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Common Best Proximity Results for Several Generalized Non-Self Alpha-Conjoint Proximal Beta-Quasi Contraction Mappings

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Abstract: Fixed point theory is a fundamental concept in mathematics, which ensures the existence and uniqueness of fixed points for self-mappings in complete metric spaces. The Banach contraction principle, explicitly stated in Banach's 1922 thesis , laid the groundwork for this field. However, for non-self mappings $H : F \to K$, where F and K are subsets of a metric space E, fixed points may not exist. This led to the development of best proximity point theory, which seeks to find elements f in F where $d(f,Hf) = d(F,K)$. This study introduces new concepts in the field of best proximity point theory, extending the existing framework with the formulation of *conjoint proximal* β*-quasi contraction*, α*-generalized conjoint proximal* β*-quasi contraction*, and *generalized conjoint* β*-quasi contraction*. These innovative constructs are utilized to investigate the existence of common best proximity points on metric spaces for both double and multiple non-self mappings. The research also presents a concrete example involving two non-self mappings, thereby illustrating the practical application of these theoretical concepts. Furthermore, the study explores various implications and consequences arising from these new formulations.

Keywords: Best proximity point, non-self mapping, P-property, approximately compact and special generalized proximal beta-quasi contraction

2020 AMS Subject Classifications: 81Q10, 81Q15, 35J10.

1 Introduction

The evolution of this field has been marked by significant contributions from various mathematicians. Fan's 1969 best approximation theorem [1] acted as a catalyst, stimulating extensive subsequent research. S. Reich's 1972 [2] work extended Fan's theorem to set-valued inward functions. Between 1988 and 1989, Sehgal and colleagues [[3]- [4]] investigated best proximity points for non-self contractions, furthering the field's scope.

Kirk and colleagues (2003) [5] investigated proximinality and best proximity point theorems in hyperconvex metric spaces within Hilbert spaces, offering optimal approximate solutions for mappings without fixed points. Sadiq Basha [6] established important existence conditions for proximal contractions of both first and second kinds. Jleli and collaborators [7] introduced a new category of non-self contractive mappings and examined the existence and uniqueness of best proximity points for these mappings.

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Recent advancements include Almeida et al.'s 2014 [8] application of P-property to locate best proximity points, and M. Ayari et al.'s findings on best proximity points for special generalized proximal beta-quasi contraction [9]. The field continues to expand, with ongoing research exploring concepts such as conjoint proximal β-quasi contraction, α-generalized conjoint β-quasi contraction, and their applications to finite numbers of non-self mappings on metric spaces.

This paper has led to a deeper understanding of fixed point theory and its extensions, providing powerful tools for analyzing diverse mathematical structures and their properties.

2 Preliminaries

This passage provides important definitions and notations. Let me summarize the key points:

Notation 1*For F and K nonempty subsets of a metric space (E,d):*

$$d(F,K) := \inf\{d(f,k) : f \in F, k \in K\},$$

$$F_0 := \{f \in F : there \quad exists \quad k \in K \quad such \quad that \quad d(f,k) = d(F,K)\},$$

$$K_0 := \{k \in K : \quad there \quad exists \quad f \in F \quad such \quad that \quad d(f,k) = d(F,K)\}.$$

Definition 1. *[6] For $H : F \to K$, an element f^* is a best proximity point of H if $d(f^*, Hf^*) = d(F,K)$.*

Definition 2. *[9] For $\chi : [0,+\infty) \to [0,+\infty)$ such that*

1.χ is non-decreasing,
2.The $\lim_{n\to\infty} \chi_\beta^n(t) = 0$ such that $t > 0$ where χ_β^n the n-th iterate of χ_β and $\chi_\beta(t) = \chi(\beta t)$,
3.$s \in (0,+\infty)$ as $\sum_{n=1}^{\infty} \chi_\beta^n(s) < \infty$ exists.

For some $\beta \in (0,+\infty)$ and χ is called β-comparison function and denoted by Ξ_β .

Example 1. [9] For $\chi : [0,+\infty) \to [0,+\infty)$ where $\chi(t) = at$, $a \in (0, \beta^{-1})$ and $\beta > 0$ then $\chi \in \Xi_\beta$. Remark that, if $a > 1$ then $\chi(t) > t$ and so $\Sigma_{n=1}^{\infty} \chi^n(t) = \infty$

Lemma 1. *[9] For $\chi \in \Xi_\beta$. We have*

1.χ_β is non decreasing,
2.$\chi_\beta(t) < t$ for all $t > 0$,
3. $\sum_{n=1}^{\infty} \chi_\beta^n(t) < \infty$ for all $t > 0$.

