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Fractional Infimal and Supremal Convolutions With Applications

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Abstract: This paper explores the ideas of *p-infimal* convolution (*p-ic*), $(h\Box_p k)(\xi) = \inf_\eta [h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta)]^p$, and *p-supremal* convolution (*p-sc*), $(h \boxtimes_p k)(\xi) = \sup_\eta [h^p(\xi-\eta)+k^p(\eta)]^{\frac{1}{p}}$ where $0 < p < 1$, as an extension of infimal and supremal convolutions, and we demonstrate that these operations are commutative and associative for any p. Meanwhile, we show that the (*p-ic*) increases with p while the (*p-sc*) decreases and notice that when applying the (*p-ic*) for a certain function several times, we get a sub-additive function, while applying the supremal convolution several times we get a super-additive function. Also, we extend the convolution of two functions to *p-convolution* (*p-c*), $(h *_p k)(t) = \left[\int_0^t (h(\tau)k(t-\tau))^{\frac{1}{p}} d\tau\right]^p$, which can calculate the Laplace transform for numerous functions, and we go on to demonstrate its practical applications. We present a new framework for solving a Volterra integral equation in the p form , $u(\xi) = h(\xi) + \lambda \left[\int_0^\xi k^{\frac{1}{p}}(\xi-t)u^{\frac{1}{p}}(t)dt\right]$, using the (*p-c*) definition.

Keywords: convolution, infimal convolution, supremal convolution.

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1 Introduction

Infimal and supremal convolutions have several applications in many fields such as convex analysis, functional analysis, optimization theory, integral equations and image processing. The convolution allows the combination of functions in a way that preserves certain properties such as convexity, the approximation of non-smooth functions and the construction of weak solutions of partial differential equations. In [1], the operation of infimal convolution, $(h\Box k)(\xi) = \inf_\eta [h(\xi-\eta)+k(\eta)]$, was introduced by Rockefeller which is a tool widely used in convex analysis and optimization theory. For more properties of this operation, see [2], [3] and [4]. In [5], Kiselman found that infimal convolution provides an effective framework for defining distances within the image plane. By using infimal convolution, he proved that a function h is sub-additive if and only if it satisfies the inequality $h\Box h \geq h$. One of his interesting results is that by repeating infimal convolution an infinite number of times for a certain function, he obtains a sub-additive function. We found that this technique can be applied to (*p-ic*), which occurs when applying (*p-ic*) to a certain function several times. In this case, we obtain a sub-additive function, whereas applying the supremal convolution several times

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yields a super-additive function. We noticed that sub-additivity (or super-additivity) can be characterized in terms of (*p-ic*) (or (*p-sc*)), so if $(h\Box_p h)^{\frac{1}{p}} \geq h^{\frac{1}{p}}$ then $h^{\frac{1}{p}}$ is sub-additive. Moreover, if $(h\boxtimes_p h)^p \leq h^p$ then h^p is super-additive. The convolution $(h*k)(t) = \int_0^t h(\tau)k(t-\tau)d\tau$ is an important operation in mathematics with many applications in diverse areas. It is an effective technique for solving differential and integral equations [6], [7], calculating probabilities [8] and defining important functions in number theory [9]. Also, it has numerous applications in signal processing [10], [11], [12], deep learning [13] and image processing such as filtering, smoothing, enhancing, and extracting features from data [14], [15], [10], [16], [17].

In this paper, we introduce new notions of convolutions such as *p-convolution* (*p-c*) $(h*_p k)(t) = \left[\int_0^t (h(\tau)k(t-\tau))^{\frac{1}{p}} d\tau\right]^p$, p-infimal convolution (*p-ic*) $(h\Box_p k)(\xi) = \inf_\eta [h^{\frac{1}{p}}(\xi-\eta) + k^{\frac{1}{p}}(\eta)]^p$ and *p-supremal* convolution (*p-sc*) $(h\boxtimes_p k)(\xi) = \sup_\eta [h^p(\xi-\eta) + k^p(\eta)]^{\frac{1}{p}}$ where $0 < p < 1$. In section two, some basic definitions are introduced. In section three, we introduce (*p-ic*) and demonstrate that its operation is both commutative and associative (Proposition 2). We prove that sub-additivity of a certain function can be characterized in terms of (*p-ic*) (Proposition 1). We notice that when applying (*p-ic*) for a certain function several times we get a sub-additive function (Theorem 1). In section four, we provide (*p-sc*) and illustrate that it has the properties of commutativity and associativity. (Proposition 4) and show that super-additivity of a certain function can be characterized in terms of (*p-sc*)(Proposition 5). We notice that when applying the (*p-sc*) for a certain function several times we get a super-additive function (Theorem 2) and then introduce an interesting relation between $(p-ic)$, $(p-sc)$, infimal convolution and supremal convolution. We notice that $(h\Box_p k)$ is increasing with p while $(h\boxtimes_p k)$ is decreasing with p (Proposition 6). In Section five, we introduce the definition of (*p-c*), followed by examples and prove the fractional convolution theorem (Theorem 4). We then provide some applications in Laplace transformations. In section six, we solve the *p*-Volterra integral equation $u(\xi) = h(\xi) + \lambda\left[\int_0^\xi k^{\frac{1}{p}}(\xi-t)u^{\frac{1}{p}}(t)dt\right]$ as an application using the definition of (*p-c*).

2 Preliminaries

We begin by outlining the core definitions and principles that underpin this study.

Definition 1 *[18], [1], [19] Let U be a vector space over the field R of real numbers. A subset X of U is said to be convex if the line segment,* $[\xi,\eta] = (1-\lambda)\xi + \lambda\eta, 0 \leq \lambda \leq 1$ *is contained in X for any given choice of* $\xi,\eta \in X$*; essentially, if* $\xi,\eta \in X$ *implies* $[\xi,\eta] \in X$.

