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## SOME FIXED POINT THEOREMS FOR $\alpha$ -ADMISSIBLE MAPPINGS IN $S$ -METRIC SPACES

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**ABSTRACT.** In this paper, we investigate the existence of fixed points for a class of mappings known as  $\alpha$ -admissible mappings within the framework of  $S$ -metric spaces, a generalization of metric spaces. Our main objective is to extend and refine existing fixed point theorems by relaxing some of the common contraction conditions typically required in classical results. Specifically, we establish new fixed point results under weaker contraction conditions, thereby broadening the applicability of these theorems to a wider class of mappings. The motivation behind considering  $\alpha$ -admissible mappings lies in their flexibility and potential to capture a larger set of functional behaviors in nonlinear analysis. Our approach not only generalizes several known results in the literature, but also unifies them under a common theoretical framework. These findings contribute meaningfully to the growing body of research in fixed point theory, particularly within the context of generalized metric structures such as  $S$ -metric spaces, and open new avenues for further investigation.

### 1. INTRODUCTION AND PRELIMINARIES

Fixed point theory represents a foundational branch of mathematical analysis focused on determining whether and when mappings have fixed points, and whether such points are unique. A cornerstone of this theory is the Banach Contraction Principle, which ensures the existence and uniqueness of a fixed point for specific self-maps within complete metric spaces. This principle has inspired a wide range of generalizations, expanding its relevance to more general mathematical frameworks.

To broaden the applicability of the Banach Contraction Principle, numerous extensions have been developed. Among these, the notion of  $\alpha$ -admissible mappings has proven to be an effective approach for deriving fixed point results under more relaxed conditions. Samet et al. [15] introduced the concepts of  $\alpha$ - $\psi$ -contractive and

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$\alpha$ -admissible mappings, establishing several fixed point theorems in the setting of complete metric spaces. Subsequently, Karapinar and Samet [6] further extended this framework by generalizing the class of  $\alpha$ - $\psi$ -contractive type mappings and obtained additional fixed point results for this broader category of contractions.

Over time, numerous researchers have investigated fixed point theory within the framework of generalized metric spaces, particularly  $S$ -metric spaces. Nabil M. Mlaiki [7] was the first to introduce the concept of  $\alpha$ - $\psi$ -contractive mappings in the context of  $S$ -metric spaces. This was later expanded upon by scholars such as Mi Zhou, N. Priyobarta, and others [21], who contributed further to this area. Building upon these foundational contributions, the present paper advances the study of  $\alpha$ -admissible mappings in  $S$ -metric spaces by establishing new results that enhance the existing body of fixed point theory and broaden its potential applications.

Fixed point theory has continued to evolve through the contributions of numerous researchers, each extending classical results to more generalized spaces and mappings. Abbas and coauthors [1, 2] initiated several studies in cone metric spaces, while Duraj and Liftaj [3], and Saluja [14] examined fixed point results in  $S$ -metric settings. Karapinar and Samet [6], and Samet et al. [15], who introduced and generalized  $\alpha$ - $\psi$  contractive mappings. Subsequent developments by Hussain et al. [5], Salimi et al. [13], Sedghi and coauthors [16, 17], and Sessa [18] enriched the theoretical framework. More recent progress includes results by Ozturk and Turkoglu [10], Zhou et al. [21] et al. Wangwe and Kumar [19], and Saadi and Hamaizia [12]. Notably, Nabil [7], Nallaselli and coauthors [4, 8, 9], and Yazici et al. [20] explored  $\alpha$ -admissibility and contraction principles in  $S$ -metric and related spaces, thereby unifying fixed point theory with diverse applications across analysis.

**Definition 1.1.** [17] *Let  $X$  be a non empty set and  $S : X^3 \rightarrow \mathbb{R}^+$ , a function that satisfies the following properties;*

- (i)  $S(x, y, z) = 0$  if and only if  $x = y = z$ ,
- (ii)  $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$  for all  $a, x, y, z \in X$  (rectangle inequality).