Definition 3. *[10] For nonempty subsets F and K of a metric space (E,d). The nonempty subsets F and K of a metric space (E,d) will have P-property if*

$$d(\xi_1, \eta_1) = d(\xi_2, \eta_2) = d(F,K) \Longrightarrow d(\xi_1, \xi_2) = d(\eta_1, \eta_2)$$

such that ξ_1, $\xi_2 \in F$ and η_1, $\eta_2 \in K$.

Definition 4. *[11] Approximately compact with respect to a nonempty subset F of a metric space (E,d) for a subset K of a metric space (E,d) iff*

$$\lim_{n\to+\infty} d(\xi, \eta_n) = d(\xi, K).$$

3 Main Consequences

3.1 Best Proximity Point for Two Mappings

Definition 5. *Consider F and K nonempty subsets of a metric space (E,d). If*

$$d(H\xi, G\eta) \leq \chi(M_{H,G}(\xi,\eta)), \quad \forall \xi,\eta \in F, \tag{1}$$

and

$$\begin{aligned} M_{H,G}(\xi,\eta) = max\{&\alpha_0 d(\xi,\eta), \alpha_1(d(\xi,H\xi)–d(F,K)),\\ &\alpha_2(d(\eta,S\eta)–d(F,K)), \alpha_3(d(\eta,H\xi)–d(F,K)),\\ &\alpha_4(d(\xi,G\eta)–d(F,K))\}. \end{aligned}$$

whenever $\chi \in \Xi_\beta$ and positive numbers α_i, $i = 0,1,2,3,4$, $\beta \in (0,+\infty)$. So, $H,G : F \to K$ are called α-conjoint proximal β-quasi contraction mappings.

Let us give an example on definition 5

Example 2. Let $E = \mathbb{R}$ with $d(\xi,\eta) = |\xi-\eta|$ and $F = [0,1]$ and $K = [0,4]$. Since K is compact, then $K = [0,4]$ is approximately compact with respect to $F = [0,1]$. Moreover, For $H,G : F \to K$ define by $H(\xi) = \frac{\xi}{2}$ and $G(\xi) = \frac{\xi}{2}+2$. Then $d(F,K) = 1$ and $F_0 = 1$, $K_0 = 2$. Thus, $H(F_0) = H(\{1\}) = \{2\} = K_0$ and $G(F_0) = G(\{1\}) = \{2\} = K_0$. H is conjoint proximal β-quasi contraction with $\chi(\zeta) = \frac{1}{2}\zeta, \beta = 2, \alpha_0 = 2$ and $\alpha_i = 0$ for $i = 1,2,3,4$.

Now, since

$$L.H.S = d(H\xi, G\eta) = |\frac{\xi}{2} - \frac{\eta}{2} - 2| \tag{2}$$

Also,

$$\begin{aligned} M_{H,G}(\xi,\eta) = max\{&\alpha_0 d(\xi,\eta), \alpha_1(d(\xi,H\xi)–d(F,K)),\\ &\alpha_2(d(\eta,S\eta)–d(F,K)), \alpha_3(d(\eta,H\xi)–d(F,K)),\\ &\alpha_4(d(\xi,G\eta)–d(F,K))\}\\ &= max\{2|\xi-\eta|,0,0,0,0\} = 2|\xi-\eta| \end{aligned}$$

Moreover,

$$R.H.S = \chi(M_{H,G}(\xi,\eta)) = \chi(2|\xi-\eta|) = \frac{2|\xi-\eta|}{2} = |\xi-\eta| \tag{3}$$

Then, from equations (2) and (3), we get

$$L.H.S = d(H\xi,G\eta) = |\frac{\xi}{2} - \frac{\eta}{2} - 2| = \frac{1}{2}|\xi-\eta-4| \leq |\xi-\eta-4| \leq |\xi-\eta| = \chi(M_{H,G}(\xi,\eta)) = R.H.S.$$

Theorem 2. *For mappings H and G on nonempty closed subsets F and K of a complete metric space (E,d) where F_0 is nonempty. Let's break down the key components of this theorem:*

1.The pair (F,K) satisfies P-property and $H(F_0) \subset K_0, G(F_0) \subset K_0$.
2.K is approximately compact with respect to F.
3.Occurring $\xi_1,\xi_2,\xi_3 \in F$ such that

$$d(\xi_2, H\xi_1) = d(\xi_3, G\xi_2) = d(F,K).$$

4.There occurs $\beta \geq max_{0\leq r\leq 3}\{\alpha_r, 2\alpha_4\}$ whereas H and G are α-conjoint proximal β-quasi contraction mappings.

