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Examining the oscillatory behavior of fourth-order neutral differential equations

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Abstract: In this paper, we establish several criteria for examining the oscillatory properties of solutions to a class of fourth-order neutral differential equations. Using the principle of comparison, we obtain sufficient conditions for oscillation based on some new monotonic properties. Furthermore, the Euler differential equation yields a sharp result when using some of the new criteria. To assess the efficacy of the new criteria, we compare them with previous studies to highlight the differences.

Keywords: Oscillation; delay differential equation; neutral delay

2020 AMS Subject Classifications: 34C10, 34K11.

1 Introduction

In mathematics, differential equations are an essential tool that is used in many different scientific fields. The basic idea is to explain the relationship between a quantity and another, typically involving time or space. Differential equations are essential for simulating and comprehending dynamic systems in many different domains. Differential equations offer vital insights into how systems change or space, whether they are used in economics to model financial markets, epidemiology to comprehend the spread of illnesses, or physics to forecast the motion of objects. From basic biological models to intricate simulations in engineering, environmental research, and climate modeling, they are incredibly versatile. They aid in the description of biological processes in medicine, including population expansion and drug absorption. They are employed in astronomy to monitor the motion of celestial bodies and comprehend astrophysical occurrences, see [1, 2].

Understanding the characteristics and behavior of differential equation solutions without necessarily solving them explicitly is the main goal of the qualitative theory of differential equations. Its goal is to investigate the long-term behavior, stability, and structure of solutions. As a part of the qualitative theory, oscillation theory is essential to comprehending the qualitative behavior of solutions, especially when identifying whether they show oscillatory or non-oscillatory behavior over time. The primary focus of this field is the study of differential equations of various orders, both linear and nonlinear. For forecasting and examining the oscillatory character of differential equations, it offers a strong framework. It is an effective instrument for theoretical investigation as well as real-world use in a wide range of engineering and scientific fields [3].

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A significant class of mathematical equations known as functional differential equations (FDEs) incorporates functions of advanced or delayed arguments to expand on the idea of ordinary differential equations (ODEs). It offers a comprehensive framework for simulating memory- or delay-effect systems. They are a dynamic field of study with important applications since they integrate theoretical analysis, computational methods, and applications [4,5].

In a set of differential equations known as delay differential equations (DDEs), the rate of change of a system at a particular point in time depends on both the system's current and previous states. They are frequently used to simulate systems with intrinsic time delays in disciplines like biology, physics, engineering, and economics. It offers a strong framework for simulating time-delayed systems, capturing dynamics that ODEs by themselves are unable to capture. They are crucial for precisely modeling and comprehending a broad variety of delayed processes in both natural and artificial systems, notwithstanding their complexity. Applications include modeling disease spread, population dynamics, and neural networks in biology; control systems with feedback delays in engineering; and systems having delayed impacts, including supply chain dynamics, in economics [6].

Derivatives of the dependent variable and its advanced (or delayed) versions are included in the significant class of functional differential equations known as neutral differential equations (NDDEs). Numerous scientific and engineering fields, including signal processing, control systems, and population dynamics, use these equations. It is a crucial and intricate field of research in mathematical modeling, with many real-world applications and substantial theoretical difficulties. As computational methods and mathematics continue to progress, so does their study [7].

Fourth-order differential equations are crucial in many scientific and technical fields, particularly when higher-order derivatives are required to describe the behavior of a system. Here's an outline of their importance: Certain sophisticated models in quantum mechanics [8], like higher-order perturbation theories, control engineering [9], complex dynamics like those found in aerospace engineering (like flight stability), and vibration analysis in mechanical systems to identify critical frequencies and amplitudes, require fourth-order differential equations to describe particle behavior under specific circumstances. Fourth-order equations are integral to modeling diffusion processes, population dynamics, and other advanced phenomena, see [10].

Consider the fourth order NDDE

$$z^{(4)}(\iota)+q(\iota)x(g(\iota))=0, \tag{1}$$

where $\iota \geq \iota_0$,

$$z(\iota)=x(\iota)+p(\iota)x(h(\iota)),$$

and we assume that

(A1) $p,q \in \mathbf{C}([\iota_0,\infty))$, $0 \leq p(\iota) < 1$, and $q(\iota) \geq 0$;
(A2) $h,g \in \mathbf{C}([\iota_0,\infty))$, $h(\iota) \leq \iota$, $g(\iota) \leq \iota$, $g'(\iota) \geq 0$, $\lim_{\iota\to\infty} h(\iota)=\infty$, and $\lim_{\iota\to\infty} g(\iota)=\infty$.

A solution to (1) is defined as a real-valued function x that is four times differentiable and satisfies (1) for all sufficiently large ι. Only those solutions of (1) that satisfy the condition $\sup\{|x(\iota)| : \iota \geq \iota_0\} > 0$ are of interest to us, for all $\iota \geq \iota_0$. A solution of Eq. (1) is called non-oscillatory if it is eventually positive or eventually negative; otherwise, it is called oscillatory. The equation is said to be oscillate, if all of the solutions are oscillatory.

