu(\mathfrak{a})\zeta^\Delta(\mathfrak{a})]+\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a})).\tag{3.3}$$

Hence,

$$\zeta^\Delta(\mathfrak{a})[1+\omega(\mathfrak{a})\mu(\mathfrak{a})]=\omega(\mathfrak{a})\zeta^\sigma(\mathfrak{a})+\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a})),\tag{3.4}$$

so we can write

$$\zeta^\Delta(\mathfrak{a})+(\ominus\,\omega)(\mathfrak{a})\zeta^\sigma(\mathfrak{a})=\frac{\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a}))}{1+\omega(\mathfrak{a})\mu(\mathfrak{a})}.\tag{3.5}$$

Multiplying (3.5) by $e_{\ominus\omega}(\mathfrak{a},\mathfrak{a}_0)$, we get

$$(\zeta e_{\ominus\omega}(\cdot,\mathfrak{a}_0))^\Delta(\mathfrak{a})=e_{\ominus\omega}(\mathfrak{a},\mathfrak{a}_0)\frac{\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a}))}{1+\mu(\mathfrak{a})\omega(\mathfrak{a})}.\tag{3.6}$$

Integrating (3.6) from $\mathfrak{a}_0$ to $\mathfrak{a}$ yields

$$\begin{aligned}&\zeta(\mathfrak{a})e_{\ominus\omega}(\mathfrak{a},\mathfrak{a}_0)-\zeta(\mathfrak{a}_0)e_{\ominus\omega}(\mathfrak{a}_0,\mathfrak{a}_0)=\\&\int_{\mathfrak{a}_\circ}^{\mathfrak{a}}e_{\ominus\omega}(z,\mathfrak{a}_0)\frac{\mathfrak{h}(z,\zeta(z))}{1+\mu(z)\omega(z)}\Delta z.\end{aligned}\tag{3.7}$$

Multiplying (3.7) by $e_\omega(\mathfrak{a},\mathfrak{a}_0)$ yields

$$\zeta(\mathfrak{a})=Ae_\omega(\mathfrak{a},\mathfrak{a}_0)+\int_{\mathfrak{a}_\circ}^{\mathfrak{a}}e_\omega(\mathfrak{a},z)\frac{\mathfrak{h}(z,\zeta(z))}{1+\mu(z)\omega(z)}\Delta z.\tag{3.8}$$

From the properties of the exponential function, it follows that

$$\zeta(\mathfrak{a})=Ae_\omega(\mathfrak{a},\mathfrak{a}_0)+\int_{\mathfrak{a}_\circ}^{\mathfrak{a}}e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z.\tag{3.9}$$

Remark 3.1 Using Lemma 3.1, the solution of Equation (1.10) in the presence of impulses can be represented by

$$\begin{aligned}\zeta(\mathfrak{a})=\ &Ae_\omega(\mathfrak{a},\mathfrak{a}_0)+\int_{\mathfrak{a}_\circ}^{\mathfrak{a}}e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z+\\&\sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}}e_\omega(\mathfrak{a},\mathfrak{a}_i)\,I_i(\zeta(\mathfrak{a}_i^-)),\quad\forall\mathfrak{a}\in\mathbb{T}.\end{aligned}\tag{3.10}$$

This formula extends the solution in the non-impulsive case by adding the sum of terms representing the cumulative effect of the impulses at points $\mathfrak{a}_i$. Each impulse contributes a jump term weighted by the exponential function e_ω.

4. Main results

This section focuses on the examination of Ulam stability for the impulsive dynamic equation (1.10), where we begin by stating some essential assumptions:

(C_1) $\omega\in\mathcal{R}^+(\mathbb{S},\mathbb{R})$.

(C_2) $\mathfrak{h} \in C_{rd}(\mathbb{S} \times \mathbb{R}, \mathbb{R})$ satisfies a Lipschitz condition with $l_{\mathfrak{h}} \in \mathcal{C}(\mathbb{S}, \mathbb{R}^+)$, i.e.,

$$|\mathfrak{h}(\alpha, \iota) - \mathfrak{h}(\alpha, \tau)| \le l_{\mathfrak{h}}(\alpha)|\iota - \tau| \quad \forall \alpha \in \mathbb{S} \text{ and } \iota, \tau \in \mathbb{R}. \tag{4.1}$$

Define $l_{\mathfrak{h}}^* := \sup_{\alpha \in \mathbb{S}} l_{\mathfrak{h}}(\alpha)$.