In Addition, either χ is continuous, or $\beta > max\{\alpha_2,\alpha_4\}$

Then under these conditions, either $d(\xi,H\xi) = d(\xi,G\xi) = d(F,K)$ or H and G have a unique common best proximity point $\xi \in F$.

Proof . There exists $\xi_1 \in F_0$ such that $H(\xi_1) \in K_0$ because $H(F_0) \subseteq K_0$. As a result of condition 3, we obtain $\xi_2 \in F_0$ such that

$$d(\xi_2, H\xi_1) = d(F,K) \tag{4}$$

From condition 3 and $G(F_0) \subseteq K_0$, we deduce that there exists $\xi_3 \in F_0$ such that

$$d(\xi_3, G\xi_2) = d(F,K) \tag{5}$$

We derive from equations (4) and (5) that:

$$d(\xi_2, H\xi_1) = d(\xi_3, G\xi_2) = d(F,K).$$

We consider a sequence $\{\xi_n\} \subseteq F_0$ by repeating this procedure where

$$d(\xi_{n+2}, H\xi_{n+1}) = d(\xi_{n+3}, G\xi_{n+2}) = d(F,K), \quad n \in \{0\} \cup \mathbb{N}. \tag{6}$$

By using P-property condition, we get

$$d(\xi_{n+2}, \xi_{n+3}) = d(H\xi_{n+1}, G\xi_{n+2}), \quad n \in \{0\} \cup \mathbb{N}. \tag{7}$$

We'll now demonstrate that $\{\xi_n\}$ is a Cauchy sequence.

Since H and G are α-conjoint proximinal β-quasi contraction, then

$$d(\xi_{n+2}, \xi_{n+3}) = d(H\xi_{n+1}, G\xi_{n+2}) \leq \chi(M_{H,G}(\xi_{n+1}, \xi_{n+2})) \tag{8}$$

And

$$\begin{aligned} M_{H,G}(\xi_{n+1}, \xi_{n+2}) &= max\{\alpha_0 d(\xi_{n+1}, \xi_{n+2}), \alpha_1(d(\xi_{n+1}, H\xi_{n+1}) - d(F,K)), \\ &\quad \alpha_2(d(\xi_{n+2}, G\xi_{n+2}) - d(F,K)), \alpha_3(d(\xi_{n+2}, H\xi_{n+1}) - d(F,K)), \\ &\quad \alpha_4(d(\xi_{n+1}, G\xi_{n+2}) - d(F,K))\} \\ &= max\{\alpha_0 d(\xi_{n+1}, \xi_{n+2}), \alpha_1(d(\xi_{n+1}, H\xi_{n+1}) - d(F,K)), \\ &\quad \alpha_2(d(\xi_{n+2}, G\xi_{n+2}) - d(F,K)), \alpha_4(d(\xi_{n+1}, G\xi_{n+2}) - d(F,K))\} \\ &\leq max\{\alpha_0 d(\xi_{n+1}, \xi_{n+2}), \alpha_1 d(\xi_{n+1}, \xi_{n+2}), \alpha_2 d(\xi_{n+2}, \xi_{n+3}), \\ &\quad \alpha_4(d(\xi_{n+1}, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}))\} \\ &\leq \beta max\{d(\xi_{n+1}, \xi_{n+2}), d(\xi_{n+2}, \xi_{n+3})\} \end{aligned}$$

By utilizing equations (6), (7), (8) as well as triangle inequality. As a result,

$$M_{H,G}(\xi_{n+1}, \xi_{n+2}) \leq \beta max\{d(\xi_{n+1}, \xi_{n+2}), d(\xi_{n+2}, \xi_{n+3})\} \tag{9}$$

whereas $\beta \geq max_{0 \leq r \leq 3}\{\alpha_r, 2\alpha_4\}$. From equations (8) and (9), we deduce

$$d(\xi_{n+2}, \xi_{n+3}) \leq \chi(\beta max\{d(\xi_{n+1}, \xi_{n+2}), d(\xi_{n+2}, \xi_{n+3})\}) = \chi_\beta(max\{d(\xi_{n+1}, \xi_{n+2}), d(\xi_{n+2}, \xi_{n+3})\})$$