Definition 2 *[18], [1], [19] Let* $U \subset R^n$ *be a nonempty convex set. A function* $h : U \to R$ *is said to be convex on U if for any* $\xi,\eta \in X$ *and* $0 \leq \lambda \leq 1$*, we have* $h((1-\lambda)\xi + \lambda\eta) \leq (1-\lambda)h(\xi) + \lambda h(\eta)$.

Definition 3 *[20], [21], [19] Let* $U \subset R^n$ *and* $0 < p < 1$*. If for each* $\xi,\eta \in U$*,* $\lambda,\mu \geq 0$ *such that* $\lambda^p + \mu^p = 1$ *,* $\lambda\xi + \mu\eta \in U$*, then U is called a p-convex set in* R^n*. The p-convexity of U can also be formalized as follows*

$$\lambda\xi + (1-\lambda^p)^{\frac{1}{p}}\eta \in U$$

for all $\xi,\eta \in U$ *and* $\lambda,\mu \in [0,1]$.

Definition 4 *[20], [21], [19] Let* $U \subset R^n$ *be a nonempty p-convex set. A function* $h : U \to R$ *is said to be p-convex on U if for any* $\xi,\eta \in U$ *and* $\lambda,\mu \in [0,1]$*, then we have*

$$h(\lambda\xi + \mu\eta) \leq \lambda h(\xi) + \mu h(\eta)$$

such that $\lambda^p + \mu^p = 1$.

Definition 5 *[18], [22], [2], [1] Let h,k be two functions which act on the set X and produce values in the extended real line $[-\infty,+\infty]$. The infimal convolution $h\Box k$ of f and k is defined by*

$$(h\Box k)(\xi) = \inf_{\eta\in X}[h(\xi-\eta)+k(\eta)]$$

Definition 6*Let h,k be two functions which act on the set X and produce values in the extended real line $[-\infty,+\infty]$. The supremal convolution $h\boxtimes k$ of h and k is defined by*

$$(h\boxtimes k)(\xi) = \sup_{\eta\in X}[h(\xi-\eta)+k(\eta)]$$

3 Fractional Infimal Convolution

We dedicate the subsequent section to a detailed presentation of the p-infimal convolution $(p-ic)$ $(h\Box_p k)$ of two functions h, k in definition 7. Moreover, we show in Proposition 2 that $(p-ic)$ is a commutative and associative operation.

Definition 7 *[23] Let h,k be two functions which act on the set X and produce values in the extended real line $[-\infty,+\infty]$. For $0<p<1$, we define the p-infimal convolution $(p-ic)$ $h\Box_p k$ of two functions h and k as follows:*

$$(h\Box_p k)(\xi) = \inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta)]^p, \xi\in X. \tag{1}$$

For $p=1$, see [24], [25], [1].

Proposition 1*A necessary condition for a function $h^{\frac{1}{p}}$ to be sub-additive (i.e., $h^{\frac{1}{p}}(\xi+\eta)\leq h^{\frac{1}{p}}(\xi)+h^{\frac{1}{p}}(\eta)$) is that $(h\Box_p h)^{\frac{1}{p}}\geq h^{\frac{1}{p}}$. Moreover, if $h^{\frac{1}{p}}$ is super-additive (i.e., $h^{\frac{1}{p}}(\xi+\eta)\geq h^{\frac{1}{p}}(\xi)+h^{\frac{1}{p}}(\eta)$) then $(h\Box_p h)^{\frac{1}{p}}\leq h^{\frac{1}{p}}$.*

Proof . Assume,

$$(h\Box_p h)^{\frac{1}{p}}(\xi)\geq h^{\frac{1}{p}}(\xi) \tag{2}$$

then from equation (1) we get,

$$\inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+h^{\frac{1}{p}}(\eta)]\geq h^{\frac{1}{p}}(\xi) \tag{3}$$

In particular, when replacing ξ by $\xi+\eta$ in equation (2) we get,

$$(h\Box_p h)^{\frac{1}{p}}(\xi+\eta)\geq h^{\frac{1}{p}}(\xi+\eta) \tag{4}$$

So from equation (1) and from the property that the infimum of a set is always less than or equal to every element of the set [26] we get,

$$h^{\frac{1}{p}}(\xi)+h^{\frac{1}{p}}(\eta)\geq \inf_{\eta\in X}[h^{\frac{1}{p}}(\xi)+h^{\frac{1}{p}}(\eta)]\geq h^{\frac{1}{p}}(\xi+\eta)$$

Hence, $h^{\frac{1}{p}}$ is sub-additive.
Similarly, let $h^{\frac{1}{p}}$ be super-additive then

$$h^{\frac{1}{p}}(\xi) = h^{\frac{1}{p}}(\xi+\eta-\eta)\geq h^{\frac{1}{p}}(\xi-\eta)+h^{\frac{1}{p}}(\eta), \forall\eta\in X$$

$$\geq \inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+h^{\frac{1}{p}}(\eta)] = (h\Box_p h)^{\frac{1}{p}}.$$

Proposition 2 *[23] Fractional infimal convolution is an operation that is both commutative and associative.*

Proof . (I) Commutativity: Assuming $\zeta=\xi-\eta$ we get from definition (7),

$$(h\Box_p k)(\xi)=\inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta)]^p=\inf_{\zeta\in X}[h^{\frac{1}{p}}(\zeta)+k^{\frac{1}{p}}(\xi-\zeta)]^p=(k\Box_p h)(\xi).$$