(C_3) $I: \mathbb{R} \to \mathbb{R}$ is such that

$$\left|I_i\left(\iota(\alpha_i^-)\right) - I_i\left(\tau(\alpha_i^-)\right)\right| \le l_{I_i}|\iota - \tau| \quad \forall \iota, \tau \in \mathbb{R} \text{ and } i \in \mathcal{N}. \tag{4.2}$$

with $l_{I_i} > 0$.

(C_4) Provided that $\Omega \in \mathcal{PC}(\mathbb{S}, \mathbb{R})$ is a nondecreasing, there exists some $l_\Omega > 0$ such that

$$\int_{\alpha_\circ}^{\alpha} \Omega(z)\Delta z \le l_\Omega \Omega(\alpha) \quad \forall \alpha \in \mathbb{S}. \tag{4.3}$$

(C_5) Let

$$0 < E_\omega = \sup_{z,\alpha \in J} |e_\omega(\alpha, \sigma(z))| < \infty. \tag{4.4}$$

(C_6) Let

$$0 < E_{\omega_i} = \sum_{\alpha_0 < \alpha_i < \alpha} |e_\omega(\alpha, \alpha_i)| < \infty. \tag{4.5}$$

Theorem 4.1 Consider the equation (1.10). Under assumptions $(C_1) - (C_6)$, the following hold:

i. If $E_\omega l_{\mathfrak{h}}^*(T - \alpha_\circ) + E_{\omega_i} \sum_{i=1}^{m} l_{I_i} < 1$, then (1.10) possesses a unique solution $\zeta \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ with initial condition $\zeta(\alpha_\circ) = A$ for any $A \in \mathbb{R}$.

ii. Equation (1.10) has Hyers-Ulam stability, and the HUS constant is $K_{\mathfrak{h},\mathcal{N}} = [E_\omega(T - \alpha_\circ) + E_{\omega_i} m] \prod_{i \in \mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T, \alpha_\circ)$.

iii. Equation (1.10) possesses Hyers-Ulam-Rassias stability with respect to (Ω, Υ), and the HURS constant is $K_{\mathfrak{h},\mathcal{N},\Omega} := (E_\omega l_\Omega + E_{\omega_i} m) \prod_{i \in \mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T, \alpha_\circ)$.

Proof. i. Let $A \in \mathbb{R}$ be fixed, and provide a definition of the operator $G: \mathcal{PC}^1(\mathbb{S}, \mathbb{R}) \to \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ by

$$G[\zeta](\alpha) := e_\omega(\alpha, \alpha_\circ)A + \int_{\alpha_\circ}^{\alpha} e_\omega(\alpha, \sigma(z))\mathfrak{h}(z, \zeta(z))\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} e_\omega(\alpha, \alpha_i) I_i(\zeta(\alpha_i^-)). \tag{4.6}$$

Based on Remark 3.1, the fixed points of G can be identified as the solutions of (1.10). We proceed to verify the existence of a fixed point by employing the contraction mapping principle. For any $\zeta, \xi \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$, it follows that

$$|G[\zeta](\alpha) - G[\xi](\alpha)| \le |e_\omega(\alpha, \alpha_\circ)||A - A| + \int_{\alpha_\circ}^{\alpha} |e_\omega(\alpha, \sigma(z))||\mathfrak{h}(z, \zeta(z)) - \mathfrak{h}(z, \xi(z))|\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} |e_\omega(\alpha, \alpha_i)||I_i(\zeta(\alpha_i^-)) - I_i(\xi(\alpha_i^-))| \tag{4.7}$$

$$\overset{(C_2),(C_3)}{\le} E_\omega \int_{\alpha_\circ}^{\alpha} l_{\mathfrak{h}}(\alpha)|\zeta(z) - \xi(z)|\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} |e_\omega(\alpha, \alpha_i)| l_{I_i} |(\zeta(\alpha_i^-)) - (\xi(\alpha_i^-))| \tag{4.8}$$

$$\le (E_\omega l_{\mathfrak{h}}(\alpha)(T - \alpha_\circ) + E_{\omega_i} \sum_{\alpha_\circ < \alpha_i < \alpha} l_{I_i}) \parallel \zeta - \xi \parallel_{\mathcal{PC}^1}. \tag{4.9}$$

Hence, for all $\zeta, \xi \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$, we get

$$\parallel G[\zeta] - G[\xi] \parallel_{\mathcal{PC}^1} \le (E_\omega l_{\mathfrak{h}}^*(T - \alpha_\circ) + E_{\omega_i} \sum_{i=1}^{m} l_{I_i}) \parallel \zeta - \xi \parallel_{\mathcal{PC}^1}. \tag{4.10}$$