Several approaches, strategies, and results about the oscillation of NDDE solutions were described and summarized by Agarwal et al. [11,12] and Gyori and Ladas [13]. Moreover, oscillation theory for second-order NDDE solutions was developed in part because of the results in [14]– [17].

Research on the sufficient conditions for the oscillatory and nonoscillatory features of fourth- and higher-order differential equations has been ongoing in recent years, see [18]– [23]. Because there is such a wide collection of relevant work on this topic, we will show some previous results in the literature.

Sufficient conditions for oscillation of solutions of the DDE

$$z^{(n)}(\iota)+q(\iota)x(g(\iota))=0, \text{ for } n \geq 2, \tag{2}$$

and specific cases of this equation have been the subject of numerous publications, see [24]– [26]. These works have an advantage over others because they considered all of the positive values of $p(\iota)$.

The oscillation of (2), was examined by Agarwal et al. [24]. They concluded that there was a new relationship between the solution x and the related function z.

Baculikova et al. [25] and Xing et al. [26] investigated the required conditions for oscillation of (2) using a different method (compared with the first-order delay equation).

Furthermore, in order to get one condition that ensures oscillation for (2), Baculikova et al. [27] devised a novel Riccati substitution.

Agarwal et al. [28] concerned the even-order NDDE (2). They obtained some sufficient conditions for the oscillation by using the Riccati transformation technique.

Grace and Lalli [29] used the weighted integral method to study the oscillatory behavior of the nth even-order nonlinear DDE

$$x^{(4)}(\iota)+q(\iota)F(x(g(\iota)))=0, \tag{3}$$

and its damped form

$$x^{(n)}(\iota)+p(\iota)\left|x^{(n-1)}(\iota)\right|^{\beta}x^{(n-1)}(\iota)+q(\iota)F(x(g(\iota)))=0,$$

for $\beta\geq 0$. They established some earlier famous oscillation criteria of (3) in its second order form (i.e. at $n=2$) to $n\geq 2$ by using the condition

$$\int_{\iota_0}^{\infty} g'(\varsigma)g^{n-2}(\varsigma)\left(\int_{\varsigma}^{\infty} q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\varsigma=\infty,$$

for $g\in\mathbf{C}([\iota_0,\infty),\mathbb{R}^+)$, $g'>0$ and $\lim_{\iota\to\infty}g(\iota)=\infty$.

Zafer [30] studied the oscillation behavior of the nonlinear NDDE

$$[x(\iota)+p(\iota)x(h(\iota))]^{(n)}+F(\iota,x(\iota),x(g(\iota)))=0, \tag{4}$$

under two conditions

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota} g^{n-1}(\iota)(1-p(g(\varsigma)))q(\varsigma)\,\mathrm{d}\varsigma>\frac{(n-1)2^{(n-1)(n-2)}}{\mathrm{e}}, \tag{5}$$

or

$$\limsup_{\iota\to\infty}\int_{g(\iota)}^{\iota} g^{n-1}(\iota)(1-p(g(\varsigma)))q(\varsigma)\,\mathrm{d}\varsigma>(n-1)2^{(n-1)(n-2)}.$$

Zhang et al. [31] considered the higher-order DDE

$$(r(\iota)\left(x^{(n-1)}(\iota)\right)^{\alpha})'+q(\iota)x^{\beta}(g(\iota))=0, \tag{6}$$

where α, β are a ration of odd integers and $0<\beta\leq\alpha$. Moreover, Zhang et al. [40] studied the oscillation of (6) and improved the results in [31].

Numerous researchers have recently examined the oscillatory behavior of solutions to a higher-order delay differential equation, see [32]– [35]. However, the study of the qualitative characteristics of fractional and difference equations reflects the advancements in the study of the qualitative behavior of differential equation solutions (see [36]– [39]).

The following lemmas are required in order to demonstrate the primary results.

Lemma 1. *[41] Suppose that x be an eventually positive and n times-differentiable solution on* $[\iota_0,\infty)$. *Then, there exists a positive integer such that*

$$x^{(i)}(\iota)>0,$$

for $i=1,2,....,\zeta$, *and*

$$x^{(\zeta+1)}(\iota)<0,$$

eventually.

Lemma 2. *[41] Suppose that x be as in Lemma 1, then there is $\iota > \iota_\lambda > \iota_0$ for every $\lambda \in (0,1)$, such that*

$$\frac{\iota^i x^{(i)}(\iota)}{i!} < \frac{1}{\lambda}\binom{\zeta}{i} x(\iota),$$

for any $i = 1,2,....,\zeta$.

2 Preliminary Results

Before analyzing the oscillatory behavior of NDDEs, identifying the signs of the derivatives of x and z plays a fundamental role. To enhance clarity, we introduce specific notations that facilitate the exposition of the results. Furthermore, supplementary lemmas are presented to provide additional support for the main theorems.