(II)Associativity:

$$\begin{aligned}(h\Box_p(k\Box_p r))(\xi)&=\inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+(k\Box_p r)^{\frac{1}{p}}(\eta)]^p \quad from\,Def.7\\
&=\inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+\inf_{\zeta\in X}(k^{\frac{1}{p}}(\eta-\zeta)+r^{\frac{1}{p}}(\zeta))]^p \quad from\,Def.7\\
&since\ the\ term\ h^{\frac{1}{p}}(\xi-\eta)\ is\ independent\ on\ \zeta,\ we\ can\ write\\
&=\inf_{\eta\in X}[\inf_{\zeta\in X}[h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta-\zeta)+r^{\frac{1}{p}}(\zeta)]^p]\\
&=\inf_{\zeta\in X}\inf_{\eta\in X}[r^{\frac{1}{p}}(\zeta)+(k^{\frac{1}{p}}(\eta-\zeta)+h^{\frac{1}{p}}(\xi-\eta))]^p\\
&the\ term\ r^{\frac{1}{p}}(\zeta)\ is\ independent\ on\ \eta\ so,\\
&=\inf_{\zeta\in X}[r^{\frac{1}{p}}(\zeta)+\inf_{\eta\in X}(k^{\frac{1}{p}}(\eta-\zeta)+h^{\frac{1}{p}}(\xi-\eta))]^p\\
&by\ adding\ and\ subtract\ \zeta\ in\ term\ of\ h^{\frac{1}{p}}(\xi-\zeta)\ we\ get,\\
&=\inf_{\zeta\in X}[r^{\frac{1}{p}}(\zeta)+\inf_{\eta-\zeta\in X}(h^{\frac{1}{p}}((\xi-\zeta)-(\eta-\zeta))+k^{\frac{1}{p}}(\eta-\zeta))]^p\\
&=\inf_{\zeta\in X}[r^{\frac{1}{p}}(\zeta)+\inf_{\tilde\eta\in X}(h^{\frac{1}{p}}((\xi-\zeta)-(\tilde\eta))+k^{\frac{1}{p}}(\tilde\eta))]^p\ where,\ \eta-\zeta=\tilde\eta.\\
&=\inf_{\zeta\in X}[r^{\frac{1}{p}}(\zeta)+(h\Box_p k)^{\frac{1}{p}}(\xi-\zeta)]^p=((h\Box_p k)\Box_p r)(\xi).\end{aligned}$$

An n-fold convolution is defined as

$$(h_1\Box_p\ldots\Box_p h_n)(\xi)=inf[\sum_{i=1}^{n}h_i^{\frac{1}{p}}(\xi^i)]^p,$$

where the infimum is taken over all choices of elements $\xi^i\in X$ for $i=1,2,\ldots n$ such that $\xi^1+\cdots+\xi^n=\xi$.

The next proposition will be used in the proof of Theorem 1.

Proposition 3*Let $h,k:X\to[0,\infty]$ be nonnegative functions and let $0<p<1$. Then the following inequality holds pointwise:*

$$(h(\xi)+k(\xi))^p\ \le\ h^p(\xi)+k^p(\xi),\quad \forall\xi\in X.$$

For a proof, see [27], [28].

Theorem 1*Let $K:X\to[0,\infty]$ be a function on X satisfying $K(0)=0$. Define a sequence of functions $(K_j)_{j=1}^{\infty}$ by putting, $K_1=K,\ K_j=(K_{j-1}^p\Box_p K^p)^{\frac{1}{p}},\quad j=2,3,\ldots$. Then the sequence (K_j^p) is decreasing and its $limK_j^p=k\ge0$ is sub-additive.*

Proof . We'll show that the sequence K_j^p is decreasing, taking $\eta=0$ in the definition of K_j we get, for $j=2$,

$$\begin{aligned}K_2^p(\xi)&=(K_1^p\Box_p K^p)(\xi)=(K^p\Box_p K^p)(\xi)\quad from\,Def.7\ we\ have,\\
&=\inf_{\eta\in X}[K(\xi-\eta)+K(\eta)]^p\quad taking\ \eta=0\ we\ get,\\
&=\inf_{\eta\in X}[K(\xi-0)+K(0)]^p\end{aligned}$$

from Proposition 3 and from the property that is the infimum of a set is always less than or equal to every element of the set we obtain,

$$\begin{aligned} K_2^p(\xi) &\leq [K^p(\xi) + K^p(0)] \\ &\leq K^p(\xi) \leq K_1^p(\xi). \end{aligned}$$

For $j = 3$,

$$\begin{aligned} K_3^p(\xi) &= (K_2^p \square_p K^p)(\xi) \\ &= \inf_{\eta \in X} [K_2(\xi - 0) + K(0)]^p \quad from\ Def.7 \\ &\leq [K_2^p(\xi) + K^p(0)] \quad from\ Prop.3 \\ &\leq K_2^p(\xi). \end{aligned}$$

For $j = j+1$,

$$\begin{aligned} K_{j+1}^p(\xi) &= (K_j^p \square_p K^p)(\xi) \\ &= \inf_{\eta \in X} [K_j(\xi - 0) + K(0)]^p \quad from\ Def.7 \\ &\leq [K_j^p(\xi) + K^p(0)] \leq K_j^p(\xi). \quad from\ Prop.3 \end{aligned}$$