*Then, the pair  $(X, S)$  is called an  $S$ -metric space.*

**Definition 1.2.** [17] *Let  $(X, S)$  be an  $S$ -metric space and  $A \subset X$ .*

- (i) *A sequence  $\{x_n\}$  in  $X$  is said to converge to  $x$  if  $S(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . In other words, for every  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $S(x_n, x_n, x) < \epsilon$ . In this case, we write  $\lim_{n \rightarrow \infty} x_n = x$  and we say that  $x$  is the limit of  $\{x_n\}$  in  $X$ .*
- (ii) *A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if for each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for each  $n, m \geq n_0$ ,  $S(x_n, x_n, x_m) < \epsilon$ .*
- (iii) *The  $S$ -metric space  $(X, S)$  is called complete if every Cauchy sequence in  $X$  converges to a limit in  $X$ .*

**Lemma 1.1.** [17] *If  $(X, S)$  is an  $S$ -metric space, then for all  $x, y \in X$ , the condition  $S(x, x, y) = S(y, y, x)$  holds.*

**Lemma 1.2.** [17] *Let  $(X, S)$  be an  $S$ -metric space. If  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $X$  converging to  $x$  and  $y$  respectively, that is,  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ , then  $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$  as  $n \rightarrow \infty$ .*

**Lemma 1.3.** [17] *Let  $(X, S)$  be an  $S$ -metric space. If the sequence  $\{x_n\}$  in  $X$  converges to  $x$ , then  $\{x_n\}$  is a Cauchy sequence.*

**Definition 1.3.** [7] Let  $(X, S)$  be a  $S$  metric space and  $T : X \rightarrow X$  be a given mapping. We say that  $T$  is  $\alpha$ -admissible if  $x, y, z \in X$ ,  $\alpha(x, y, z) \geq 1$  implies that  $\alpha(Tx, Ty, Tz) \geq 1$ .

**Example 1.1.** Let  $X = [0, \infty)$ . Define the mapping  $T : X \rightarrow X$  and  $\alpha : X \times X \times X \rightarrow [0, \infty)$  by  $Tx = x^2$  and

$$\alpha(x, y, z) = \begin{cases} 2 & \text{if } x \geq y \geq z, \\ 0 & \text{if otherwise.} \end{cases} \quad (1)$$

Then,  $T$  is  $\alpha$ -admissible.

**Proof.** If  $\alpha(x, y, z) \geq 1$  then  $x \geq y \geq z$ . Then  $Tx = x^2 \geq y^2 = Ty \geq z^2 = Tz$ . That is  $Tx \geq Ty \geq Tz \implies \alpha(Tx, Ty, Tz) = 2 \geq 1$ . So  $T$  is  $\alpha$ -admissible.

## 2. MAIN RESULTS

In this section, we establish several fixed point theorems for  $\alpha$ -admissible mappings in the setting of  $S$ -metric spaces.

**Theorem 2.1.** Let  $(X, S)$  be a complete  $S$ -metric space and  $T : X \rightarrow X$  be an  $\alpha$ -admissible mapping. Assume that there exists a function  $\beta : [0, \infty) \rightarrow [0, 1]$  such that, for any bounded sequence  $\{t_n\}$  of positive reals,  $\beta(t_n) \rightarrow 1 \implies t_n \rightarrow 0$  and

$$(S(Tx, Ty, Tz) + l)^{\alpha(x, x, Tx)\alpha(y, y, Ty)\alpha(z, z, Tz)} \leq \beta(S(x, y, z))S(x, y, z) + l \quad (2)$$

for all  $x, y, z \in X$  where  $l \geq 1$ . Suppose that either

- (i)  $T$  is continuous, or
- (ii) If  $\{x_n\}$  is a sequence in  $X$  such that  $\{x_n\} \rightarrow x$ ,  $\alpha(x_n, x_n, x_{n+1}) \geq 1$  for all  $n$ , then  $\alpha(x, x, Tx) \geq 1$ . If there exists  $x_0 \in X$  such that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point.

**Proof.** Consider the sequence  $\{x_n\}$  defined by  $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_n = Tx_{n-1} = T^n x_0, \dots$ . By assumption we know that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , since  $T$  is  $\alpha$ -admissible, therefore,  $\alpha(x_1, x_1, x_2) \geq 1$ . So, using the fact that  $T$  is  $\alpha$ -admissible and by induction on  $n$ , we conclude that

$$\alpha(x_n, x_n, x_{n+1}) \geq 1.$$

Now, since for  $n \in \mathbb{N}$  we have,

$$\begin{aligned} & S(Tx_{n-1}, Tx_{n-1}, Tx_n) + l \\ & \leq (S(Tx_{n-1}, Tx_{n-1}, Tx_n) + l)^{\alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_n, x_n, Tx_n)} \\ & \leq \beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n) + l \\ & \implies S(x_n, x_n, x_{n+1}) \leq \beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n), \end{aligned} \quad (3)$$

which implies  $S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n)$ .