If for some n, we have $d(\xi_{n+1}, \xi_{n+2}) \leq d(\xi_{n+2}, \xi_{n+3})$ and from lemma 1. It follows that $d(\xi_{n+2}, \xi_{n+3}) \leq \chi_\beta(d(\xi_{n+2}, \xi_{n+3})) < d(\xi_{n+2}, \xi_{n+3})$ which gives a contradiction.

Then for all $n \in \{0\} \cup \mathbb{N}$, we have $d(\xi_{n+2}, \xi_{n+3}) < d(\xi_{n+1}, \xi_{n+2})$ and it follows that

$$d(\xi_{n+2}, \xi_{n+3}) \leq \chi_\beta(d(\xi_{n+1}, \xi_{n+2})) \quad n \in \{0\} \cup \mathbb{N} \tag{10}$$

Then, by establishing an inductive relationship for the distances between successive terms of a sequence ξ_n:

$$d(\xi_{n+2}, \xi_{n+3}) \leq \chi_\beta^n(d(\xi_2, \xi_1)) \quad n \in \{0\} \cup \mathbb{N} \tag{11}$$

Let $\varepsilon > 0$ be fixed. Since $\Sigma_{n=1}^{+\infty}\chi_\beta^n(d(\xi_2,\xi_1))$ converges, occurs $n_0 \in \mathbb{Z}^+$ whereas $\Sigma_{n\geq n_0}^{+\infty}\chi_\beta^n(d(\xi_2,\xi_1)) < \varepsilon$. For $m > n > n_0$ and equation (11), we obtain

$$\begin{aligned} d(\xi_n,\xi_m) &\leq \Sigma_{k=n}^{m-1} d(\xi_k,\xi_{k+1}) \\ &\leq \Sigma_{k=n}^{m-1}\chi_\beta^k(d(\xi_2,\xi_1)). \end{aligned}$$

Since $\Sigma_{n=1}^{+\infty}\chi_\beta^n(t)$ converges for all $t \geq 0$; as a result

$$\Sigma_{k=n}^{m-1}\chi_\beta^k(d(\xi_2,\xi_1)) \to 0 \quad as \quad n,m \to +\infty.$$

It then proves that $\{\xi_n\}$ is Cauchy using F is closed in a complete metric (E,d), so the sequence $\{\xi_n\}$ converges to some element $\xi \in F$.

Moreover,

$$\begin{aligned} d(\xi,K) &\leq d(\xi,H\xi_{n+1}) \\ &\leq d(\xi,\xi_{n+2}) + d(\xi_{n+2},H\xi_{n+1}) \\ &= d(\xi,\xi_{n+2}) + d(F,K) \\ &\leq d(\xi,\xi_{n+2}) + d(\xi,K). \end{aligned}$$

If $n \to \infty$, so $d(\xi,H\xi_{n+1})$ converges to $d(\xi,K)$. According to condition (2), then there exists a sub sequence $\{\xi_{n(k)}\}$ of $\{\xi_n\}$ for which $\{H\xi_{n(k)}\}$ converges to some $\eta \in K$. Similarly, we may deduce that $\{\xi_{n(k)}\}$ is a sub sequence of $\{\xi_n\}$ such that $\{G\xi_{n(k)}\}$ converges to $\eta^* \in K$.

So,

$$\begin{aligned} d(F,K) &\leq d(\xi,\eta) \\ &\leq d(\xi,\xi_{n_{k+1}}) + d(\xi_{n_{k+1}},H\xi_{n_k}) + d(H\xi_{n_k},\eta) \\ &= d(\xi,\xi_{n_{k+1}}) + d(F,K) + d(H\xi_{n_k},\eta). \end{aligned}$$

While $n \to \infty$, so $d(\xi,\eta) = d(F,K)$ and $\xi \in F_0$.

Since $G(F_0) \subset K_0$, there exists $u \in F$ where $d(u,G\xi) = d(F,K)$. Therefore, we find $d(u,G\xi) = d(\xi_{n+2},H\xi_{n+1}) = d(F,K)$. According to P-property, we conclude that $d(u,\xi_{n+2}) = d(G\xi,H\xi_{n+1})$.