Since

$$(E_\omega l_{\mathfrak{h}}^*(T - \alpha_\circ) + E_{\omega_i} \sum_{i=1}^{m} l_{I_i}) < 1, \tag{4.11}$$

then the operator G is a contraction on $\mathcal{PC}^1(\mathbb{S}, \mathbb{R})$. Accordingly, G possesses a unique fixed point $\zeta^* \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$, which is the unique solution of (1.1) satisfying $\zeta^*(\alpha_\circ) = A$.

ii. Let $\xi \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ fulfill (2.3) and let $\zeta \in \mathcal{PC}^1(\mathbb{S}, \mathbb{R})$ represent the unique solution of (1.10) with $\zeta(\alpha_\circ) = \xi(\alpha_\circ) = A$. By assumption (C_1) and Remark 3.1, ζ can be expressed as

$$\zeta(\alpha) = e_\omega(\alpha, \alpha_\circ)\xi(\alpha_\circ) + \int_{\alpha_\circ}^{\alpha} e_\omega\left(\alpha, \sigma(z)\right)\mathfrak{h}\left(z, \zeta(z)\right)\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} e_\omega(\alpha, \alpha_i) I_i(\zeta(\alpha_i^-)), \quad \forall \varsigma \in \mathbb{S}. \tag{4.12}$$

Now, since ξ fulfills (2.3), by Remark 2.1 it is possible to write

$$\xi^\Delta(\alpha) - \omega(\alpha)\xi(\alpha) = \mathfrak{h}(\alpha, \xi(\alpha)) + g(\alpha), \quad \forall \alpha \in \mathbb{S}^k \backslash \{\alpha_i\}, \tag{4.13}$$

and

$$\xi(\alpha_i^+) - \xi(\alpha_i^-) = I_i(\xi(\alpha_i^-)) + g_i, \quad i \in \mathcal{N}, \tag{4.14}$$

where

$$|g(\alpha)| \le \delta, \quad \forall \alpha \in \mathbb{S} \text{ and } |g_i| \le \delta, \quad \forall i \in \mathcal{N}. \tag{4.15}$$

Thus

$$\xi(\alpha) = e_\omega(\alpha, \alpha_\circ)\xi(\alpha_\circ) + \int_{\alpha_\circ}^{\alpha} e_\omega\left(\alpha, \sigma(z)\right)\left[\mathfrak{h}\left(z, \xi(z)\right) + g(z)\right]\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} e_\omega(\alpha, \alpha_i)(I_i(\xi(\alpha_i^-)) + g_i) \tag{4.16}$$

$$= e_\omega(\alpha, \alpha_\circ)\xi(\alpha_\circ) + \int_{\alpha_\circ}^{\alpha} e_\omega(\alpha, \sigma(z))\mathfrak{h}(z, \xi(z))\Delta z + \int_{\alpha_\circ}^{\alpha} e_\omega(\alpha, \sigma(z))g(z)\Delta z + \sum_{\alpha_\circ < \alpha_i < \alpha} e_\omega(\alpha, \alpha_i)(I_i(\xi(\alpha_i^-)) + \sum_{\alpha_\circ < \alpha_i < \alpha} e_\omega(\alpha, \alpha_i) g_i. \tag{4.17}$$

This gives

$$|\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| \quad \textbf{(4.18)}$$

$$\leq \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} |e_\omega(\mathfrak{a},\sigma(z))||g(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} |e_\omega(\mathfrak{a},\mathfrak{a}_i)||g_i| \quad \textbf{(4.19)}$$

$$\leq E_\omega\delta \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} \Delta z + E\omega_i \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} |g_i| \quad \textbf{(4.20)}$$

$$\leq E_\omega\delta(\mathfrak{a}-\mathfrak{a}_\circ) + E\omega_i m\delta \quad \textbf{(4.21)}$$

$$\leq \delta[E_\omega(\mathfrak{a}-\mathfrak{a}_\circ) + E_{\omega_i}m]. \quad \textbf{(4.22)}$$