Notation 1*To keep things brief, we define the functions*

$$Q(\iota) := q(\iota)(1 - p(g(\iota))),$$

and

$$\hat{Q}(\iota) := \int_\iota^\infty \int_u^\infty Q(s)\,\mathrm{d}s\mathrm{d}u.$$

The following lemma, which is immediately deduced from [42], is presented for the classification of positive solutions to equation (1). Thus,

Lemma 3. *For (1), let x be a positive solution. Then z satisfies one of the following cases:*

$$\mathbf{C}_1 : z(\iota) > 0,\ z'(\iota) > 0,\ z''(\iota) > 0,\ z'''(\iota) > 0$$

and

$$\mathbf{C}_2 : z(\iota) > 0,\ z'(\iota) > 0,\ z''(\iota) < 0,\ z'''(\iota) > 0.$$

Lemma 4. *Let x be a positive solution of (1) and satifies case $\mathbf{C}_1$. If*

$$\int_{\iota_0}^\infty g^3(s)\,Q(s)\,\mathrm{d}s = \infty, \tag{7}$$

then $\frac{z''(\iota)}{\iota}$ is decreasing and tends to zero.

Proof . For (1), let x be a positive solution and satisfies case $\mathbf{C}_1$ for $\iota \geq \iota_0$. Then,

$$\begin{aligned}\left(\iota^2\left[\frac{z''}{\iota}\right]'\right)' &= \left(\iota z''' - z''\right)' \\ &= \iota z^{(4)} + z''' - z''' \\ &= \iota z^{(4)} \\ &= -\iota q(\iota)x(g(\iota)).\end{aligned} \tag{8}$$

Since $z'(\iota) > 0$ and $h(\iota) \leq \iota$, we have that

$$\begin{aligned}x(\iota) &= z(\iota) - p(\iota)x(h(\iota)) \\ &\geq z(\iota) - p(\iota)z(h(\iota)) \\ &= (1 - p(\iota))z(\iota)\end{aligned}$$

Thus, (8) becomes

$$\left(\iota^2\left[\frac{z''}{\iota}\right]'\right)' \leq -\iota Q(\iota) z(g(\iota)). \tag{9}$$

Since $z^i(\iota) > 0$ for $i = 0,1,2,3$ and $z^{(4)}(\iota) < 0$, it follows from Lemmas 1 and 2 that

$$z(\iota) > \frac{\lambda}{6}\iota^2 z''(\iota), \tag{10}$$

for $\lambda \in (0,1)$, which with (9) gives

$$\left(\iota^2\left[\frac{z''}{\iota}\right]'\right)' \leq -\frac{\lambda}{6}\iota g^2(\iota) Q(\iota) z''(g(\iota)).$$

Integration of (9) from ι_0 to ι yields

$$\begin{aligned}\iota^2\left(\frac{z''}{\iota}\right)' &\leq -L - \int_{\iota_0}^{\iota} s g^2(s) Q(s) z''(g(s))\, \mathrm{d}s \\ &\leq -L - z''(g(\iota_0)) \int_{\iota_0}^{\iota} s g^2(s) Q(s)\, \mathrm{d}s\end{aligned} \tag{11}$$

where

$$L := -\left[\iota^2\left(\frac{z''}{\iota}\right)'\right]_{\iota=\iota_0}.$$

Using (7) gives

$$\left(\frac{z''}{\iota}\right)' \leq 0,$$

and $L > 0$.

Setting $w = \frac{z''}{\iota}$. Since w is positive and decreasing, we get that $\lim_{\iota\to\infty} w(\iota) = w_0 \geq 0$. Suppose that $w_0 > 0$. From (11), we obtain

$$w'(\iota) + \frac{1}{\iota^2}\int_{\iota_0}^{\iota} s g^2(s) Q(s) z''(g(s))\, \mathrm{d}s \leq -\frac{L}{\iota^2} < 0,$$

or

$$w'(\iota) + \frac{1}{\iota^2}\int_{\iota_0}^{\iota} s g^3(s) Q(s) w(g(s))\, \mathrm{d}s \leq -\frac{L}{\iota^2} < 0, \tag{12}$$

Integration of (12) from ι_0 to ∞ yields

$$\begin{aligned} w(\iota_0) &> w_0 + \frac{\lambda}{6} w_0 \int_{\iota_0}^{\infty} \frac{1}{u^2}\int_{\iota_0}^{u} s g^3(s) Q(s)\, \mathrm{d}s\mathrm{d}u \\ &= w_0 + \frac{\lambda}{6} w_0 \int_{\iota_0}^{\infty} g^3(s) Q(s)\, \mathrm{d}s.\end{aligned}$$

This a contradiction with (7). Therefore, $w_0 = 0$.