Since H,G are α-conjoint proximal β-quasi contraction mappings, we find

$$\begin{aligned} d(u,\xi_{n+2}) &= d(H\xi_{n+1},G\xi) \\ &\leq \chi(M_{H,G}(\xi_{n+1},\xi)) \quad \forall n \in \{0\}\cup\mathbb{N}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} M_{H,G}(\xi_{n+1},\xi) = max\{&\alpha_0 d(\xi_{n+1},\xi), \alpha_1(d(\xi_{n+1},H\xi_{n+1}) - d(F,K)), \\ &\alpha_2(d(\xi,G\xi) - d(F,K)), \alpha_3(d(\xi,H\xi_{n+1}) - d(F,K)), \\ &\alpha_4(d(\xi_{n+1},G\xi) - d(F,K))\}. \end{aligned} \tag{13}$$

Further, from (6),

$$\begin{aligned} M_{H,G}(\xi_{n+1},\xi) = max\{&\alpha_0 d(\xi_{n+1},\xi), \alpha_1 d(\xi_{n+1},\xi_{n+2}), \\ &\alpha_2(d(\xi,G\xi) - d(F,K)), \alpha_3 d(\xi,\xi_{n+2}), \\ &\alpha_4(d(\xi_{n+1},\xi) + d(\xi,G\xi) - d(F,K))\}. \end{aligned} \tag{14}$$

From triangle inequality and equation (6), we get

$$\begin{aligned} d(\xi,G\xi) &\leq d(\xi,\xi_{n+2}) + d(\xi_{n+2},H\xi_{n+1}) + d(H\xi_{n+1},G\xi) \\ &= d(\xi,\xi_{n+2}) + d(F,K) + d(H\xi_{n+1},G\xi) \quad \forall n \in \{0\}\cup\mathbb{N}. \end{aligned} \tag{15}$$

From (15) and (12), then

$$d(\xi,G\xi)-d(\xi,\xi_{n+2})-d(F,K)\leq\chi(M_{H,G}(\xi_{n+1},\xi)),\quad\forall n\in\{0\}\cup\mathbb{N}.\tag{16}$$

The proof then considers $f=d(\xi,G\xi)–d(F,K)$, and let $n\to+\infty$ in (14), we find

$$\lim_{n\to+\infty}M_{H,G}(\xi_{n+1},\xi)\leq max\{\alpha_2,\alpha_4\}f\tag{17}$$

Say $f\geq 0$, χ continuous and non decreasing. Also, $n\to+\infty$ in (16), we get

$$f\leq\chi(max\{\alpha_2,\alpha_4\}f)\leq\chi(\beta f)=\chi_\beta(f)<f,$$

when $\beta>max\{\alpha_2,\alpha_4\}$a contradiction occurs. We further argue $f=0$. Suppose that $f>0$.

From (17) then, there exist $\varepsilon>0$ and $N>0$ such that, for all $n>N$, we have $M_{H,G}(\xi_{n+1},\xi)<(max\{\alpha_2,\alpha_4\}+\varepsilon)f$. Since χ is non decreasing, from (17), resulting

$$\begin{aligned}d(\xi,G\xi)-d(\xi,\xi_{n+2})-d(F,K)&\leq\chi(M_{H,G}(\xi_{n+1},\xi))\\&\leq\chi((max\{\alpha_2,\alpha_4\}+\varepsilon)f)\\&=\chi_\beta(\frac{max\{\alpha_2,\alpha_4\}+\varepsilon}{\beta}f)\\&<\frac{max\{\alpha_2,\alpha_4\}+\varepsilon}{\beta}f<f\end{aligned}\tag{18}$$

For $n\to+\infty$ in (18), we have

$$f<\frac{max\{\alpha_2,\alpha_4\}+\varepsilon}{\beta}f<f$$

This result causes a contradiction. Wherefore, f must be zero or $d(\xi,G\xi)–d(F,K)=0$, As a response $d(\xi,G\xi)=d(F,K)$, proving that ξ is the best proximity point of G.

Similarly, we can get that ξ is the best proximity point of H or $d(\xi,G\xi)=d(\xi,H\xi)=d(F,G)$. In another word, H and G have a common best proximity point $\xi\in F$. Whereas $d(\xi,H\xi)=d(\xi,G\xi)=d(F,K)$.