Now, for $\mathfrak{a} \in \mathbb{S}$, we may write

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| = |\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\zeta(\mathfrak{a}_i^-)) + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-)) - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| \quad \textbf{(4.23)}$$

$$\leq |\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| + |\sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\zeta(\mathfrak{a}_i^-)) - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| + |\int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z| \quad \textbf{(4.24)}$$

$$\overset{(4.22)}{\leq} \delta E_\omega(\mathfrak{a}-\mathfrak{a}_\circ) + E_{\omega_i}m] + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} |e_\omega(\mathfrak{a},\sigma(z))||\mathfrak{h}(z,\xi(z)) - \mathfrak{h}(z,\zeta(z))|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} |e_\omega(\mathfrak{a},\mathfrak{a}_i)||I_i(\zeta(\mathfrak{a}_i^-)) - I_i(\xi(\mathfrak{a}_i^-))| \quad \textbf{(4.25)}$$

$$\overset{(C_2),(C_3)}{\leq} \delta[E_\omega(\mathfrak{a}-\mathfrak{a}_\circ) + E_{\omega_i}m] + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} l_{\mathfrak{h}}(z)|e_\omega(\mathfrak{a},\sigma(z))||\xi(z) - \zeta(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} l_{I_i}|e_\omega(\mathfrak{a},\mathfrak{a}_i)||\zeta(\mathfrak{a}_i^-) - \xi(\mathfrak{a}_i^-)| \quad \textbf{(4.26)}$$

$$\leq \delta[E_\omega(\mathfrak{a}-\mathfrak{a}_\circ) + E_{\omega_i}m] + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} E_\omega l_{\mathfrak{h}}(z)|\xi(z) - \zeta(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} E_{\omega_i} l_{I_i}|\zeta(\mathfrak{a}_i^-) - \xi(\mathfrak{a}_i^-)|. \quad \textbf{(4.27)}$$

In light of [40, Theorem 3.1], it follows from (4.18) that

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq \delta[E_\omega(\mathfrak{a}-\mathfrak{a}_\circ) + E_{\omega_i}m] \prod_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}}(\mathfrak{a},\mathfrak{a}_\circ) \quad \textbf{(4.28)}$$

$$\leq \delta[E_\omega(T-\mathfrak{a}_\circ) + E_{\omega_i}m] \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ) \quad \textbf{(4.29)}$$

$$\leq \delta K_{\mathfrak{h},\mathcal{N}}, \quad \textbf{(4.30)}$$

Where $K_{\mathfrak{h},\mathcal{N}} = [E_\omega(T-\mathfrak{a}_\circ) + E_{\omega_i}m] \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ)$. Then (1.10) is Ulam-Hyers stable.

iii. Let $\xi \in \mathcal{PC}^1(\mathbb{S},\mathbb{R})$ fulfill (2.9), and let $\zeta \in \mathcal{PC}^1(\mathbb{S},\mathbb{R})$ represent the unique solution of (1.10) that satisfies $\zeta(\mathfrak{a}_\circ) = \xi(\mathfrak{a}_\circ) = A$. Then in view of (C_1), Remark 3.1 enables us to express

$$\zeta(\mathfrak{a}) = e_\omega(\mathfrak{a},\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\zeta(\mathfrak{a}_i^-)). \quad \textbf{(4.31)}$$

Given that, $\xi \in \mathcal{PC}^1(\mathbb{S},\mathbb{R})$ fulfills (2.5) by Remark 2.2, it is possible to write

$$\xi^\Delta(\mathfrak{a}) - \omega(\mathfrak{a})\xi(\mathfrak{a}) = \mathfrak{h}(\mathfrak{a},\xi(\mathfrak{a})) + w(\mathfrak{a}) \quad \forall \mathfrak{a} \in \mathbb{S}^k\backslash\{\mathfrak{a}_i\}, \quad \textbf{(4.32)}$$

and

$$\xi(\mathfrak{a}_i^+) - \xi(\mathfrak{a}_i^-) = I_i(\xi(\mathfrak{a}_i^-)) + w_i, \quad i \in \mathcal{N}, \quad \textbf{(4.33)}$$

Where

$$|w(\mathfrak{a})| \leq \delta\Omega(\mathfrak{a}), \quad \forall \mathfrak{a} \in \mathbb{S}, \text{ and } |w_i| \leq \delta\Upsilon, \quad \forall i \in \mathcal{N}, \quad \textbf{(4.34)}$$

so

$$\xi(\mathfrak{a}) = e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))[\mathfrak{h}(z,\xi(z)) + w(z)]\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)\big(I_i\big(\xi(\mathfrak{a}_i^-)\big) + w_i\big) \quad \textbf{(4.35)}$$

$$= e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))w(z)\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-)) + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)w_i. \quad \textbf{(4.36)}$$

so

$$|\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| \leq \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} |e_\omega(\mathfrak{a},\sigma(z))||w(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} |e_\omega(\mathfrak{a},\mathfrak{a}_i)||w_i| \quad \textbf{(4.37)}$$

$$\leq E_\omega\delta \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} \Omega(z)\Delta z + E_{\omega_i} \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} \delta\Upsilon \quad \textbf{(4.38)}$$

$$\overset{C_4}{\leq} E_\omega\delta l_\Omega\Omega(\varsigma) + mE_{\omega_i}\delta\Upsilon \quad \textbf{(4.39)}$$

$$= \delta(E_\omega l_\Omega\Omega(\varsigma) + E_{\omega_i}m\Upsilon). \quad \textbf{(4.40)}$$

