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VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-CORDIAL LABELING OF F-TREE, Y-TREE, KEY GRAPH AND SPIDER GRAPH

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Abstract. Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors x and y by $< x, y >$. Let $\phi : V(G) \to S$ be a function.For edge uv assign the label $< \phi(u), \phi(v) >$. Then ϕ is called a vector basis S-cordial labeling of G (VB S-cordial labeling) if $|\phi_x - \phi_y| \leq 1$ and $|\gamma_i - \gamma_j| \leq 1$ where ϕ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph (VB S-cordial graph). In this paper, we find the existence of VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of Y-tree, F-tree, key graph, generalized key graph, spider graph, H_n and $K_{1,m}@2P_n$.

1. Introduction

In this paper, we consider only a finite, simple and undirected graph. The cardinality of the vertex set of a graph G is called the order of G and is denoted by p. The cardinality of its edge set is called the size of G and is denoted by q. The paper written by Leonhard Euler on the Seven Bridges of Konigsberg and published in 1736 is regarded as the first paper in the history of graph theory. Most graph labeling trace their origins to labeling presented by Alexander Rosa in his 1967 paper [15]. The terminology and definitions of graph and vector were followed in [7, 8]. The importance of graph labeling includes its numerous applications in many areas like circuit design, radar, coding theory, model of surveillance, security system in civil engineering and urban planning, circuit design, communication networks, electrical switchboards, demand and supply scenario, matching or assignment of resources and persons, ad-hoc networks and satellite communication.
Lucas graceful labeling for some graphs were studied in [11]. The notion of prime labeling was introduced in [17] and prime labeling of some special class of graphs like firecracker graphs, spider graphs, scorpion graphs, flower graphs, splitting graphs of stars, bistars, friendship graph and twig graph were examined in [9, 18]. Even vertex odd edge root

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square mean labeling were discussed in [1]. Cordial labeling in graph theory was introduced by I. Cahit [5] in 1987 as a weaker version of graceful and harmonious labeling. The idea of prime cordial labeling of graphs was introduced in [16]. Prime cordial labeling of several classes of graphs were discussed in [2, 3, 4, 10].
The concept of vector basis S-cordial labeling of graphs was presented in [12]. The vector basis (1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)-cordial labeling behaviour of path, cycle, star, comb, complete graph, generalized friendship graph, tadpole graph and gear graph and thorn related graphs have been investigated in [12, 13, 14]. A dynamic survey on graph labeling is given by Gallian [6]. In this paper, we examine the VB {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling behavior of Y-tree, F-tree, key graph, generalized key graph, spider graph, H_n and $K_{1,m}@2P_n$.

2. Vector Basis S-Cordial Labeling

Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors x and y by $< x, y >$. Let $\phi : V(G) \to S$ be a function.For edge uv assign the label $< \phi(u), \phi(v) >$. Then ϕ is called a vector basis S-cordial labeling of G (VB S-cordial labeling) if $|\phi_x - \phi_y| \leq 1$ and $|\gamma_i - \gamma_j| \leq 1$ where ϕ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph (VB S-cordial graph).
An example of VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial graph is given in Figure 1.

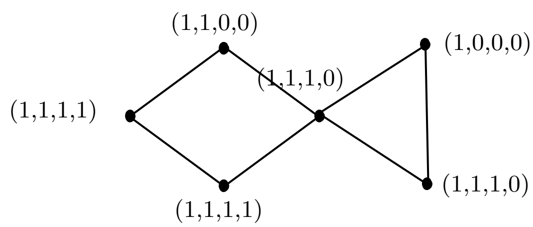

Figure 1. VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-Cordial Graph

In this paper, we consider the inner product space R^n and the standard inner product $< x, y >= x_1y_1 + x_2y_2 + \cdots + x_ny_n$ where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$, $x_i, y_i \in R$.

3. Main Results

In this section, we find the existence of VB {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling of Y-tree, F-tree, key graph, generalized key graph, spider graph, H_n and $K_{1,m}@2p_n$.