This rigorous proof establishes the existence of a best proximity point for the mapping G under the given conditions. The approach could likely be adapted to prove a similar result for H, leading to the conclusion of a common best proximity point for both H and G.

To prove the uniqueness, let ξ and θ two separate best proximity points such that $d(\xi,H\xi)=d(\xi,G\xi)=d(\theta,H\theta)=d(\theta,G\theta)=d(F,K)$. From P-property condition, we found $d(\xi,\theta)=d(H\xi,G\theta)$.

Further, since H and G are α-conjoint proximal β-quasi contraction mappings, thus we conclude that

$$d(\xi,\theta)=d(H\xi,G\theta)\leq\chi(M_{H,G}(\xi,\theta)),$$

Where

$$M_{H,G}(\xi,\theta)=max\{\alpha_0 d(\xi,\theta),\alpha_3(d(\theta,H\xi)–d(F,K)),\alpha_4(d(\xi,G\theta)–d(F,K))\}.$$

We drive from triangular inequalities in $M_{H,G}(\xi,\theta)$ and ξ and θ are best proximity points for H and G that:

$$M_{H,G}(\xi,\theta)\leq max\{\alpha_0,\alpha_3,\alpha_4\}d(\xi,\theta).$$

Let's say $r=d(\xi,\theta)$. Since χ is non decreasing, we observe

$$r\leq\chi(max\{\alpha_0,\alpha_3,\alpha_4\}r)\leq\chi(\beta r)=\chi_\beta(r)<r,$$

That make a discrepancy. So, $\xi=\theta$. Then H and G have a unique common best proximity point $\xi\in F$ such that $d(\xi,H\xi)=d(\xi,G\xi)=d(F,K)$.

Remark. If we put in the contraction $G = H$ in the last theorem, we get the results in Mohamed Jleli and Ayari [[7], [11]]as a spacial case.

Example 3. Let $E = \mathbb{R}$ with $d(\xi,\eta) = |\xi - \eta|$ and $F = [0,1]$ and $K = [0,4]$. Since K is compact, then $K = [0,4]$ is approximately compact with respect to $F = [0,1]$. Moreover, For $H,G : F \to K$ define by $H(\xi) = \frac{\xi}{2}$ and $G(\xi) = \frac{\xi}{2} + 2$. Then $d(F,K) = 1$ and $F_0 = 1$, $K_0 = 2$. Thus, $H(F_0) = H(\{1\}) = \{2\} = K_0$ and $G(F_0) = G(\{1\}) = \{2\} = K_0$. H is conjoint proximal β-quasi contraction with $\chi(\zeta) = \frac{1}{2}\zeta, \beta = 2, \alpha_0 = 2$ and $\alpha_i = 0$ for $i = 1,2,3,4$.

Since

$$d(H\xi,G\eta) = |\frac{\xi}{2} - \frac{\eta}{2} - 2|$$

Also,

$$\begin{aligned} M_{H,G}(\xi,\eta) = max\{&\alpha_0 d(\xi,\eta), \alpha_1(d(\xi,H\xi)–d(F,K)),\\ &\alpha_2(d(\eta,S\eta)–d(F,K)), \alpha_3(d(\eta,H\xi)–d(F,K)),\\ &\alpha_4(d(\xi,G\eta)–d(F,K))\}\\ &= max\{2|\xi-\eta|,0,0,0,0\} = 2|\xi-\eta| \end{aligned}$$

And

$$\chi(M_{H,G}(\xi,\eta)) = \chi(2|\xi-\eta|) = \frac{2|\xi-\eta|}{2} = |\xi-\eta|$$

So,

$$d(H\xi,G\eta) = |\frac{\xi}{2} - \frac{\eta}{2} - 2| = \frac{1}{2}|\xi-\eta-4| \leq |\xi-\eta| = M_{H,G}(\xi,\eta) = \frac{1}{2}max\{2d(\xi,\eta),0,0,0,0\}$$

Since $\chi(\zeta)$ is continuous mapping and $\beta = 2 \geq max\{\alpha_2,\alpha_4\} = 0$. Also, by Theorem 2, we deduce that H and G have a unique common best proximity point which is $\xi^* = 1$. That is,
$d(\xi^*,H\xi^*) = d(1,H(1)) = d(1,2) = d(\xi^*,G\xi^*) = d(1,G(1)) = d(1,2) = d(F,K) = 1.$