Now, for $\mathfrak{a} \in \mathbb{S}$, we can write

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| = |\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega\big(\mathfrak{a},\sigma(z)\big)\mathfrak{h}\big(z,\zeta(z)\big)\Delta z + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega\big(\mathfrak{a},\sigma(z)\big)\mathfrak{h}\big(z,\xi(z)\big)\Delta z - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega\big(\mathfrak{a},\sigma(z)\big)\mathfrak{h}\big(z,\xi(z)\big)\Delta z - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\zeta(\mathfrak{a}_i^-)) + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-)) - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| \quad \textbf{(4.41)}$$

$$\leq |\xi(\mathfrak{a}) - e_\omega(\mathfrak{a},\mathfrak{a}_\circ)\xi(\mathfrak{a}_\circ) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega\big(\mathfrak{a},\sigma(z)\big)\mathfrak{h}\big(z,\xi(z)\big)\Delta z$$

$$-\sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| + |\sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\zeta(\mathfrak{a}_i^-)) - \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} e_\omega(\mathfrak{a},\mathfrak{a}_i)I_i(\xi(\mathfrak{a}_i^-))| + |\int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\xi(z))\Delta z - \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\mathfrak{h}(z,\zeta(z))\Delta z| \quad \textbf{(4.42)}$$

$$\overset{(4.40)}{\leq} \delta(E_\omega l_\Omega \Omega(\mathfrak{a}) + E_{\omega_i} m\Upsilon) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} |e_\omega(\mathfrak{a},\sigma(z))||\mathfrak{h}(z,\xi(z)) - \mathfrak{h}(z,\zeta(z))|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} |e_\omega(\mathfrak{a},\mathfrak{a}_i)||I_i(\zeta(\mathfrak{a}_i^-)) - I_i(\xi(\mathfrak{a}_i^-))| \quad \textbf{(4.43)}$$

$$\overset{C_2,C_3}{\leq} \delta(E_\omega l_\Omega \Omega(\mathfrak{a}) + E_{\omega_i} m\Upsilon) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} l_f(z)|e_\omega(\mathfrak{a},\sigma(z))||\xi(z) - \zeta(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} l_{I_i}|e_\omega(\mathfrak{a},\mathfrak{a}_i)||\zeta(\mathfrak{a}_i^-) - \xi(\mathfrak{a}_i^-)| \quad \textbf{(4.44)}$$

$$\leq \delta(E_\omega l_\Omega \Omega(\mathfrak{a}) + E_{\omega_i} m\Upsilon) + \int_{\mathfrak{a}_\circ}^{\mathfrak{a}} E_\omega l_{\mathfrak{h}}(z)|\zeta(z) - \xi(z)|\Delta z + \sum_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} E_{\omega_i} l_{I_i}|\zeta(\mathfrak{a}_i^-) - \xi(\mathfrak{a}_i^-)|. \quad \textbf{(4.45)}$$

According to [40, Theorem 3.1] we can write for all $\mathfrak{a} \geq \mathfrak{a}_\circ$

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq \delta(E_\omega l_\Omega \Omega(\mathfrak{a}) + E_{\omega_i} m\Upsilon) \prod_{\mathfrak{a}_\circ<\mathfrak{a}_i<\mathfrak{a}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}}(\mathfrak{a},\mathfrak{a}_\circ) \quad \textbf{(4.46)}$$

$$\leq (E_\omega l_\Omega + E_{\omega_i} m)(\Omega(\mathfrak{a}) + \Upsilon) \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ) \quad \textbf{(4.47)}$$

$$\leq K_{\mathfrak{h},\mathcal{N},\Omega}(\Omega(\mathfrak{a}) + \Upsilon)\delta, \quad \textbf{(4.48)}$$

where
$K_{\mathfrak{h},\mathcal{N},\Omega} := (E_\omega l_\Omega + E_{\omega_i} m) \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ)$,
which establishes that equation (1.10) exhibits Hyers-Ulam-Rassias stability relative to (Ω,Υ).