So, we get $K_2^p \leq K_1^p$, $K_3^p \leq K_2^p$ and $K_{j+1}^p \leq K_j^p$. Therefore, the sequence (K_j^p) is decreasing. Next, we shall prove that the limit of the sequence (K_j^p) is sub-additive (i.e. $k(\xi + \eta) \leq k(\xi) + k(\eta)$). Since the sequence is decreasing, then $\inf_j K_j^p(\xi) = k(\xi)$. Let ξ, η be given with $k(\xi), k(\eta) < +\infty$ and consider a fixed positive number ε. Then we can find numbers j, r such that $K_j^p(\xi) < k(\xi) + \varepsilon$ and $K_r^p(\xi) < k(\eta) + \varepsilon$. Replacing ξ by $\xi + \eta$ in the case of K_{j+1}^p we obtain,

$$K_{j+1}^p(\xi + \eta) \leq K_j^p(\xi) + K^p(\eta),$$

$$\begin{aligned} K_{j+2}^p(\xi + \eta) = K_{j+2}^p(\xi + \eta + \zeta - \zeta) &\leq K_{j+1}^p(\xi + \zeta) + K^p(\eta - \zeta) \\ &\leq K_j^p(\xi) + K^p(\zeta) + K^p(\eta - \zeta) \\ &\leq K_j^p(\xi) + \inf_{\zeta \in X} [K^p(\eta - \zeta) + K^p(\zeta)] \\ &\leq K_j^p(\xi) + K_2^p(\eta). \end{aligned}$$

Similarly we get,

$$K_{j+r}^p(\xi + \eta) \leq K_j^p(\xi) + K_r^p(\eta),$$

$$k(\xi + \eta) \leq K_{j+r}^p(\xi + \eta) \leq K_j^p(\xi) + K_r^p(\eta) < k(\xi) + k(\eta) + 2\varepsilon.$$

Since ε is arbitrary, the inequality $k(\xi + \eta) \leq k(\xi) + k(\eta)$ follows. Therefore, the limit k is sub-additive.

For $p = 1$ see, [5].

4 Fractional Supremal Convolution

In this section, we present the p-supremal convolution $(p-sc)$ $(h \boxtimes_p k)$ of the two functions h, k in definition 8. Moreover, in proposition 4 $(p-sc)$ is shown to be commutative and associative operations.

Definition 8 *Consider two functions h, k defined on the domain X and mapping to the extended real line $[0, +\infty]$. For $0 < p < 1$, we define the p-supremal convolution $(p-sc)$ $(h \boxtimes_p k)$ of two functions h and k as follows:*

$$(h \boxtimes_p k)(\xi) = \sup_{\eta \in X} [h^p(\xi - \eta) + k^p(\eta)]^{\frac{1}{p}}, \xi \in X.$$

Proposition 4 *Fractional supremal convolution is an operation that is both commutative and associative.*

Proof . (I) Commutativity: By setting $\zeta = \xi - \eta$ we obtain from definition (8),

$$(h \boxtimes_p k)(\xi) = \sup_{\eta \in X} [h^p(\xi - \eta) + k^p(\eta)]^{\frac{1}{p}} = \sup_{\zeta \in X} [h^p(\zeta) + k^p(\xi - \zeta)]^{\frac{1}{p}} = (k \boxtimes_p h)(\xi).$$

(II)Associativity:

$$\begin{aligned}
[h \boxtimes_p (k \boxtimes_p r)](\xi) &= \sup_{\eta \in X} [h^p(\xi - \eta) + (k \boxtimes_p r)^p(\eta)]^{\frac{1}{p}} \quad from\, Def.8 \\
&= \sup_{\eta \in X} [h^p(\xi - \eta) + \sup_{\zeta \in X} (k^p(\eta - \zeta) + r^p(\zeta))]^{\frac{1}{p}} \quad from\, Def.8 \\
& \ since\ the\ term\ h^p(\xi - \eta)\ is\ independent\ on\ \zeta,\ we\ can\ write \\
&= \sup_{\eta \in X} [\sup_{\zeta \in X} [h^p(\xi - \eta) + k^p(\eta - \zeta) + r^p(\zeta)]^{\frac{1}{p}}] \\
&= \sup_{\zeta \in X} \sup_{\eta \in X} [r^p(\zeta) + (k^p(\eta - \zeta) + h^p(\xi - \eta))]^{\frac{1}{p}} \\
& the\ term\ r^p(\zeta)\ is\ independent\ on\ \eta\ so, \\
&= \sup_{\zeta \in X} [r^p(\zeta) + \sup_{\eta \in X} (k^p(\eta - \zeta) + h^p(\xi - \eta))]^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& by\ adding\ and\ subtracting\ \zeta\ in\ term\ of\ h^p(\xi - \zeta)\ we\ get, \\
&= \sup_{\zeta \in X} [r^p(\zeta) + \sup_{\eta - \zeta \in X} (h^p((\xi - \zeta) - (\eta - \zeta)) + k^p(\eta - \zeta))]^{\frac{1}{p}} \\
&= \sup_{\zeta \in X} [r^p(\zeta) + \sup_{\tilde{\eta} \in X} (h^p((\xi - \zeta) - (\tilde{\eta})) + k^p(\tilde{\eta}))]^{\frac{1}{p}} \quad where, \eta - \zeta = \tilde{\eta} \\
&= \sup_{\zeta \in X} [r^p(\zeta) + (h \boxtimes_p k)^p(\xi - \zeta)]^{\frac{1}{p}} \\
&= [(h \boxtimes_p k) \boxtimes_p r](\xi).
\end{aligned}$$

Proposition 5 *A necessary condition for a function h^p to be super-additive is that $(h \boxtimes_p h)^p \leq h^p$. Moreover, if h^p is sub-additive then $(h \boxtimes_p h)^p \geq h^p$.*

Proof . Assume,

$$(h \boxtimes_p h)^p(\xi) \leq h^p(\xi)$$

i.e.