It follows that the sequence  $S(x_n, x_n, x_{n+1})$  is decreasing. Thus, there exists  $\alpha \in \mathbb{R}$ , such that  $\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = \alpha$ . We will prove that  $\alpha = 0$ .

From (3) we have

$$\frac{S(x_n, x_n, x_{n+1})}{S(x_{n-1}, x_{n-1}, x_n)} \leq \beta(S(x_{n-1}, x_{n-1}, x_n)) \leq 1,$$

which implies  $\lim_{n \rightarrow \infty} \beta(S(x_{n-1}, x_{n-1}, x_n)) = 1$ . Using the property of the function  $\beta$ , we conclude that  $\lim_{n \rightarrow \infty} S(x_{n-1}, x_{n-1}, x_n) = 0$ . Hence,

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = 0. \quad (4)$$

Next, we will prove that  $\{x_n\}$  is a Cauchy sequence. Suppose, to the contrary, that  $\{x_n\}$  is not a Cauchy sequence. Then there is  $\epsilon > 0$  and sequences  $\{m(k)\}$  and  $\{n(k)\}$  such that, for all positive integers  $k$ , we have  $n(k) > m(k) > k$ ,  $S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \epsilon$  and  $S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}) < \epsilon$ . Then,

$$\begin{aligned} \epsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &= S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\ &\leq S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}), \end{aligned}$$

$k \in \mathbb{N}$ . Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using (4), we get

$$\epsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) < 2(0) + \epsilon = \epsilon.$$

Which is a contradiction. Hence,

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \epsilon. \quad (5)$$

Again,

$$\begin{aligned} &S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \\ &\leq S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{n(k)}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k+1)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) \\ &\quad + S(x_{n(k)}, x_{n(k)}, x_{m(k)}). \end{aligned}$$

Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using equations (4) and (5), we get

$$S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \leq 2(0) + 0 + 0 + \epsilon = \epsilon.$$

Hence,

$$\lim_{n \rightarrow \infty} S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) = \epsilon. \quad (6)$$

From (1.1), (5) and (6) we have

$$\begin{aligned} &S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) + l \\ &\leq (S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) + l)^{\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, x_{m(k)}, Tx_{m(k)})} \\ &\leq (S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)}) + l)^{\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, x_{m(k)}, Tx_{m(k)})} \\ &\leq \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)}))S(x_{n(k)}, x_{n(k)}, x_{m(k)}) + l. \end{aligned}$$

Hence,

$$\frac{S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})}{S(x_{n(k)}, x_{n(k)}, x_{m(k)})} \leq \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) \leq 1,$$

letting  $k \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) = 1$ .

That is,  $\lim_{n \rightarrow \infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = 0 < \epsilon$ , which is a contradiction. Hence  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete, then there is  $z \in X$  such that  $x_n \rightarrow z$ . Since  $T$  is continuous, then we have

$$Tz = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = z.$$

So,  $z$  is a fixed point of  $T$ . Next we suppose that (ii) holds. Then  $\alpha(z, z, Tz) \geq 1$ . Now

$$\begin{aligned} S(Tz, Tz, x_{n+1}) + l &\leq (S(Tz, Tz, Tx_n) + l)^{\alpha(z, z, Tz)\alpha(z, z, Tz)\alpha(x_n, x_n, Tx_n)} \\ &\leq \beta(S(z, z, x_n))S(z, z, x_n) + l \end{aligned}$$

This implies that,

$$S(Tz, Tz, x_{n+1}) \leq \beta(S(z, z, x_n))S(z, z, x_n).$$

Hence, we get

$$\begin{aligned} S(Tz, Tz, z) &\leq S(Tz, Tz, x_{n+1}) + S(Tz, Tz, x_{n+1}) + S(z, z, x_{n+1}) \\ &= 2S(Tz, Tz, x_{n+1}) + S(z, z, x_{n+1}) \\ &\leq 2\beta(S(z, z, x_n))S(z, z, x_n) + S(z, z, x_{n+1}), \end{aligned}$$

letting  $n \rightarrow \infty$ , we get  $S(Tz, Tz, z) \leq 2\beta(S(z, z, z))S(z, z, z) + S(z, z, z)$ . This implies that,  $S(Tz, Tz, z) \leq 0$ . But  $S(Tz, Tz, z) \geq 0$ . That is  $Tz = z$ .