Corollary 4.1 Suppose the equation (1.10). In light of the assumptions $(C_1) - (C_6)$, equation (1.10) exhibits generalized Hyers-Ulam stability.

Proof. According to the proof of Theorem 4.1, we have HUS constant as

$$K_{\mathfrak{h},\mathcal{N}} = [E_\omega(T - \mathfrak{a}_\circ) + E_{\omega_i} m] \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ), \quad \textbf{(4.49)}$$

if we take $\Xi_{\mathfrak{h},\mathcal{N}} = \delta K_{\mathfrak{h},\mathcal{N}}$, so

$$\Xi_{\mathfrak{h},\mathcal{N}}(\delta) = \delta[E_\omega(T - \mathfrak{a}_\circ) + E_{\omega_i} m] \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ), \quad \textbf{(4.50)}$$

thereby concluding the proof.

Corollary 4.2 Suppose the equation (1.10). In light of the assumptions $(C_1) - (C_6)$, equation (1.10) exhibits generalized Hyers-Ulam-Rassias stability relative to (Ω,Υ), where the GHURS constant is given by

$$(E_\omega l_\Omega + E_{\omega_i} m) \prod_{i\in\mathcal{N}} (1 + E_{\omega_i} l_{I_i}) e_{E_\omega l_{\mathfrak{h}}^*}(T,\mathfrak{a}_\circ).$$

Proof. Choosing $\delta = 1$ in the proof of part iii of Theorem 4.1 immediately yields the stated conclusion.

5. Application

This section presents an example to demonstrate the main results obtained in our study.

Example 5.1. Consider $\mathbb{T} = [0,2] \cup [3,4]$ and $\mathfrak{a}_\circ = 0$, $T = 4$, $\mathfrak{a}_1 = \frac{7}{2}$, and $\mathfrak{a}_2 = 3$. Then, take $\mathbb{S} := [0,4]_{\mathbb{T}}$. Let us study the impulsive dynamic problem

$$\zeta^\Delta(\mathfrak{a}) - \zeta(\mathfrak{a}) = \frac{1}{3e^9}(\zeta^2(\mathfrak{a}) + 3)^{\frac{1}{2}} + \mathfrak{a},$$
$$\mathfrak{a} \in [0,4]\mathbb{T}^k \backslash \{\mathfrak{a}_1,\mathfrak{a}_2\}, \qquad \zeta(\mathfrak{a}_k^+) - \zeta(\mathfrak{a}_k^-) = \frac{1}{5}\zeta(\mathfrak{a}_k^-),$$
$$k = 1,2, \zeta(0) = 0, \quad \textbf{(5.1)}$$

as well as its related inequality

$$|\xi^\Delta(\mathfrak{a}) - \xi(\mathfrak{a}) - \tfrac{1}{3e^9}(\xi^2(\mathfrak{a}) + 3)^{\frac{1}{2}} - \mathfrak{a}| \leq \epsilon, \mathfrak{a} \in [0,4]_{\mathbb{T}}^k \backslash \{\mathfrak{a}_1,\mathfrak{a}_2\},$$
$$|\xi(\mathfrak{a}_k^+) - \xi(\mathfrak{a}_k^-) - \tfrac{1}{5}\xi(\mathfrak{a}_k^-)| < \epsilon, \quad k = 1,2. \quad \textbf{(5.2)}$$

Here, $\omega(\mathfrak{a}) \equiv 1$, for which $1 + \mu(\mathfrak{a})\omega(\mathfrak{a}) > 0$, $\mathfrak{h}(\mathfrak{a},\zeta(\mathfrak{a})) = \frac{1}{3e^9}(\zeta^3(\mathfrak{a}) + 3)^{\frac{1}{2}} + \mathfrak{a}$, that verifies (C_2) with $l_{\mathfrak{h}}^* = \frac{1}{3e^9}$ and $I_k(\zeta(\mathfrak{a}_k^-)) = \frac{1}{5}\zeta(\mathfrak{a}_k^-)$ that verifies (C_3) with $l_{I_k} = \frac{1}{5}$. With these values, we obtain

$$E_\omega = \sup_{z,\mathfrak{a}\in[0,4]_{\mathbb{T}}} |e_\omega(\mathfrak{a},\sigma(z))| = e^4, \quad \textbf{(5.3)}$$

$$e_\omega(T,\mathfrak{a}_1) = e_\omega(4,\tfrac{7}{2}) = \sqrt{e}, \quad \textbf{(5.4)}$$