$$\sup_{\eta\in X}[h^p(\xi-\eta)+h^p(\eta)] \leq h^p(\xi)$$

In particular,

$$(h\boxtimes_p h)^p(\xi+\eta) \leq h^p(\xi+\eta)$$

so,

$$h^p(\xi)+h^p(\eta) \leq \sup_{\eta\in X}[h^p(\xi)+h^p(\eta)] \leq h^p(\xi+\eta)$$

Hence, h^p is super-additive.
Similarly, let h^p is sub-additive then

$$h^p(\xi) = h^p(\xi+\eta-\eta) \leq h^p(\xi-\eta)+h^p(\eta),\ \forall\eta\in X$$

$$\leq \sup_{\eta\in X}[h^p(\xi-\eta)+h^p(\eta)] = (h\boxtimes_p h)^p.$$

An example can be constructed using Proposition 4 for a p norm $\parallel \xi \parallel$. If we take $h^p(\xi) = \parallel \xi \parallel^p$ so we get,

$$\parallel \xi \parallel^p \leq \parallel \xi-\eta \parallel^p + \parallel \eta \parallel^p = h^p(\xi-\eta)+h^p(\eta), \forall\eta\in X$$

Then,

$$\parallel \xi \parallel^p \leq \sup_{\eta\in X}[h^p(\xi-\eta)+h^p(\eta)]$$

i.e. $(h\boxtimes_p h)^p \geq h^p$.

Theorem 2*Let $K: X\to[0,\infty]$ be a function on X satisfying $K(0)=0$. Define a sequence of functions $(K_j)_{j=1}^{\infty}$ by putting, $K_1 = K,\ K_j = (K_{j-1}^{\frac{1}{p}}\boxtimes_p K^{\frac{1}{p}})^p,\ j=2,3,\ldots$. Then the sequence $(K_j^{\frac{1}{p}})$ is increasing and its lim$K_j^{\frac{1}{p}} = k\geq 0$ is super-additive.*

Proof . This proof establishes that the sequence $K_j^{\frac{1}{p}}$ is increasing, taking $\eta=0$ in the definition of K_j we get, for $j=2$,

$$\begin{aligned}
K_2^{\frac{1}{p}}(\xi) &= (K_1^{\frac{1}{p}}\boxtimes_p K^{\frac{1}{p}})(\xi) = (K^{\frac{1}{p}}\boxtimes_p K^{\frac{1}{p}})(\xi)\\
&= \sup_{\eta\in X}[K(\xi-0)+K(0)]^{\frac{1}{p}} \quad from\ Def.8\\
&\geq [K(\xi-0)+K(0)]^{\frac{1}{p}} \quad since\ \frac{1}{p}>1\ then,\\
&\geq [K^{\frac{1}{p}}(\xi)+K^{\frac{1}{p}}(0)]\\
&\geq K^{\frac{1}{p}}(\xi) \geq K_1^{\frac{1}{p}}(\xi).
\end{aligned}$$

For $j=3$,

$$\begin{aligned}
K_3^{\frac{1}{p}}(\xi) &= (K_2^{\frac{1}{p}}\boxtimes_p K^{\frac{1}{p}})(\xi)\\
&\overset{\text{def8}}{=} \sup_{\eta\in X}[K_2(\xi-0)+K(0)]^{\frac{1}{p}}\\
&\geq [K_2^{\frac{1}{p}}(\xi)+K^{\frac{1}{p}}(0)] \geq K_2^{\frac{1}{p}}(\xi).
\end{aligned}$$

For $j = j+1$,

$$\begin{aligned} K_{j+1}^{\frac{1}{p}}(\xi) &= (K_j^{\frac{1}{p}} \boxtimes_p K^{\frac{1}{p}})(\xi) \\ &= \sup_{\eta \in X}[K_j(\xi - 0) + K(0)]^{\frac{1}{p}} \\ &\geq [K_j^{\frac{1}{p}}(\xi) + K^{\frac{1}{p}}(0)] \geq K_j^{\frac{1}{p}}(\xi). \end{aligned}$$

So, we get $K_2^{\frac{1}{p}} \geq K_1^{\frac{1}{p}}$, $K_3^{\frac{1}{p}} \geq K_2^{\frac{1}{p}}$ and $K_{j+1}^{\frac{1}{p}} \geq K_j^{\frac{1}{p}}$.Therefore, the sequence $(K_j^{\frac{1}{p}})$ is increasing. Next, we shall prove that $k(\xi + \eta) \geq k(\xi) + k(\eta)$. Let $\sup_j K_j^{\frac{1}{p}}(\xi) = k(\xi)$ and let ξ, η be given with $k(\xi), k(\eta) < +\infty$ and suppose ε is a positive real number. Then we can find numbers j, r such that $K_j^{\frac{1}{p}}(\xi) > k(\xi) - \varepsilon$ and $K_r^{\frac{1}{p}}(\xi) > k(\eta) - \varepsilon$. We have,

$$K_{j+1}^{\frac{1}{p}}(\xi + \eta) \geq K_j^{\frac{1}{p}}(\xi) + K^{\frac{1}{p}}(\eta),$$

$$\begin{aligned} K_{j+2}^{\frac{1}{p}}(\xi + \eta) &\geq K_{j+1}^{\frac{1}{p}}(\xi + \zeta) + K^{\frac{1}{p}}(\eta - \zeta) \\ &\geq K_j^{\frac{1}{p}}(\xi) + K^{\frac{1}{p}}(\zeta) + K^{\frac{1}{p}}(\eta - \zeta) \\ &\geq K_j^{\frac{1}{p}}(\xi) + \sup_{\zeta \in X}[K^{\frac{1}{p}}(\eta - \zeta) + K^{\frac{1}{p}}(\zeta)] \\ &\geq K_j^{\frac{1}{p}}(\xi) + K_2^{\frac{1}{p}}(\eta). \end{aligned}$$