Theorem 3.1. *The graph H_n is a VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial for all $n \geq 3$.*

Proof. consider the graph H_n, $n \geq 3$. Let $V(H_n) = \{u_i, v_j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n-2\}$ and $E(H_n) = \{u_iu_{i+1}, u_jv_j \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n-2\}$. Then $p = |V(H_n)| = 2n-2$ and $q = |E(H_n)| = 2n - 3$. Assign the vectors to the vertices in the following order. $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}$.
Case (i): $n \equiv 0 \pmod 4$
Let $n = 4t$, $t > 0$. Then $p = 8t - 2$ and $q = 8t - 3$. Assign the vectors (1,1,1,1) to the first $2t$ vertices and (1,1,1,0) to the next $2t$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t - 1$ vertices and (1,0,0,0) to the next $2t - 1$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = 2t$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t - 1$. Clearly $\gamma_4 = \gamma_2 = \gamma_1 = 2t - 1$ and $\gamma_3 = 2t$.
Case (ii): $n \equiv 1 \pmod 4$

Let $n = 4t + 1$, $t > 0$. Then $p = 8t$ and $q = 8t - 1$. Assign the vectors (1,1,1,1) to the first $2t$ vertices and (1,1,1,0) to the next $2t$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t$ vertices and (1,0,0,0) to the remaining $2t$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t$. Clearly $\gamma_4 = 2t - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = 2t$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4t + 2$, $t > 0$. Then $p = 8t + 2$ and $q = 8t + 1$. Assign the vectors (1,1,1,1) to the first $2t + 1$ vertices and (1,1,1,0) to the next $2t + 1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t$ vertices and (1,0,0,0) to the remaining $2t$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = 2t + 1$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t$. Clearly $\gamma_4 = \gamma_2 = \gamma_1 = 2t$ and $\gamma_3 = 2t + 1$.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4t + 3$, $t \geq 0$. Then $p = 8t + 4$ and $q = 8t + 3$. Assign the vectors (1,1,1,1) to the first $2t + 1$ vertices and (1,1,1,0) to the next $2t + 1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t + 1$ vertices and (1,0,0,0) to the remaining $2t + 1$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t + 1$ and $2t + 1$. Clearly $\gamma_4 = 2t$ and $\gamma_3 = \gamma_2 = \gamma_1 = 2t + 1$.

Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of H_n. □

Example 3.0. *A VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of H_5 is given in Figure 2.*

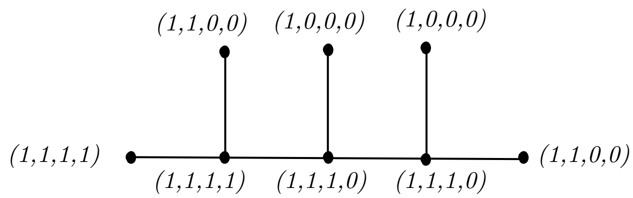

Figure 2. *VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of H_5*

Theorem 3.2. *The Y-tree Y_n is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial for all $n \geq 3$.*

Proof. consider the Y-tree Y_n, $n \geq 3$. Let $V(Y_n) = \{u_i \mid 1 \leq i \leq n\}$ and $E(Y_n) = \{u_i u_{i+1}, u_{n-2}u_n \mid 1 \leq i \leq n-2\}$. Then $p = |V(Y_n)| = n$ and $q = |E(Y_n)| = n - 1$. Assign the vectors to the vertices in the following order. $u_1, u_2, \ldots, u_n$.

Case (i): $n \equiv 0 \pmod 4$

Let $n = 4t$, $t > 0$. Then $p = 4t$ and $q = 4t - 1$. Assign the vectors (1,1,1,1) to the first t vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = t - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = t$.

Case (ii): $n \equiv 1 \pmod 4$

Let $n = 4t + 1$, $t > 0$. Then $p = 4t + 1$ and $q = 4t$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = t + 1$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = t$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4t + 2$, $t > 0$. Then $p = 4t + 2$ and $q = 4t + 1$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next $t + 1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = t + 1$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = \gamma_2 = \gamma_1 = t$ and $\gamma_3 = t + 1$.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4t + 3$, $t \geq 0$. Then $p = 4t + 3$ and $q = 4t + 2$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next $t + 1$ vertices. Thereafter assign the

vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = t+1$ and $\phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = \gamma_1 = t$ and $\gamma_3 = \gamma_2 = t+1$.
Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of Y_n. □