$$e_\omega(T,\mathfrak{a}_2) = e_\omega(4,3) = e. \quad \textbf{(5.5)}$$

This leads to

$$E_\omega l_{\mathfrak{h}}^*(T - \mathfrak{a}_\circ) + |e_\omega(T,\mathfrak{a}_1)|l_{I_1} + |e_\omega(T,\mathfrak{a}_2)|l_{I_2} = e^4 \frac{1}{3e^9}(4 - 0) + \frac{\sqrt{e}}{5} + \frac{e}{5} = \frac{4}{3e^5} + \frac{\sqrt{e}}{5} + \frac{e}{5} < 1 \quad \textbf{(5.6)}$$

Therefore, all the assumptions of Theorem 4.1 are fulfilled. Hence, (5.1), possesses a unique solution, which can be represented as follows

$$\zeta(\mathfrak{a}) = \int_0^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))(\tfrac{1}{3e^9}(\zeta^2(\mathfrak{a}) + 3)^{\frac{1}{2}} + z)\Delta z + e_\omega(\mathfrak{a},\tfrac{7}{2})\frac{\zeta(\frac{7}{2}^-)}{5} + e_\omega(\mathfrak{a},3)\frac{\zeta(3^-)}{5}, \quad \mathfrak{a} \in [0,4]_{\mathbb{T}}^k. \quad \textbf{(5.7)}$$

Next, let $\xi \in \mathcal{PC}^1(\mathbb{S},\mathbb{R})$ act as a solution to (5.2). According to Remark 2.1, it follows that there exists $g \in \mathcal{PC}^1(\mathbb{S},\mathbb{R})$ and $g_1, g_2 \in \mathbb{R}$ with $|g(\mathfrak{a})| \leq \delta$ and $|g_1| \leq \delta$, $|g_2| \leq \delta$ such that

$$\xi^\Delta(\mathfrak{a}) - \xi(\mathfrak{a}) = \frac{1}{3e^9}(\xi^2(\mathfrak{a}) + 3)^{\frac{1}{2}} + \mathfrak{a} + g(\mathfrak{a}),$$
$$\mathfrak{a} \in [0,4]_{\mathbb{T}}^k\{\mathfrak{a}_1,\mathfrak{a}_2\}, \xi(\mathfrak{a}_k^+) - \xi(\mathfrak{a}_k^-) = \frac{\xi(\mathfrak{a}_k^-)}{5} + g_k,$$

$$k = 1,2. \quad \textbf{(5.8)}$$

In view of Remark 3.1, the unique solution of (5.8) is represented by

$$\xi(\mathfrak{a}) = \int_0^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\left(\frac{1}{3e^9}(\xi^2(z)+3)^{\frac{1}{2}} + z + g(z)\right)\Delta z$$
$$+e_\omega(\mathfrak{a},\tfrac{7}{2})\left(\frac{\xi(\frac{7}{2}^-)}{5} + g_1\right) + e_\omega(\mathfrak{a},3)\left(\frac{\xi(3^-)}{5} + g_2\right), \quad \mathfrak{a} \in [0,4]_{\mathbb{T}}^{k}. \quad \textbf{(5.9)}$$

Now, from (5.7) and (5.9), it follows that

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| = \Big| \int_0^{\mathfrak{a}} e_\omega(\mathfrak{a},\sigma(z))\Big[\Big(\frac{1}{3e^9}(\xi^2(z)+3)^{\frac{1}{2}} + z\Big) -$$
$$\Big(\frac{1}{3e^9}(\zeta^2(z)+3)^{\frac{1}{2}} + z\Big)\Big]\Delta z + e_1(\mathfrak{a},\tfrac{7}{2})\Big[\frac{\xi(\frac{7}{2}^-)}{5} - \frac{\zeta(\frac{7}{2}^-)}{5} + g_1\Big] +$$
$$e_1(\mathfrak{a},3)\Big[\frac{\xi(3^-)}{5} - \frac{\zeta(3^-)}{5} + g_2\Big] + \int_0^{\mathfrak{a}} e_1(\mathfrak{a},\sigma(z))g(z)\Delta z\Big| \quad \textbf{(5.10)}$$