**Theorem 2.2.** Let  $(X, S)$  be a complete  $S$ -metric space and  $T : X \rightarrow X$  be an  $\alpha$ -admissible mapping. Assume that there exists a function  $\beta : [0, \infty) \rightarrow [0, 1]$  such that, for any bounded sequence  $\{t_n\}$  of positive reals,  $\beta(t_n) \rightarrow 1 \implies t_n \rightarrow 0$  and

$$(\alpha(x, x, Tx)\alpha(y, y, Ty)\alpha(z, z, Tz) + 1)^{S(Tx, Ty, Tz)} \leq 2^{\beta(S(x, y, z))S(x, y, z)} \quad (7)$$

for all  $x, y, z \in X$ . Suppose that either

- (i)  $T$  is continuous, or
- (ii) If  $\{x_n\}$  is a sequence in  $X$  such that  $\{x_n\} \rightarrow x$ ,  $\alpha(x_n, x_n, x_{n+1}) \geq 1$  for all  $n$ , then  $\alpha(x, x, Tx) \geq 1$ . If there exists  $x_0 \in X$  such that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point.

**Proof.** Consider the sequence  $\{x_n\}$  defined by  $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_n = Tx_{n-1} = T^n x_0, \dots$ . By assumption we know that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , since  $T$  is  $\alpha$ -admissible, therefore,  $\alpha(x_1, x_1, x_2) \geq 1$ . So, using the fact that  $T$  is  $\alpha$ -admissible and by induction on  $n$ , we conclude that

$$\alpha(x_n, x_n, x_{n+1}) \geq 1.$$

Now, since for  $n \in \mathbb{N}$  we have,

$$\begin{aligned} &2^{S(Tx_{n-1}, Tx_{n-1}, Tx_n)} \\ &\leq (\alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_n, x_n, Tx_n) + 1)^{S(Tx_{n-1}, Tx_{n-1}, Tx_n)} \\ &\leq 2^{\beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n)} \\ &\implies S(x_n, x_n, x_{n+1}) \leq \beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n), \end{aligned} \quad (8)$$

which implies  $S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n)$ .

It follows that the sequence  $S(x_n, x_n, x_{n+1})$  is decreasing. Thus, there exists  $\alpha \in \mathbb{R}$ , such that  $\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = \alpha$ . We will prove that  $\alpha = 0$ .

From (8) we have

$$\frac{S(x_n, x_n, x_{n+1})}{S(x_{n-1}, x_{n-1}, x_n)} \leq \beta(S(x_{n-1}, x_{n-1}, x_n)) \leq 1,$$

which implies  $\lim_{n \rightarrow \infty} \beta(S(x_{n-1}, x_{n-1}, x_n)) = 1$ . Using the property of the function  $\beta$ , we conclude that  $\lim_{n \rightarrow \infty} S(x_{n-1}, x_{n-1}, x_n) = 0$ . Hence,

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = 0. \quad (9)$$

Next, we will prove that  $\{x_n\}$  is a Cauchy sequence. Suppose, to the contrary, that  $\{x_n\}$  is not a Cauchy sequence. Then there is  $\epsilon > 0$  and sequences  $\{m(k)\}$  and  $\{n(k)\}$  such that, for all positive integers  $k$ , we have  $n(k) > m(k) > k$ ,  $S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \epsilon$  and  $S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}) < \epsilon$ . Then,

$$\begin{aligned} \epsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &= S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\ &\leq S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}), \end{aligned}$$

$k \in \mathbb{N}$ . Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using (9), we get

$$\epsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) < 2(0) + \epsilon = \epsilon.$$

Which is a contradiction. Hence,

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \epsilon. \quad (10)$$

Again,

$$\begin{aligned} &S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \\ &\leq S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{n(k)}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k+1)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) \\ &\quad + S(x_{n(k)}, x_{n(k)}, x_{m(k)}). \end{aligned}$$

Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using (9) and (10), we get

$$S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \leq 2(0) + 0 + 0 + \epsilon = \epsilon.$$

Hence,

$$\lim_{n \rightarrow \infty} S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) = \epsilon. \quad (11)$$

From (7), (10) and (11) we have  $2^{S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})}$

$$\begin{aligned} &\leq (\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, x_{m(k)}, Tx_{m(k)}))^{S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)})} \\ &\leq 2^{\beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)}))S(x_{n(k)}, x_{n(k)}, x_{m(k)})} \end{aligned}$$

Hence,

$$\frac{S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})}{S(x_{n(k)}, x_{n(k)}, x_{m(k)})} \leq \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) \leq 1,$$

letting  $k \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) = 1$ .