3.2 Best Proximity Point for Several Mappings

Theorem 3. *For of nonempty closed subsets F and K of a complete metric space (E, d) be whereas F_0 is nonempty. For which the system of non-self-mappings H_i: $F \to K$ and $i = 1,2,...,n$ where:*

1.$H_i(F_0) \subset K_0$ for each $i = 1,2,...n$ and (F,K) structure the P-property,
2.K is approximately compact with respect to F,
3.occurring $\xi_i, \xi_{i+1}, \xi_{i+2} \in F$ whereas

$$d(\xi_{i+1},H_i\xi_i) = d(\xi_{i+2},H_{i+1}\xi_{i+1}) = d(F,K)$$

for each $i = 1,2,...n$,
4.There $\beta \geq max_{0\leq h\leq 3}\{\alpha_h, 2\alpha_4\}$ whereas H_i and H_{i+1} are conjoint proximal β-quasi contraction for each $i = 1,2,...n$.

Moreover, either one χ is continuous or $\beta > max\{\alpha_2,\alpha_4\}$.

Then the mappings H_1, H_2, ..., H_n have a unique common best proximity point $\xi \in F$ whereas $d(\xi,H_1\xi) = d(\xi,H_2\xi) = ... = d(\xi,H_n\xi) = d(F,K)$.

Proof . If we start with $i = 1$ then all conditions in theorem 2 are hold. So, there exists $\theta_1 \in F_0$ such that θ_1 is a unique common best proximity point between H_1 and H_2.i.e

$$d(\theta_1,H_1\theta_1) = d(\theta_1,H_2\theta_1) = d(F,K). \tag{19}$$

Similarly, if we but $i = 2$ then there exists $\theta_2 \in F_0$ such that $\theta_2$is a unique common best proximity point between H_2 and H_3. i.e

$$d(\theta_2, H_2\theta_2) = d(\theta_2, H_3\theta_2) = d(F, K). \tag{20}$$

Now, we want to prove that $\theta_1 = \theta_2$. From (19), (20) and using P-property, we get

$$d(\theta_1, \theta_2) = d(H_1\theta_1, H_2\theta_2) \tag{21}$$

So, from (19)-(21), triangle inequality, condition (4) in theorem 3 and the definition of α-conjoint proximal β-quasi contraction mappings for H_1 and H_2, we have

$$\begin{aligned}
d(\theta_1, \theta_2) &= d(H_1\theta_1, H_2\theta_2) \\
&\le \chi(M_{H_1,H_2}(\theta_1, \theta_2)) \\
&= \chi(max\{\alpha_0 d(\theta_1, \theta_2), \alpha_1(d(\theta_1, H_1\theta_1) – d(F, K))), \\
&\alpha_2(d(\theta_2, H_2\theta_2) – d(F, K)), \alpha_3(d(\theta_2, H_1\theta_1) – d(F, K)), \\
&\alpha_4(d(\theta_1, H_2\theta_2) – d(F, K))\} \\
&\le \chi(max\{\alpha_0 d(\theta_1, \theta_2), \alpha_3 d(\theta_1, \theta_2), \alpha_4 d(\theta_1, \theta_2)\}) \\
&\le \chi(d(\theta_1, \theta_2) max\{\alpha_0, \alpha_3, \alpha_4\}) \\
&\le \chi(\beta d(\theta_1, \theta_2)) = \chi_\beta(d(\theta_1, \theta_2)) \le d(\theta_1, \theta_2).
\end{aligned}$$

which gives a contradiction. Then $\theta_1 = \theta_2 = \theta$. i.e

$$d(\theta, H_1\theta) = d(\theta, H_2\theta) = d(\theta, H_3\theta) = d(F, K).$$

So, occurs a unique common best proximity point θ between H_1, H_2 and H_3.

By repeating this manner for i=3. We will find θ_3 such that:

$$d(\theta_3, H_3\theta_3) = d(\theta_3, H_4\theta_3) = d(H, K).$$

and θ_3=θ_2=θ.

Hence,

$$d(\theta, H_1\theta) = d(\theta, H_2\theta) = d(\theta, H_3\theta) = d(\theta, H_4\theta) = d(F, K).$$

Then occurs a unique common best proximity point θ between H_1, H_2, H_3 and H_4.