Lemma 5. *Let x be a positive solution of (1) and and satifies case $\mathbf{C}_2$. If*

$$\int_{\iota_0}^{\infty} g(s)\hat{Q}(s)\, \mathrm{d}s = \infty, \tag{13}$$

then then $\frac{z(\iota)}{\iota}$ is decreasing and tends to zero.

Proof . For (1), let x be a positive solution and satisfies case $\mathbf{C}_2$ for $\iota \geq \iota_0$,

$$x(\iota) > z(\iota)(1-p(\iota))$$

which with (1) yields

$$0 \geq z^{(4)}(\iota) + Q(\iota) z(g(\iota)) \tag{14}$$

Integrating (14) from ι to ∞, we obtain

$$-z'''(\iota) + z(g(\iota)) \int_{\iota}^{\infty} Q(s)\,\mathrm{d}s \leq 0. \tag{15}$$

Integrating (15) from ι to ∞, we have

$$z''(\iota) + \int_{\iota}^{\infty} z(g(u)) \int_{\iota}^{\infty} Q(s)\,\mathrm{d}s\mathrm{d}u \leq 0,$$

yields,

$$z''(\iota) + \hat{Q}(\iota) z(g(\iota)) \leq 0. \tag{16}$$

After that, we see that

$$\begin{aligned}\left(\iota^2 \left[\frac{z(\iota)}{\iota}\right]'\right)' &= \left(\iota z'(\iota) - z'(\iota)\right)' \\ &= \iota z''(\iota) \\ &= -\iota \hat{Q}(\iota) z(g(\iota)).\end{aligned} \tag{17}$$

After integrating (17) through $[\iota_0, \iota]$, we get

$$\iota^2 \left[\frac{z(\iota)}{\iota}\right]' - \iota_0^2 \left[\frac{z(\iota_0)}{\iota_0}\right]' \leq -\int_{\iota_0}^{\iota} s\hat{Q}(\iota) z(g(s))\,\mathrm{d}s.$$

After taking $K = \iota_0^2 \left[\frac{z(\iota_0)}{\iota_0}\right]'$, we find that

$$\iota^2 \left[\frac{z(\iota)}{\iota}\right]' \leq K - \int_{\iota_0}^{\iota} s\hat{Q}(s) z(g(s))\,\mathrm{d}s. \tag{18}$$

Hence,

$$\iota^2 \left[\frac{z(\iota)}{\iota}\right]' \leq K - z(g(\iota_0)) \int_{\iota_0}^{\iota} s\hat{Q}(s)\,\mathrm{d}s.$$

We determine that $\frac{z(\iota)}{\iota}$ is decreasing using (13), indicating that K is positive.

Setting $\bar{w} = \frac{z}{\iota}$. Since $\bar{w}$ is positive and decreasing, we get that $\lim_{\iota\to\infty} \bar{w}(\iota) = \bar{w}_0 \geq 0$. Assume that $\bar{w}_0 > 0$. From (18), we obtain

$$\bar{w}' + \frac{1}{\iota^2} \int_{\iota_0}^{\iota} sg(s) \hat{Q}(s) \bar{w}(g(s))\,\mathrm{d}s \leq -\frac{K}{\iota^2} < 0. \tag{19}$$

By integrating (19) from ι_0 to ∞, we have

$$\begin{aligned}\bar{w}(\iota_0) &> \bar{w}_0 + \int_{\iota_0}^{\infty} \frac{1}{u^2} \int_{\iota_0}^{\iota} sg(s) \hat{Q}(s) \bar{w}(g(s))\,\mathrm{d}s\mathrm{d}u \\ &> \bar{w}_0 + \bar{w}_0 \int_{\iota_0}^{\infty} g(s) \hat{Q}(s)\,\mathrm{d}s,\end{aligned}$$

which is in conflict with (13). Consequently, $\bar{w}_0 = 0$. The proof is finished.

3 Main Results

Based on the results obtained in the preceding section, we establish new oscillation criteria for Eq. (1), which are formulated in the subsequent theorems.

3.1 Oscillation Criteria

Theorem 1. *Assume that (7) and (13) hold. If*

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota}\frac{1}{u^2}\int_0^u sg^3(s)\,Q(s)\,\mathrm{d}s\mathrm{d}u>\frac{6}{\lambda \mathrm{e}}, \tag{20}$$

for any $\lambda\in(0,1)$, *and*

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota}\frac{1}{u^2}\int_0^u sg(s)\,\hat{Q}(s)\,\mathrm{d}s\mathrm{d}u>\frac{1}{\mathrm{e}}, \tag{21}$$

then equation (1) is oscillatory.