Example 3.0. *A VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of Y_5 is given in Figure 3.*

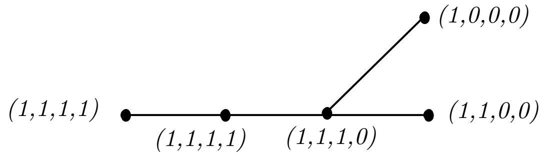

Figure 3. *VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of Y_5*

Theorem 3.3. *The F-tree F_n is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial for all $n \geq 3$.*

Proof. consider the F-tree F_n, $n \geq 3$. Let $V(F_n) = \{u, v, u_i \mid 1 \leq i \leq n\}$ and $E(F_n) = \{u_iu_{i+1}, uu_{n-1}, vu_n \mid 1 \leq i \leq n-1\}$. Then $p = |V(F_n)| = n+2$ and $q = |E(F_n)| = n+1$. Assign the vectors to the vertices in the following order. $u_1, u_2, \ldots, u_n, u, v$.
Case (i): $n \equiv 0 \pmod 4$
Let $n = 4t$, $t > 0$. Then $p = 4t+2$ and $q = 4t+1$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = t+1$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = \gamma_2 = \gamma_1 = t$ and $\gamma_3 = t+1$.
Case (ii): $n \equiv 1 \pmod 4$
Let $n = 4t+1$, $t > 0$. Then $p = 4t+3$ and $q = 4t+2$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $t+1$ vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = t+1$ and $\phi_{(1,0,0,0)} = t$. Clearly $\gamma_4 = \gamma_1 = t$ and $\gamma_3 = \gamma_2 = t+1$.
Case (iii): $n \equiv 2 \pmod 4$
Let $n = 4t+2$, $t > 0$. Then $p = 4t+4$ and $q = 4t+3$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $t+1$ vertices and (1,0,0,0) to the next $t+1$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t+1$. Clearly $\gamma_4 = t$ and $\gamma_3 = \gamma_2 = \gamma_1 = t+1$.
Case (iv): $n \equiv 3 \pmod 4$
Let $n = 4t+3$, $t \geq 0$. Then $p = 4t+5$ and $q = 4t+3$. Assign the vectors (1,1,1,1) to the first $t+2$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $t+1$ vertices and (1,0,0,0) to the next $t+1$ vertices. We have $\phi_{(1,1,1,1)} = t+2$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t+1$. Clearly $\gamma_4 = \gamma_1 = \gamma_3 = \gamma_2 = t+1$.
Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of F_n. □

Example 3.0. *A VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of F_4 is given in Figure 4.*

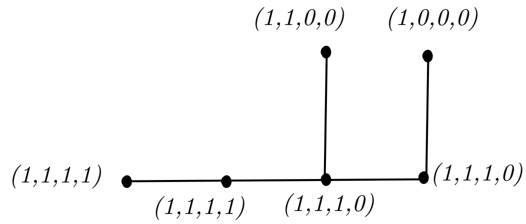

Figure 4. *VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of* F_4

Theorem 3.4. *The key graph* KY_n *is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial for all* $n \geq 1$.