Similarly, we get

$$K_{j+r}^{\frac{1}{p}}(\xi + \eta) \geq K_j^{\frac{1}{p}}(\xi) + K_r^{\frac{1}{p}}(\eta),$$

$$\begin{aligned} k(\xi + \eta) &\geq K_{j+r}^{\frac{1}{p}}(\xi + \eta) \\ &\geq K_j^{\frac{1}{p}}(\xi) + K_r^{\frac{1}{p}}(\eta) \\ &> k(\xi) + k(\eta) - 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the inequality $k(\xi + \eta) \geq k(\xi) + k(\eta)$ follows. Therefore, the limit k is super-additive.

We will introduce an interesting relation between $(p - ic), (p - sc)$, infimal convolution and supremal convolution. We notice that $(h \square_p k)$ increases with p, while $(h \boxtimes_p k)$ decreases with p. More precisely, we get the following proposition.

Before proceeding, we state the following theorem, which will be used in the proof of the next proposition.

Theorem 3 (Jensen's inequality)*Let $a_1, a_2, \ldots, a_n$ be real or complex numbers. If $0 < p < q$, then*

$$\left(\sum_{i=1}^{n} |a_i|^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}.$$

For a proof see, [29], [30], [31].

Proposition 6*For $0 < p < q < 1$ we get*

$$(h \square_p k) \leq (h \square_q k) \leq (h \square k) \leq (h \boxtimes k) \leq (h \boxtimes_q k) \leq (h \boxtimes_p k).$$

Proof . From Definition 7, we get,

$$(h\square_p k)(\xi) = \inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta)]^p$$

From Jensen's inequality, Proposition 3 and for $p<q$ we have,

$$(h\square_p k)(\xi) = \inf_{\eta\in X}[h^{\frac{1}{p}}(\xi-\eta)+k^{\frac{1}{p}}(\eta)]^p \leq \inf_{\eta\in X}[h^{\frac{1}{q}}(\xi-\eta)+k^{\frac{1}{q}}(\eta)]^q \leq$$
$$\leq \inf_{\eta\in X}[h(\xi-\eta)+k(\eta)] \leq \sup_{\eta\in X}[h(\xi-\eta)+k(\eta)] = h\boxtimes k.$$

Similarly,

$$(h\boxtimes_p k)(\xi) = \sup_{\eta\in X}[h^p(\xi-\eta)+k^p(\eta)]^{\frac{1}{p}} \geq \sup_{\eta\in X}[h^q(\xi-\eta)+k^q(\eta)]^{\frac{1}{q}} \geq$$
$$\geq \sup_{\eta\in X}[h(\xi-\eta)+k(\eta)] \geq \inf_{\eta\in X}[h(\xi-\eta)+k(\eta)] = h\square k.$$

Remark 1

$$\sup_p (h\square_p k)(\xi) = \inf_p (h\boxtimes_p k)(\xi) = (h\square k)(\xi).$$

5 Fractional convolution

In this section, we present the (p-c) $(h *_p k)$ of the two functions h,k in definition 9. Moreover, we introduce in theorem 4 Laplace transformation for (p-c).

Definition 9*The fractional convolution of two functions $h(t)$ and $k(t)$ defined for $t>0$, $h,k\in L^p$, is given by the integral*

$$(h *_p k)(t) = \left[\int_0^t \left(h(\tau)k(t-\tau)\right)^{\frac{1}{p}} d\tau\right]^p$$

which exists if h and k are piece-wise continuous [32].

Substituting $u = t-\tau$ gives

$$(h *_p k)(t) = \left[\int_0^t \left(h(t-u)k(u)\right)^{\frac{1}{p}} du\right]^p = (k *_p h)(t)$$

then, the fractional convolution is commutative. In addition, the p-convolution exhibits these fundamental properties: :
$(i) c(h *_p k) = ch *_p k = h *_p ck, c$ constant;
$(ii) h *_p (k *_p r) = (h *_p k) *_p r$(associative property);
$(iii) h *_p (k+r) = (h *_p k)+(h *_p r)$ (distributive property).
Properties (i) and (iii) are routinely verified. As for (ii),

$$
\begin{aligned}
[h*_p(k*_p r)](t) &= \left[\int_0^t \Big(h(\tau)(k*_p r)(t-\tau)\Big)^{\frac{1}{p}} d\tau\right]^p \\
&= \left[\int_0^t h^{\frac{1}{p}}(\tau)\left(\int_0^{t-\tau} k^{\frac{1}{p}}(\xi) r^{\frac{1}{p}}(t-\tau-\xi)d\xi\right)d\tau\right]^p \\
&= \left[\int_0^t h^{\frac{1}{p}}(\tau)\int_\tau^t k^{\frac{1}{p}}(u-\tau) r^{\frac{1}{p}}(t-u)dud\tau\right]^p, (\xi = u-\tau) \\
&= \left[\int_0^t \left(\int_0^u h^{\frac{1}{p}}(\tau) k^{\frac{1}{p}}(u-\tau)d\tau\right) r^{\frac{1}{p}}(t-u)du\right]^p \\
&= [(h*_p k)*_p r](t)
\end{aligned}
$$

For $p = 1$ see, [33], [32], [34].