Proof . Let x be a positive solution of (1). We have two cases from Lemma 3. The case $\mathbf{C}_1$:

From Lemma 4, we can infer that (12) holds. Consequently,

$$\begin{aligned} w'(\iota) &\le -\frac{1}{\iota^2}\left[L+\frac{\lambda}{6}\int_{\iota_1}^{\iota} sg^3(s)\,Q(s)\,w(g(s))\,\mathrm{d}s\right]\\ &\le -\frac{1}{\iota^2}\left[L+\frac{\lambda}{6}w(g(\iota))\int_{\iota_1}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s\right]\\ &\le -\frac{1}{\iota^2}\left[L+\frac{\lambda}{6}w(g(\iota))\int_{0}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s-\frac{\lambda}{6}w(g(\iota))\int_0^{\iota_0} sg^3(s)\,Q(s)\,\mathrm{d}s\right]\\ &= -\frac{1}{\iota^2}\left[L+\frac{\lambda}{6}w(g(\iota))\int_{0}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s-Mw(g(\iota))\right], \end{aligned} \tag{22}$$

where $M=\frac{\lambda}{6}\int_0^{\iota_0} sg^3(s)\,Q(s)\,\mathrm{d}s$. There is $\iota_1\ge\iota_0$ such that $L-M\,w(g(\iota))\ge 0$ since $w(\iota)$ converges to zero. Thus, (22) turns into

$$w'(\iota)+\frac{\lambda}{6}\frac{1}{\iota^2}w(g(\iota))\int_0^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s\le 0. \tag{23}$$

The existence of a positive solution $w(\iota)$ to inequality (23) defies condition (20), as stated in [43, Theorem 2.1.1]. The case $\mathbf{C}_2$:

Lemma 5 indicates that (19) is true. As in the last case, condition (21), we likewise obtain a contradiction. As a result, all solutions oscillates.

3.2 Improved criteria for oscillation

3.2.1 Part one

This part involves enhancing the monotonic properties of the functions w and $\bar{w}$, followed by an examination of the impact on the oscillation criterion. For convenience, the following symbols are also defined:

$$\eta(\iota)=\exp\left(\frac{\lambda}{6}\int_{\iota_0}^{\iota}\frac{1}{u^2}\int_{\iota_1}^{u} sg^3(s)\,Q(s)\,\mathrm{d}s\mathrm{d}u\right)$$

and

$$\mu(\iota)=\exp\left(\int_{\iota_0}^{\iota}\frac{1}{u^2}\int_{\iota_0}^{u} sg(s)\,\hat{Q}(s)\,\mathrm{d}s\mathrm{d}u\right).$$

Lemma 6. *Let x be a positive solution of (1), (7) and (13) hold. Then, the functions ηw and $\mu \bar{w}$ are decreasing for $\lambda \in (0,1)$.*

Proof . For (1), let x be a positive solution. The case $\mathbf{C}_1$:

From Lemma 4, we can infer that (12) holds. Consequently,

$$w'(\iota)+\frac{\lambda}{6}\frac{1}{\iota^2}w(\iota)\int_{\iota_1}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s<0,$$

hence,

$$\begin{aligned}(\eta(\iota)w(\iota))' &= \eta(\iota)w'(\iota)+w(\iota)\eta'(\iota)\\ &= \eta(\iota)w'(\iota)+w(\iota)\eta(\iota)\left(\frac{\lambda}{6}\frac{1}{\iota^2}w(\iota)\int_{\iota_1}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s\right)\\ &= \eta(\iota)\left(w'(\iota)+w(\iota)\frac{\lambda}{6}\frac{1}{\iota^2}w(\iota)\int_{\iota_1}^{\iota} sg^3(s)\,Q(s)\,\mathrm{d}s\right)\\ &<0.\end{aligned}$$

Similarly, The case $\mathbf{C}_2$, from Lemma 5, we find that (19) holds. Consequently,

$$\bar{w}'+\frac{1}{\iota^2}\bar{w}(\iota)\int_{\iota_0}^{\iota} sg(s)\,\hat{Q}(s)\,\mathrm{d}s<0,$$

hence,

$$\begin{aligned}(\mu(\iota)\bar{w}(\iota))' &= \mu(\iota)\bar{w}'(\iota)+\bar{w}(\iota)\mu'(\iota)\\ &= \mu(\iota)\bar{w}'(\iota)+\bar{w}(\iota)\mu(\iota)\left(\frac{1}{\iota^2}\int_{\iota_0}^{u} sg(s)\,\hat{Q}(s)\,\mathrm{d}s\right)\\ &= \mu(\iota)\left(\bar{w}'(\iota)+\frac{1}{\iota^2}\bar{w}(\iota)\int_{\iota_0}^{u} sg(s)\,\hat{Q}(s)\,\mathrm{d}s\right)\\ &<0.\end{aligned}$$

The proof is finished.