Proof. consider the key graph KY_n, $n \geq 1$. Let $V(KY_n) = \{u_1, u_2, u_3, u_4, u_5, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(KY_n) = \{u_iu_{i+1}, u_5u_1, u_1v_1, v_jv_{j+1} \mid 1 \leq i \leq 4 \,\text{and}\, 1 \leq j \leq n-1\} \cup \{v_iw_i \mid 1 \leq i \leq n\}$. Then $p = |V(KY_n)| = 2n+5$ and $q = |E(KY_n)| = 2n+5$. Assign the vectors to the vertices in the following order: $u_1, u_2, \ldots, u_5, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$.
Case (i): $n \equiv 0 \pmod 4$
Let $n = 4t$, $t > 0$. Then $p = 8t+5$ and $q = 8t+5$. Assign the vectors (1,1,1,1) to the first $2t+2$ vertices and (1,1,1,0) to the next $2t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t+1$ vertices and (1,0,0,0) to the next $2t+1$ vertices. We have $\phi_{(1,1,1,1)} = 2t+2$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t+1$. If $2t+2 < 5$, $\gamma_4 = \gamma_2 = \gamma_1 = 2t+1$ and $\gamma_3 = 2t+2$. If $2t+2 \geq 5$, $\gamma_4 = 2t+2$ and $\gamma_3 = \gamma_2 = \gamma_1 = 2t+1$.
Case (ii): $n \equiv 1 \pmod 4$
Let $n = 4t+1$, $t \geq 0$. Then $p = 8t+7$ and $q = 8t+7$. Assign the vectors (1,1,1,1) to the first $2t+2$ vertices and (1,1,1,0) to the next $2t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t+2$ vertices and (1,0,0,0) to the next $2t+2$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,0,0)} = 2t+2$ and $\phi_{(1,1,1,0)} = \phi_{(1,0,0,0)} = 2t+1$. If $2t+2 < 5$, $\gamma_4 = 2t+1$ and$\gamma_3 = \gamma_2 = \gamma_1 = 2t+2$. If $2t+2 \geq 5$, $\gamma_4 = \gamma_2 = \gamma_1 = 2t+2$ and $\gamma_3 = 2t+1$.
Case (iii): $n \equiv 2 \pmod 4$
Let $n = 4t+2$, $t \geq 0$. Then $p = 8t+9$ and $q = 8t+9$. Assign the vectors (1,1,1,1) to the first $2t+3$ vertices and (1,1,1,0) to the next $2t+2$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t+2$ vertices and (1,0,0,0) to the next $2t+2$ vertices. We have $\phi_{(1,1,1,1)} = 2t+3$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t+2$. If $2t+2 < 5$, $\gamma_4 = \gamma_2 = \gamma_1 = 2t+2$ and$\gamma_3 = 2t+3$. If $2t+2 \geq 5$, $\gamma_4 = 2t+3$ and $\gamma_3 = \gamma_2 = \gamma_1 = 2t+2$.
Case (iv): $n \equiv 3 \pmod 4$
Let $n = 4t+3$, $t \geq 0$. Then $p = 8t+11$ and $q = 8t+11$. Assign the vectors (1,1,1,1) to the first $2t+3$ vertices and (1,1,1,0) to the next $2t+2$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $2t+3$ vertices and (1,0,0,0) to the next $2t+3$ vertices. We have $\phi_{(1,1,1,1)} = 2t+3$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 2t+2$. If $2t+2 < 5$, $\gamma_4 = 2t+2$ and$\gamma_3 = \gamma_2 = \gamma_1 = 2t+3$. If $2t+2 \geq 5$, $\gamma_4 = \gamma_2 = \gamma_1 = 2t+3$ and $\gamma_3 = 2t+2$.
Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling of KY_n. □

Theorem 3.5. *The generalized key graph* $KY_{n,n}$ *is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only of* $n \equiv 1, 2, 3 \pmod 4$.

Proof. consider the generalized key graph $KY_{n,n}$, $n \geq 3$. Let $V(KY_{n,n}) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(KY_{n,n}) = \{u_iu_{i+1}, u_1u_n, u_1v_1, v_iv_{i+1}, v_jw_j \mid 1 \leq i \leq n-1 \,\text{and}\, 1 \leq j \leq n\}$. Then $p = |V(KY_{n,n})| = 3n$ and $q = |E(KY_{n,n})| = 3n$. Assign the vectors to the vertices in the following order: $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$.
Case (i): $n \equiv 0 \pmod 4$
Let $n = 4t$, $t \geq 1$. Then $p = 12t$ and $q = 12t$. From $3t$ vertices with vertex label (1,1,1,1) we get $3t-1$ edges with edge label 4. This leads to a contradiction.
Case (ii): $n \equiv 1 \pmod 4$
Let $n = 4t+1$, $t > 0$. Then $p = 12t+3$ and $q = 12t+3$. Assign the vectors (1,1,1,1)

to the first $3t+1$ vertices and (1,1,1,0) to the next $3t$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $3t+1$ vertices and (1,0,0,0) to the next $3t+1$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 3t+1$ and $\phi_{(1,1,1,0)} = 3t$. Thus $\gamma_4 = 3t$ and $\gamma_3 = \gamma_2 = \gamma_1 = 3t+1$.