Example 1 $h(t) = e^{\frac{t}{2}}, k(t) = \sqrt{t}$, *then for* $p = 1/2$,

$$
(h*_p k)(t) = \left[\int_0^t e^{\tau}(t-\tau)d\tau\right]^{\frac{1}{2}} = \left[te^{\tau}\Big|_0^t - (\tau e^{\tau} - e^{\tau})\Big|_0^t\right]^{\frac{1}{2}} = \left[te^t - t - te^t + e^t - e^0\right]^{\frac{1}{2}} = \left[e^t - t - 1\right]^{\frac{1}{2}}.
$$

First, we want to define H_p- transform which has a form

$$
H_p(s) = \int_0^\infty h^{\frac{1}{p}}(t).e^{-st}dt
$$

Let, $\mathscr{L}(h(t)) = H_p(s)$, we can obtain the table of H_p-integral transforms as follows (Table 1);

$h(t)$	$H_{\frac{1}{2}}(s)$	$H_{\frac{1}{3}}(s)$	$H_p(s)$
1	$\frac{1}{s}$	$\frac{1}{s}$	$\frac{1}{s}$
t	$\frac{2}{s^3}$	$\frac{6}{s^4}$	$\frac{(1/p)!}{s^{(1/p)+1}}$
t^n	$\frac{(2n)!}{s^{2n+1}}$	$\frac{(3n)!}{s^{3n+1}}$	$\frac{(n/p)!}{s^{(n/p)+1}}$
e^{at}	$\frac{1}{s-2a}$	$\frac{1}{s-3a}$	$\frac{p}{sp-a}$
$sin(wt)$	$\frac{2w^2}{4sw^2+s^2}$	$\frac{6w^3}{(w^2+s^2)(9w^2+s^2)}$	$\frac{p\blacksquare(1+1/p)}{2w}\left(\frac{1}{(s-i)^{1+1/p}}+\frac{1}{(s+i)^{1+1/p}}\right)$
$cos(wt)$	$\frac{2w^2+s^2}{4sw^2+s^2}$	$\frac{s(7w^2+s^2)}{(w^2+s^2)(9w^2+s^2)}$	$-\frac{ip\blacksquare(1+1/p)}{2w}\left(\frac{1}{(s-i)^{1+1/p}}-\frac{1}{(s+i)^{1+1/p}}\right)$

Table 1: *Table of H_p- integral transforms*

We introduce Laplace transform of the $(p\text{-}c)$ of two functions by using the following theorem.

Theorem 4 *(Fractional Convolution Theorem). If h and k are functions in L^p, piece-wise continuous on $[0,\infty]$ and of exponential order α, $\mathscr{L}(h(t)) = H_p(s), \mathscr{L}(k(t)) = K_p(s)$ then*

$$
\mathscr{L}(h*_p k)^{\frac{1}{p}}(t) = H_p(s).K_p(s)
$$

where, $(h*_p k)(t) = \left[\int_0^t (h(\tau)k(t-\tau))^{\frac{1}{p}} d\tau\right]^p$ *and* $H_p(s) = \int_0^\infty h^{\frac{1}{p}}(t).e^{-st}dt, K_p(s) = \int_0^\infty k^{\frac{1}{p}}(t).e^{-st}dt.$

Proof . Let's begin with the product

$$\begin{aligned} H_p(s).K_p(s) &= \int_{\tau=0}^{\infty} h^{\frac{1}{p}}(\tau).e^{-s\tau}d\tau \int_{u=0}^{\infty} k^{\frac{1}{p}}(u).e^{-su}du \\ &= \int_{\tau=0}^{\infty} h^{\frac{1}{p}}(\tau)d\tau \int_{u=0}^{\infty} k^{\frac{1}{p}}(u).e^{-su}.e^{-s\tau}du \\ &= \int_{\tau=0}^{\infty}\left(\int_{u=0}^{\infty} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(u).e^{-s(u+\tau)}du\right)d\tau. \end{aligned}$$

Since τ is treated as a constant in the interior integral, we can substitute $t=\tau+u$, to find that $du=dt$. This yields

$$H_p(s).K_p(s) = \int_{\tau=0}^{\infty}\left(\int_{\tau}^{\infty} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(t-\tau).e^{-st}dt\right)d\tau$$

.

If we define $k^{1/p}(t)=0$ for $t<0$, then $k^{1/p}(t-\tau)=0$ for $t<\tau$ and we can write

$$H_p(s).K_p(s) = \int_{\tau=0}^{\infty}\int_{t=0}^{\infty} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(t-\tau).e^{-st}dtd\tau.$$

Now we can reverse the order of integration, $\tau: 0\to\infty, t: 0\to\infty$ reverse $\tau: 0\to t, t: 0\to\infty$, so that

$$\begin{aligned} H_p(s).K_p(s) &= \int_{t=0}^{\infty}\int_{\tau=0}^{\infty} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(t-\tau).e^{-st}d\tau dt \\ &= \int_{t=0}^{\infty}\left(\int_{\tau=0}^{t} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(t-\tau).e^{-st}d\tau\right)dt \\ &= \int_{t=0}^{\infty}\left(\int_{\tau=0}^{t} h^{\frac{1}{p}}(\tau)k^{\frac{1}{p}}(t-\tau)d\tau\right).e^{-st}dt \\ &= \mathscr{L}(h*_p k)^{\frac{1}{p}}(t). \end{aligned}$$

for $p=1$ see, [32], [34]

Example 2 *let* $f(t)=\sqrt{t}$, $k(t)=\sqrt{sin(t)}$ *then for* $p=1/2$ *the* $1/2$*-convolution will be*

$$\begin{aligned} (h*_{1/2}k)^2(t) &= \int_{\tau=0}^{t}\Big(h(\tau)k(t-\tau)\Big)^2 d\tau = \int_{\tau=0}^{t}\tau sin(t-\tau)d\tau = \\ &= \Big[\tau cos(t-\tau)+sin(t-\tau)\Big]_0^t = t-sint \end{aligned}$$