Theorem 2. *Assume that (7) and (13) hold. If*

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota}\frac{\eta(g(u))}{u^2}\int_0^u\frac{sg^3(s)\,Q(s)}{\eta(g(s))}\,\mathrm{d}s\mathrm{d}u>\frac{6}{\lambda\mathrm{e}},\tag{24}$$

for any $\lambda \in (0,1)$, and

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota}\frac{\mu(g(u))}{u^2}\int_0^u\frac{sg(s)\,\hat{Q}(s)}{\mu(g(s))}\,\mathrm{d}s\mathrm{d}u>\frac{1}{\mathrm{e}},\tag{25}$$

then equation (1) is oscillatory.

Proof . For (1), let x be a positive solution. We have two cases from Lemma 3. The case C_1:

From Lemma 4, we can infer that (12) holds. Given that ηw is decreasing, It is evident from (12) that

$$w'(\iota)+\frac{\lambda}{6}\frac{\eta(g(\iota))}{\iota^2}w(g(\iota))\int_{\iota_1}^{\iota}\frac{sg^3(s)\,Q(s)}{\eta(g(s))}\,\mathrm{d}s<0.\tag{26}$$

The existence of a positive solution $w(\iota)$ to inequality (26) defies condition (24), as stated in [43, Theorem 2.1.1].

The case C_2:

Lemma 5 indicates that (19) is true. As in the last case, condition (25), we likewise obtain a contradiction. As a result, all solutions oscillates.

3.2.2 Part two

In this part, we will improve the monotonic features using a different approach. Next, we assess how the additional features affect the oscillation criterion in equation (1). The following symbols are also provided for convenience:

$$\Psi(\iota) = \iota^3 + \frac{\lambda_1}{6}\int_{\iota_1}^{\iota} s^3 g^3(s) Q(s)\,\mathrm{d}s$$

and

$$\hat{\Psi}(\iota) = \iota + \lambda_3 \int_{\iota_1}^{\iota} s g(s) \hat{Q}(s)\,\mathrm{d}s.$$

Theorem 3. *For any $\lambda_i \in (0,1)$ and $\iota_1 \geq \iota_0$. If*

$$\liminf_{\iota\to\infty} \int_{g(\iota)}^{\iota} Q(s)\Psi(g(s))\,\mathrm{d}s > \frac{6}{\lambda_2 \mathrm{e}}, \tag{27}$$

and

$$\liminf_{\iota\to\infty} \int_{g(\iota)}^{\iota} \hat{Q}(s)\hat{\Psi}(g(s))\,\mathrm{d}s > \frac{1}{\lambda_3 \mathrm{e}}, \tag{28}$$

then equation (1) is oscillatory.

Proof . For (1), let x be a positive solution. We have two cases from Lemma 3. The case $\mathbf{C}_1$:

Let $F = z'$, then $F^{(j)} > 0$ for $j = 0,1,2$ and $F^{(3)} < 0$. Once we apply Lemmas 1 and 2 to z and once to F, we obtain

$$z(\iota) > \frac{\lambda_1}{6}\iota^3 z'''(\iota)$$

and

$$\frac{1}{\lambda_2}F > \frac{\iota^2}{2}F'' \to z'(\iota) > \frac{\lambda_2}{2}\iota^2 z'''(\iota), \tag{29}$$

for $\lambda_i \in (0,1)$. Equation (1) therefore reduces to

$$z^{(4)}(\iota) \leq -\frac{\lambda_1}{6} Q(\iota) g^3(\iota) z'''(g(\iota)). \tag{30}$$

Utilizing (29) and (30), we arrive at

$$\begin{aligned}\left(z(\iota) - \frac{\lambda_2}{6}\iota^3 z'''(\iota)\right)' &= z'(\iota) - \frac{\lambda_2}{6}\left(\iota^3 z^{(4)}(\iota) + 3\iota^2 z'''(\iota)\right)\\ &> -\frac{\lambda_2}{6}\iota^3 z^{(4)}(\iota)\\ &> \frac{\lambda_1\lambda_2}{36}\iota^3 g^3(\iota) Q(\iota) z'''(g(\iota)).\end{aligned} \tag{31}$$

After integrating (31) from ι_1 to ι, we get

$$z(\iota) - \frac{\lambda_2}{6}\iota^3 z'''(\iota) > \frac{\lambda_1\lambda_2}{36}\int_{\iota_1}^{\iota} s^3 g^3(s) Q(s) z'''(g(s))\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned} z(\iota) &> \frac{\lambda_2}{6}\iota^3 z'''(\iota) + \frac{\lambda_1\lambda_2}{36}\int_{\iota_1}^{\iota} s^3 g^3(s) Q(s) z'''(g(s))\,\mathrm{d}s\\ &> \frac{\lambda_2}{6} z'''(\iota)\left[\iota^3 + \frac{\lambda_1}{6}\int_{\iota_1}^{\iota} s^3 g^3(s) Q(s)\,\mathrm{d}s\right]\\ &= \frac{\lambda_2}{6} z'''(\iota)\Psi(\iota).\end{aligned} \tag{32}$$

When we set $z'''(\imath) = \Omega(\imath)$ and combine (1) with (32), we discover that $\Omega(\imath)$ is a positive solution of

$$\Omega'(\imath) + \frac{\lambda_2}{6} Q(\imath) \Psi(g(\imath)) \Omega(g(\imath)) \leq 0.$$

There is a contradiction with (27), according to [43, Theorem 2.1.1].