Case (iii): $n \equiv 2 \pmod 4$

Let $n = 4t+2$, $t > 0$. Then $p = 12t+6$ and $q = 12t+6$. Assign the vectors (1,1,1,1) to the first $3t+2$ vertices and (1,1,1,0) to the next $3t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $3t+2$ vertices and (1,0,0,0) to the next $3t+1$ vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,0,0)} = 3t+2$ and $\phi_{(1,1,1,0)} = \phi_{(1,0,0,0)} = 3t+1$. Thus $\gamma_4 = \gamma_1 = 3t+1$ and$\gamma_3 = \gamma_2 = 3t+2$.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4t+3$, $t > 0$. Then $p = 12t+9$ and $q = 12t+9$. Assign the vectors (1,1,1,1) to the first $3t+3$ vertices and (1,1,1,0) to the next $3t+2$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $3t+2$ vertices and (1,0,0,0) to the next $3t+2$ vertices. We have $\phi_{(1,1,1,1)} = 3t+3$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = 3t+2$. Thus $\gamma_4 = \gamma_2 = \gamma_1 = 3t+2$ and$\gamma_3 = 3t+3$.

Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0),(1,0,0,0)}-cordial labeling of $KY_{n,n}$. □

Example 3.0. *A VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $KY_{5,5}$ is given in Figure 5.*

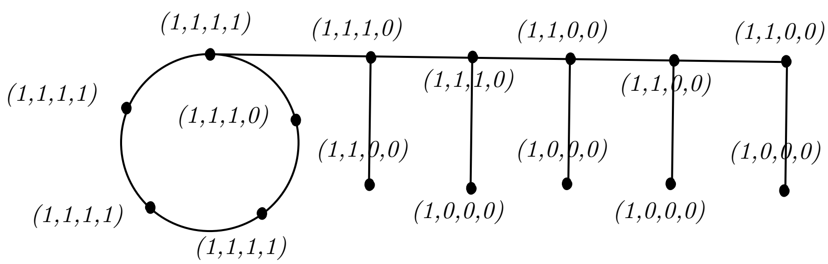

Figure 5. *VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $KY_{5,5}$*

Theorem 3.6. *The spider graph $S_{m,n}$ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only of $m, n \geq 1$.*

Proof. consider the spider graph $S_{m,n}$, $m, n \geq 1$. Let $V(S_{m,n}) = \{u_0, u_{i,j} \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ and $E(S_{m,n}) = \{u_0 u_{i,1}, u_{i,j} u_{i,j+1} \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n-1\}$. Then $p = |V(S_{m,n})| = mn+1$ and $q = |E(S_{m,n})| = mn$. Assign the vectors to the vertices in the following order: $u_0, u_{1,1}, u_{2,1}, \ldots, u_{m,1}, u_{1,2}, u_{2,2}, \ldots, u_{m,2}, \ldots, u_{1,n}, u_{2,n} \ldots, u_{m,n},$.

Case (i): $p \equiv 0 \pmod 4$

Let $p = 4t$, $t > 1$. Then $q = 4t-1$. Assign the vectors (1,1,1,1) to the first t vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = t-1$ and$\gamma_3 = \gamma_2 = \gamma_1 = t-1$.

Case (ii): $p \equiv 1 \pmod 4$

Let $p = 4t+1$, $t > 0$. Then $q = 4t$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = t+1$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = t$ and $\gamma_3 = \gamma_2 = \gamma_1 = t$.

Case (iii): $p \equiv 2 \pmod 4$

Let $p = 4t+2$, $t \geq 0$. Then $q = 4t+1$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = t+1$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = \gamma_2 = \gamma_1 = t$ and$\gamma_3 = t+1$.