That is,  $\lim_{n \rightarrow \infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = 0 < \epsilon$ , which is a contradiction. Hence  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete, then there is  $z \in X$  such that  $x_n \rightarrow z$ . Since  $T$  is continuous, then we have

$$Tz = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = z.$$

So,  $z$  is a fixed point of  $T$ . Next we suppose that (ii) holds. Then  $\alpha(z, z, Tz) \geq 1$ . Now

$$\begin{aligned} 2^{S(Tz, Tz, x_{n+1})} &\leq (\alpha(z, z, Tz)\alpha(z, z, Tz)\alpha(x_n, x_n, Tx_n) + 1)^{S(Tz, Tz, Tx_n)} \\ &\leq 2^{\beta(S(z, z, x_n))S(z, z, x_n)}. \end{aligned}$$

Hence, we get

$$S(Tz, Tz, z) \leq \beta(S(z, z, x_n))S(z, z, x_n),$$

letting  $n \rightarrow \infty$ , we get  $S(Tz, Tz, z) \leq \beta(S(z, z, z))S(z, z, z) \implies S(Tz, Tz, z) \leq 0$ . But  $S(Tz, Tz, z) \geq 0$ . That is  $Tz = z$ .

**Theorem 2.3.** *Let  $(X, S)$  be a complete  $S$ -metric space and  $T : X \rightarrow X$  be an  $\alpha$ -admissible mapping. Assume that there exists a function  $\beta : [0, \infty) \rightarrow [0, 1]$  such that, for any bounded sequence  $\{t_n\}$  of positive reals,  $\beta(t_n) \rightarrow 1 \implies t_n \rightarrow 0$  and*

$$\alpha(x, x, Tx)\alpha(y, y, Ty)\alpha(z, z, Tz)S(Tx, Ty, Tz) \leq \beta(S(x, y, z))S(x, y, z) \quad (12)$$

for all  $x, y, z \in X$ . Suppose that either

- (i)  $T$  is continuous, or
- (ii) If  $\{x_n\}$  is a sequence in  $X$  such that  $\{x_n\} \rightarrow x$ ,  $\alpha(x_n, x_n, x_{n+1}) \geq 1$  for all  $n$ , then  $\alpha(x, x, Tx) \geq 1$ . If there exists  $x_0 \in X$  such that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point.

**Proof.** Consider the sequence  $\{x_n\}$  defined by  $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_n = Tx_{n-1} = T^n x_0, \dots$ . By assumption we know that  $\alpha(x_0, x_0, Tx_0) \geq 1$ , since  $T$  is  $\alpha$ -admissible, therefore,  $\alpha(x_1, x_1, x_2) \geq 1$ . So, using the fact that  $T$  is  $\alpha$ -admissible and by induction on  $n$ , we conclude that

$$\alpha(x_n, x_n, x_{n+1}) \geq 1.$$

Now, since for  $n \in N$  we have,

$$\begin{aligned} \alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_{n-1}, x_{n-1}, Tx_{n-1})\alpha(x_n, x_n, Tx_n)S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ \leq \beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n) \\ \implies S(x_n, x_n, x_{n+1}) \leq \beta(S(x_{n-1}, x_{n-1}, x_n))S(x_{n-1}, x_{n-1}, x_n), \end{aligned} \quad (13)$$

which implies  $S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n)$ .

It follows that the sequence  $S(x_n, x_n, x_{n+1})$  is decreasing. Thus, there exists  $\alpha \in \mathbb{R}$ , such that  $\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = \alpha$ . We will prove that  $\alpha = 0$ .