In the case $\mathbf{C}_2$ we have Lemma 5 implies that (16) holds. According to Lemmas 1 and 2, we find that

$$z(\imath) > \lambda_3 \imath z'(\imath),$$

for $\lambda_3 \in (0,1)$. After that, from (16), we get

$$\begin{aligned}\left(z(\imath) - \lambda_3 \imath z'(\imath)\right)' &= z'(\imath) - \lambda_3 \left(\imath z''(\imath) + z'(\imath)\right) \\ &= (1-\lambda_3) z'(\imath) - \lambda_3 \imath z''(\imath) \\ &> -\lambda_3 \imath z''(\imath) \\ &> \lambda_3 \imath \hat{Q}(\imath) z(g(\imath)) \\ &> \lambda_3^2 \imath g(\imath) \hat{Q}(\imath) z'(g(\imath)).\end{aligned} \tag{33}$$

After integrating (33) from $\imath_1$ to $\imath$, we obtain

$$\begin{aligned}z(\imath) &> \lambda_3 \imath z'(\imath) + \lambda_3^2 \int_{\imath_1}^{\imath} s g(s) \hat{Q}(s) z'(g(s)) \mathrm{d}s \\ &> \lambda_3 z'(\imath) \left[\imath + \lambda_3 \int_{\imath_1}^{\imath} s g(s) \hat{Q}(s) \mathrm{d}s\right] \\ &= \lambda_3 z'(\imath) \hat{\Psi}(\imath).\end{aligned}$$

When we set $F(\imath) = z'(\imath)$ and combine (1) with (32), we discover that $F(\imath)$ is a positive solution of

$$F'(\imath) + \lambda_3 \hat{Q}(\imath) \hat{\Psi}(g(\imath)) F(g(\imath)) \leq 0.$$

There is a contradiction with (28), according to [43, Theorem 2.1.1]. The proof is so finished.

4 Examples and Discussions

In the following, we apply these results to a classical form of Euler-type differential equations to assess their practical applicability and identify the criteria that provide the sharpest bounds. Furthermore, we present a numerical comparison with earlier studies to emphasize the improvements attained through our approach.

Example 1. For $\imath \geq 1$. Consider NDDE

$$(x(\imath) + p_0 x(\delta \imath))'''' + \frac{q_0}{\imath^4} x(\xi \imath) = 0, \tag{34}$$

where $\delta, \xi \in (0,1)$ and $q_0 > 0$. We note that $p(\imath) = p_0$, $h(\imath) = \delta \imath$, $q(\imath) = q_0/\imath^4$ and $g(\imath) = \xi \imath$. It is simple to confirm that

$$Q(\imath) = \frac{q_0}{\imath^4}(1-p_0),$$

and

$$\begin{aligned}\hat{Q}(\imath) \ :&= \int_{\imath}^{\infty} \int_{u}^{\infty} \frac{q_0}{s^4}(1-p_0) \mathrm{d}s \mathrm{d}u \\ &= \frac{q_0}{\imath^2}(1-p_0).\end{aligned}$$

Before using the following Theorems, we must confirm that

$$\begin{aligned}\int_{\iota_0}^{\infty} g^3(s)\,Q(s)\,\mathrm{d}s &= \int_{\iota_0}^{\infty} \xi^3 s^3 \frac{q_0}{s^4}(1-p_0)\,\mathrm{d}s\\ &= \xi^3 q_0(1-p_0)\int_{\iota_0}^{\infty}\frac{\mathrm{d}s}{s}\\ &= \infty,\end{aligned}$$

and

$$\begin{aligned}\int_{\iota_0}^{\infty} g(s)\,\hat{Q}(s)\,\mathrm{d}s &= \int_{\iota_0}^{\infty} \xi s \frac{q_0}{s^2}(1-p_0)\,\mathrm{d}s\\ &= \xi q_0(1-p_0)\int_{\iota_0}^{\infty}\frac{\mathrm{d}s}{s}\\ &= \infty\end{aligned}$$

are fulfilled. Thus, the Conditions (20) and (21) are satisfied if

$$q_0 > \frac{6}{\lambda\xi^3(1-p_0)\ln(1/\xi)\,\mathrm{e}},$$

and

$$q_0 > \frac{1}{\xi(1-p_0)\ln(1/\xi)\,\mathrm{e}},$$

respectively. Therefore, by using Theorem 1, we see that (34) is oscillatory if

$$\begin{aligned}q_0 &> \max\left\{\frac{6}{\lambda\xi^3(1-p_0)\ln(1/\xi)\,\mathrm{e}},\frac{1}{\xi(1-p_0)\ln(1/\xi)\,\mathrm{e}}\right\}\\ &= \frac{6}{\lambda\xi^3(1-p_0)\ln(1/\xi)\,\mathrm{e}}\end{aligned} \tag{35}$$