Case (iv): $p \equiv 3 \pmod 4$

Let $p = 4t + 3,\ t > 0$. Then $q = 4t + 2$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next $t + 1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $t+1$ vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = t + 1$ and $\phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = \gamma_1 = t$ and$\gamma_3 = \gamma_2 = t + 1$.
Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0),(1,0,0,0)}-cordial labeling of $S_{m,n}$. □

Example 3.0. *A VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $S_{3,4}$ is given in Figure 6.*

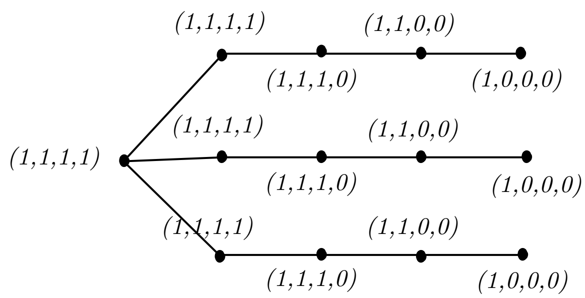

Figure 6. *VB {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $S_{3,5}$*

Theorem 3.7. *The graph $K_{1,m}@2P_n$ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial if and only of $m, n \geq 1$.*

Proof. consider the graph $K_{1,m}@2P_n$, $m, n \geq 1$. Let $V(K_{1,m}@2P_n) = \{u_0, u_i, u_{i,j}, u_{i,j}' \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ and $E(K_{1,m}@2P_n) = \{u_0u_i, u_iu_{i,1}, u_iu_{i,1}', u_{i,j}u_{i,j+1}, u_{i,j}'u_{i,j+1}' \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n-1\}$. Then $p = |V(K_{1,m}@2P_n)| = m(2n+1)+1$ and $q = |E(K_{1,m}@2P_n)| = m(2n+1)$. Assign the vectors to the vertices in the following order: $u_0, u_1, u_2, \ldots, u_m, u_{1,1}, u_{1,1}', u_{2,1}, u_{2,1}', \ldots, u_{m,1}, u_{m,1}', u_{1,2}, u_{1,2}', u_{2,2}, u_{2,2}', \ldots, u_{m,2}, u_{m,2}', \ldots, u_{1,n}, u_{1,n}', u_{2,n}, u_{2,n}', \ldots, u_{m,n}, u_{m,n}'$.
Case (i): $p \equiv 0 \pmod 4$
Let $p = 4t,\ t > 1$. Then $q = 4t - 1$. Assign the vectors (1,1,1,1) to the first t vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices.We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = t - 1$ and $\gamma_3 = \gamma_2 = \gamma_1 = t$.
Case (ii): $p \equiv 1 \pmod 4$
Let $p = 4t + 1,\ t > 0$. Then $q = 4t$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next t vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = t + 1$ and $\phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = t$.
Case (iii): $p \equiv 2 \pmod 4$
Let $p = 4t+2,\ t \geq 0$. Then $q = 4t+1$. Assign the vectors (1,1,1,1) to the first $t+1$ vertices and (1,1,1,0) to the next $t+1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next t vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = t + 1$ and $\phi_{(1,1,0,0)} = \phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = \gamma_2 = \gamma_1 = t$ and$\gamma_3 = t + 1$.
Case (iv): $p \equiv 3 \pmod 4$
Let $p = 4t + 3,\ t > 0$. Then $q = 4t + 2$. Assign the vectors (1,1,1,1) to the first $t + 1$ vertices and (1,1,1,0) to the next $t + 1$ vertices. Thereafter assign the vectors (1,1,0,0) to the next $t+1$ vertices and (1,0,0,0) to the next t vertices. We have $\phi_{(1,1,1,1)} = \phi_{(1,1,1,0)} = \phi_{(1,1,0,0)} = t + 1$ and $\phi_{(1,0,0,0)} = t$. Thus $\gamma_4 = \gamma_1 = t$ and$\gamma_3 = \gamma_2 = t + 1$.
Hence ϕ is a VB {(1,1,1,1), (1,1,1,0), (1,1,0,0),(1,0,0,0)}-cordial labeling of $K_{1,m}@2P_n$. □

4. Conclusion

In this paper, we have discussed the existence of VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of Y-tree, F-tree, key graph, generalized key graph, spider graph, H_n and $K_{1,m}@2P_n$. Investigation of the VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling behaviour of some graphs like Mongolian tent, Tensor grid, planar grid, step ladder graph, generalized petersen graph, generalised web graph, generalized prism graph, generalized antiprism graph, Mongolian tent, Tensor grid, planar grid, step ladder graph and generalized Jahangir graph are open problems.

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