$$\leq \int_0^{\mathfrak{a}} |e_1(\mathfrak{a},\sigma(z))|\frac{1}{3e^9}|\xi(z)-\zeta(z)|\Delta z + |e_1(\mathfrak{a},\tfrac{7}{2})||\xi(\tfrac{7}{2}^-) -$$
$$\zeta(\tfrac{7}{2}^-)| + |e_1(\mathfrak{a},3)||\xi(3^-) - \zeta(3^-)| + |e_1(\mathfrak{a},\tfrac{7}{2})||g_1| +$$
$$|e_1(\mathfrak{a},3)||g_2| + \int_0^{\mathfrak{a}} |e_1(\mathfrak{a},\sigma(z))||g(z)|\Delta z \quad \textbf{(5.11)}$$

$$\leq \int_0^{\mathfrak{a}} \frac{1}{3e^5}|\xi(z)-\zeta(z)|\Delta z + \sqrt{e}|\xi(\tfrac{7}{2}^-) - \zeta(\tfrac{7}{2}^-)| +$$
$$e|\xi(3^-) - \zeta(3^-)| + \delta\sqrt{e} + e\delta + \delta\int_0^{\mathfrak{a}} |e_\omega(\mathfrak{a},\sigma(z))|\Delta z \quad \textbf{(5.12)}$$

$$\leq \delta\sqrt{e} + e\delta + \delta e^4\mathfrak{a} + \int_0^{\mathfrak{a}} \frac{1}{3e^5}|\xi(z)-\zeta(z)|\Delta z +$$
$$\sqrt{e}|\xi(\tfrac{7}{2}^-) - \zeta(\tfrac{7}{2}^-)| + e|\xi(3^-) - \zeta(3^-)| \quad \textbf{(5.13)}$$

According to [40, Theorem 3.1], we have with $a(\mathfrak{a}) = (\sqrt{e} + e + e^4\mathfrak{a})\delta$, $\omega(z) = \frac{1}{3e^5}$, and $b_i = e^{\frac{i}{2}}$, $i = 1,2$, so we have that

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq \delta(\sqrt{e} + e + e^4\mathfrak{a})(1+\sqrt{e})(1 +$$
$$e)e_{\frac{1}{3e^5}}(\mathfrak{a},0), \quad \mathfrak{a} \geq 0. \quad \textbf{(5.14)}$$

Hence,

$$|\xi(\mathfrak{a}) - \zeta(\mathfrak{a})| \leq \delta(\sqrt{e} + e + 4e^4)(1+\sqrt{e})(1 +$$
$$e)e_{\frac{1}{3e^5}}(4,0), \quad \mathfrak{a} \in [0,4]_{\mathbb{T}}, \quad \textbf{(5.15)}$$

this implies that (5.1) possesses Hyers-Ulam stability with HUS constant $(\sqrt{e} + e + 4e^4)(1+\sqrt{e})(1+e)e_{\frac{1}{3e^5}}(4,0)$.

6. Conclusion

This work presents a thorough investigation into Ulam-type stability with respect to a class of first-order nonlinear dynamic equations with impulses, defined over finite intervals of time scales. By employing fixed point theory particularly the Banach contraction principle alongside generalized integral inequality techniques on, we establish the existence as well as the uniqueness of solutions under appropriate conditions. Additionally, we derive explicit estimates that describe the stability behavior of approximate solutions relative to exact ones. These results offer a solid theoretical foundation for understanding the stability characteristics of impulsive dynamic systems and contribute valuable analytical tools for investigating hybrid models that exhibit both continuous and discrete dynamics with instantaneous changes.

Future research could extend these findings in several directions. One promising avenue is the study of impulsive dynamic systems with time delays or state-dependent impulses, which frequently arise in biological and engineering applications. Another direction involves exploring Ulam-type stability in the context of Banach and Hilbert spaces, allowing the treatment of infinite-dimensional systems such as partial differential equations with impulses. Furthermore, the development of numerical algorithms that effectively capture the stability properties established in theory would bridge the gap between analytical results and computational practice. Finally, investigating the interplay between Ulam stability and other stability notions such as Lyapunov stability or Mittag-Leffler stability could lead to a more unified framework for analyzing complex dynamic systems on time scales.

Author Contributions

Conceptualization, resources, and methodology: G.A.M.E., H.M.R., A.A.-E.D., and A.A.S.Z.; investigation and supervision: H.M.R., A.A.-E.D., and A.A.S.Z.; data curation: G.A.M.E., H.M.R., A.A.-E.D., and A.A.S.Z.; writing – original draft preparation: G.A.M.E., H.M.R., and A.A.-E.D.; writing – review and editing: G.A.M.E., H.M.R., A.A.-E.D., and A.A.S.Z. All authors have read and agreed to the published version of the manuscript.

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