Likewise, by using Theorem 2, we see that (34) is oscillatory if

$$\begin{aligned}q_0 &> \max\left\{\frac{-\lambda\xi^3(1-p_0)+1}{\lambda\xi^3(1-p_0)\ln(1/\xi)\,\mathrm{e}},\frac{-\xi(1-p_0)+1}{\xi(1-p_0)\ln(1/\xi)\,\mathrm{e}}\right\}\\ &> \frac{-\lambda\xi^3(1-p_0)+1}{\lambda\xi^3(1-p_0)\ln(1/\xi)\,\mathrm{e}}.\end{aligned} \tag{36}$$

From Theorem 3, we obtain that (34) is oscillatory if

$$q_0 > \frac{6}{\mathrm{e}\lambda_2\xi^3(1-p_0)\left(1+\frac{\lambda_1}{18}\xi^3 q_0(1-p_0)\right)\ln(1/\xi)}. \tag{37}$$

Remark. However, by using Theorem 2 in [30], equation (34) is oscillatory if

$$q_0 > \frac{192}{\mathrm{e}\xi^3(1-p_0)\ln(1/\xi)} \tag{38}$$

Table Table 1 provides a comparative analysis of the effectiveness and originality of our results compared to criteria (38) mentioned in previous studies and also compares the Theorem 1 criteria with the Theorem 3 criteria. Specifically, the table shows the lower bounds of the coefficient q_0 corresponding to different values p_0, δ and ξ.

We can notice from Table 1 that

1. Theorem 3 improves Theorem 1 and provides a better criterion for testing the oscillation.
2. By comparing our results with those of [30], it is observed that our results improve upon and complement the related previous works.

Table 1: Comparison of new results with previous relevant findings in the literature.

	(35)	(37)	(38)
$p_0 = 1/2, \delta = 1/2, \xi = 0.9$	$q_0 > 57.4$	$q_0 > 34.02$	$q_0 > 1839.2$
$p_0 = 1/3, \delta = 1/3, \xi = 0.8$	$q_0 > 28.98$	$q_0 > 1.64$	$q_0 > 927.35$
$p_0 = 1/4, \delta = 1/4, \xi = 0.7$	$q_0 > 24.05$	$q_0 > 1.203$	$q_0 > 769.8$

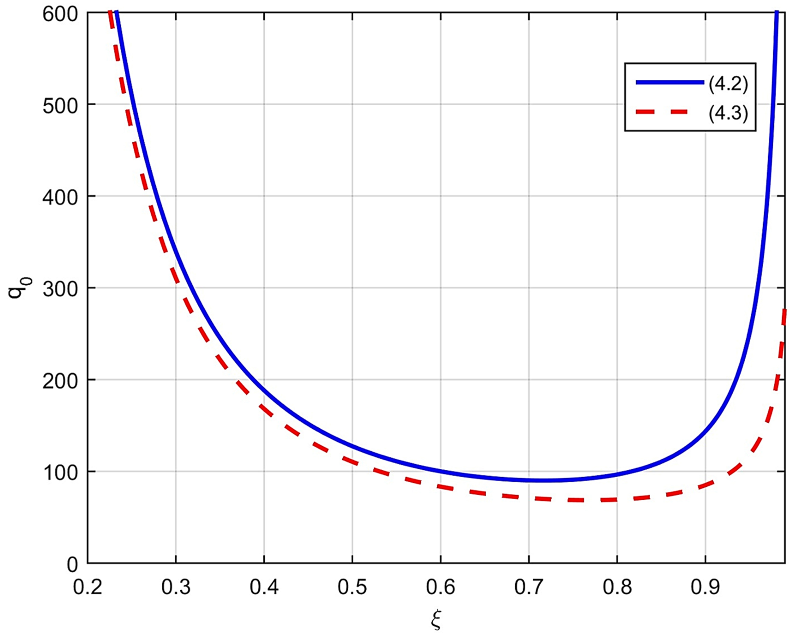

Fig. 1: Comparison between criteria (36) and (37).

5 Conclusion

In this work, the oscillatory behavior of a fourth-order NDDE in the standard case is studied. Classifying positive solutions according to the sign of their derivatives always comes first when examining the oscillations of NDDEs. In the oscillation theory of NDDEs, finding the relationships between the solution and its corresponding function is crucial. We then used the new relationships and properties to derive a set of oscillation criteria. By comparing our results with those of previous studies such as [30], the value of the new criteria was evaluated numerically. The results demonstrate that our criteria are more accurate and general, providing significant improvements in predicting oscillation behavior across different test cases. We suggest applying our results to higher-order non-standards, as the number of derivative signs increases in this case and thus becomes an interesting research topic.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

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