From equation (13) we have

$$\frac{S(x_n, x_n, x_{n+1})}{S(x_{n-1}, x_{n-1}, x_n)} \leq \beta(S(x_{n-1}, x_{n-1}, x_n)) \leq 1,$$

which implies  $\lim_{n \rightarrow \infty} \beta(S(x_{n-1}, x_{n-1}, x_n)) = 1$ . Using the property of the function  $\beta$ , we conclude that  $\lim_{n \rightarrow \infty} S(x_{n-1}, x_{n-1}, x_n) = 0$ . Hence,

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = 0. \quad (14)$$

Next, we will prove that  $\{x_n\}$  is a Cauchy sequence. Suppose, to the contrary, that  $\{x_n\}$  is not a Cauchy sequence. Then there is  $\epsilon > 0$  and sequences  $\{m(k)\}$  and  $\{n(k)\}$  such that, for all positive integers  $k$ , we have

$n(k) > m(k) > k, S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \epsilon$  and  $S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}) < \epsilon$ .

Then,

$$\begin{aligned} \epsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &= S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\ &\leq S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{m(k)}, x_{m(k)}, x_{m(k-1)}) + S(x_{n(k)}, x_{n(k)}, x_{m(k-1)}), \end{aligned}$$

$k \in \mathbb{N}$ . Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using (14), we get

$$\epsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) < 2(0) + \epsilon = \epsilon.$$

Which is a contradiction. Hence,

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \epsilon. \quad (15)$$

Again,

$$\begin{aligned} &S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \\ &\leq S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{n(k+1)}, x_{n(k+1)}, x_{n(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{n(k)}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k+1)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) + S(x_{m(k+1)}, x_{m(k+1)}, x_{m(k)}) \\ &\quad + S(x_{n(k)}, x_{n(k)}, x_{m(k)}). \end{aligned}$$

Taking the limit as  $k \rightarrow +\infty$  in the above inequality and using (14) and (15), we get

$$S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \leq 2(0) + 0 + 0 + \epsilon = \epsilon.$$

Hence,

$$\lim_{n \rightarrow \infty} S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) = \epsilon. \quad (16)$$

From (12), (15) and (16) we have

$$\begin{aligned} &S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)}) \\ &\leq (\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{n(k)}, x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, x_{m(k)}, Tx_{m(k)})S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)})) \\ &\leq \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})S(x_{n(k)}, x_{n(k)}, x_{m(k)})) \end{aligned}$$

Hence,

$$\frac{S(x_{n(k+1)}, x_{n(k+1)}, x_{m(k+1)})}{S(x_{n(k)}, x_{n(k)}, x_{m(k)})} \leq \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) \leq 1,$$

letting  $k \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \beta(S(x_{n(k)}, x_{n(k)}, x_{m(k)})) = 1$ .

That is,  $\lim_{n \rightarrow \infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = 0 < \epsilon$ , which is a contradiction. Hence  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete, then there is  $z \in X$  such that  $x_n \rightarrow z$ . Since  $T$  is continuous, then we have

$$Tz = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = z.$$

So,  $z$  is a fixed point of  $T$ . Next we suppose that (ii) holds. Then  $\alpha(z, z, Tz) \geq 1$ .

Now

$$\begin{aligned} S(Tz, Tz, x_{n+1}) &\leq (\alpha(z, z, Tz)\alpha(z, z, Tz)\alpha(x_n, x_n, Tx_n)S(Tz, Tz, Tx_n)) \\ &\leq \beta(S(z, z, x_n))S(z, z, x_n). \end{aligned}$$

Hence, we get

$$S(Tz, Tz, z) \leq \beta(S(z, z, x_n))S(z, z, x_n),$$

letting  $n \rightarrow \infty$ , we get  $S(Tz, Tz, z) \leq \beta(S(z, z, z))S(z, z, z) \implies S(Tz, Tz, z) \leq 0$ .

But  $S(Tz, Tz, z) \geq 0$ . That is  $Tz = z$ .

### 3. CONCLUSION AND FUTURE WORKS

In conclusion, the fixed point theorems presented in this work for  $\alpha$ -admissible mappings in  $S$ -metric spaces provide significant extensions and generalizations of several classical results by incorporating weaker contractive conditions. This approach broadens the applicability of fixed point theory to more general settings. Furthermore, the unification of existing results within a single theoretical framework demonstrates the robustness and versatility of the proposed method. These findings not only strengthen the theoretical foundation of fixed point theory in  $S$ -metric spaces but also create new avenues for exploration.

As future work, the authors intend to extend the current results to partially ordered  $S$ -metric spaces and to investigate their applications to coupled and tripled fixed point problems. Such directions may further enhance the depth and scope of fixed point theory in generalized metric structures.

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