By taking Laplace transformation of $1/2$*-convolution we have,*

$$\begin{aligned} \mathscr{L}((h*_{1/2}k)^2(t)) &= \mathscr{L}(t-sint) = \mathscr{L}(t)-\mathscr{L}(sint) = \\ &= \frac{1}{s^2}-\frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} = \\ &= H_{1/2}(s).K_{1/2}(s) \end{aligned}$$

.

where, $H_{1/2}(s)=\int_0^\infty h^2(t).e^{-st}dt, K_{1/2}(s)=\int_0^\infty k^2(t).e^{-st}dt.$

Example 3*let $h(t)=\sqrt{cos(t)}$, $k(t)=\sqrt{sin(t)}$ then for p = 1/2 the 1/2-convolution will be*

$$\begin{aligned}
(h *_{1/2} k)^2(t) &= \int_{\tau=0}^{t}\Big(h(\tau)k(t-\tau)\Big)^2 d\tau \\
&= \int_{\tau=0}^{t} cos(\tau)sin(t-\tau)d\tau \\
&= \int_{\tau=0}^{t} cos(\tau)(sint.cos\tau - cost.sin\tau)d\tau \\
&= \int_{\tau=0}^{t} (1+cos(2\tau))d\tau - cost\left[\frac{sin^2\tau}{2}\right]_0^t \\
&= \frac{1}{2}sint\left[\tau+\frac{sin2\tau}{2}\right]_0^t - \frac{1}{2}cost.sin^2 t \\
&= \frac{sint}{2}(t+\frac{sin2t}{2} - \frac{1}{2}(cost.sin^2 t) \\
&= \frac{tsint}{2} + \frac{2sin^2 tcost}{4} - \frac{costsin^2 t}{2} = \frac{1}{2}tsint.
\end{aligned}$$

By taking Laplace transformation of 1/2-convolution we have,

$$\begin{aligned}
\mathscr{L}((h *_{1/2} k)^2(t)) = \mathscr{L}(\frac{1}{2}tsint) = \frac{1}{2}\mathscr{L}(tsint) = -\frac{1}{2}\frac{d}{ds}H(s) = \\
= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{s}{(s^2+1)^2} = H_{1/2}(s).K_{1/2}(s)
\end{aligned}$$

where,$H_{1/2}(s)=\int_0^\infty h^2(t).e^{-st}dt, K_{1/2}(s)=\int_0^\infty k^2(t).e^{-st}dt.$

In the next section, we present the application of the $(p-c)$ in the integral equations.

6 p-Volterra integral equation of p-convolution type

A p-volterra integral equation is

$$u(\xi) = h(\xi) + \lambda\left[\int_0^\xi k^{\frac{1}{p}}(\xi-t)u^{\frac{1}{p}}(t)dt\right]$$

where h is a given function and u is an unknown function, u, h *and* k are functions in L^p. The function $k^{1/p}$ is called the kernel. Taking the Laplace transform for both sides;

$$\mathscr{L}\Big(u(\xi)\Big) = \mathscr{L}\Big(h(\xi)\Big) + \lambda\mathscr{L}\left(\int_0^\xi k^{\frac{1}{p}}(\xi-t)u^{\frac{1}{p}}(t)dt\right)$$

From Definition 9, we get

$$\mathscr{L}\Big(u(\xi)\Big) = \mathscr{L}\Big(h(\xi)\Big) + \lambda\mathscr{L}\Big((k *_p u)^{\frac{1}{p}}(\xi)\Big)$$

From Theorem 4, we have

$$\mathscr{L}\Big(u(\xi)\Big) = \mathscr{L}(h(\xi)) + \lambda\left[\mathscr{L}\Big(k(\xi)\Big).\mathscr{L}\Big(u(\xi)\Big)\right]$$

$$\mathscr{L}\Big(u(\xi)\Big)\left[1-\lambda\mathscr{L}\Big(k(\xi)\Big)\right] = \mathscr{L}\Big(h(\xi)\Big)$$

$$\mathscr{L}\Big(u(\xi)\Big) = \frac{\mathscr{L}\Big(h(\xi)\Big)}{1-\lambda\mathscr{L}\Big(k(\xi)\Big)} = \mathscr{L}\Big(h(\xi)\Big).\mathscr{L}\Big(\tau(\xi)\Big) = \mathscr{L}(h *_p \tau)^{\frac{1}{p}}(\xi)$$

where

$$\mathscr{L}\Big(\tau(\xi)\Big) =; \frac{1}{1-\lambda\mathscr{L}\Big(k(\xi)\Big)}, \quad \tau \in L^p$$

Taking Laplace inverse for both sides, then we get the solution of p-Volterra integral equation.

$$u(\xi) = (h *_p \tau)^{\frac{1}{p}} = \int_0^{\xi} h^{\frac{1}{p}}(\xi - t).\tau^{\frac{1}{p}}(t)dt \quad from\, Def.9$$

for $p=1$ see, [35], [36], [37], [38].

7 Conclusion

We generalize infimal and supremal convolutions to fractional infimal and supremal convolutions and study the properties of these operations. We notice that when applying the fractional infimal (or supremal) convolution for a certain function several times we get a sub-additive (or super-additive) function. Also, we extend the notion of convolution to fractional convolution and present its applications in Laplace transformation and Volterra integral equations.

Conflict of Interest

The authors declare that he has no competing interests.

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All references are listed at the end of the paper; no additional data was used.

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