Hence, for finite number n. We reach to, occurs a unique common best proximity point θ between $H_1, H_2, H_3, ..., H_n$.

4 Repercussions

Many of the repercussions of section 3 examined in this section.

We suggest the following results in the situation of two functions.

Definition 6. *Let F and K be two nonempty subsets of a metric space (E, d). Two non-self-mappings $H, G : F \to K$ are said to be conjoint proximal β-quasi contraction mappings iff there exist $\chi \in \Xi_\beta$ such that:*

$$d(H\xi, G\eta) \le \chi(M^*_{H,G}(\xi, \eta)), \quad \forall \xi, \eta \in F, \tag{22}$$

where

$$\begin{aligned}
M^*_{H,G}(\xi, \eta) = max\{&d(\xi, \eta), (d(\xi, H\xi) – d(F, K)), \\
&(d(\eta, G\eta) – d(F, K)), (\frac{(d(\eta, H\xi) + d(\xi, G\eta))}{2} – d(F, K))\}.
\end{aligned}$$

whenever $\beta \in (0, +\infty)$.

In the next corollary, we investigate the presence of a common best proximity point in the case of *conjoint proximal β-quasi-contraction mappings* in our main theorem 2 and $\alpha_i = 1$ for each $i = 0,1,2,3,4$ as:

Corollary 1. *For non empty closed subsets F,K of a complete metric space (E,d) and F_0 is nonempty. For non-self mappings $H,G : F \to K$ whereas:*

1.Let (F,K) have P-property and $H(F_0) \subset K_0, G(F_0) \subset K_0$,
2.K is approximately compact with respect to F,
3.Occurring $\xi_1, \xi_2, \xi_3 \in F$ whereas

$$d(\xi_2, H\xi_1) = d(\xi_3, G\xi_2) = d(F,K).$$

4.Occurs $\chi \in \Xi_2$ whereas

$$d(H\xi, G\eta) \leq \chi(M^*_{H,G}(\xi,\eta)), \quad \forall \xi,\eta \in F. \tag{23}$$

Then H,G a unique best proximity point $\xi \in F$ such that $d(\xi,H\xi) = d(\xi,G\xi) = d(F,K)$.

Proof . Since

$$\begin{aligned}M^*_{H,G}(\xi,\eta) = max\{&d(\xi,\eta),(d(\xi,H\xi)–d(F,K)),\\ &(d(\eta,G\eta)–d(F,K)),(\frac{(d(\eta,H\xi)+d(\xi,G\eta))}{2}–d(F,K))\}\\ &\leq max\{d(\xi,\eta),(d(\xi,H\xi)–d(F,K)),\\ &\quad (d(\eta,G\eta)–d(F,K)),(d(\eta,H\xi)–d(F,K)),\\ &\quad d(\xi,H\eta)–d(F,K)\}\\ &= M_{H,G}(\xi,\eta).\end{aligned}$$

Also, the comparison function $\chi \in \Xi_2$ where $\beta \geq 2 > max\{\alpha_2,\alpha_4\} = 1$. From theorem 2, H and G a unique common proximity point in F.

When we put $\alpha_i = 1$ in theorem 2 for each $i = 0,1,2,3,4$ and replace χ by some $q \in [0,1)$ by some way we found that:

Corollary 2. *For a nonempty closed subsets F,K of a complete metric space (E,d) whereas F_0 is nonempty. And $H,G : F \to K$ where:*

1.$H(F_0) \subset K_0, G(F_0) \subset K_0$ and (F,K) holds the P-property,
2.K is approximately compact with respect to F,
3.occurring $\xi_1,\xi_2,\xi_3 \in F$ whereas

$$d(\xi_2, H\xi_1) = d(\xi_3, G\xi_2) = d(F,K).$$

4.occurs $q \in [0,1)$ whereas

$$d(H\xi, G\eta) \leq qM^-_{H,G}(\xi,\eta) \quad \forall \xi,\eta \in F,$$

where

$$\begin{aligned}M^-_{H,G}(\xi,\eta) = max\{&d(\xi,\eta),(d(\xi,H\xi)–d(F,K)),\\ &(d(\eta,G\eta)–d(F,K)),(d(\eta,H\xi)–d(F,K)),(d(\xi,G\eta)–d(F,K))\}.\end{aligned}$$

Then H,G have a unique best proximity point $\xi \in F$ whereas $d(\xi,H\xi) = d(\xi,G\xi) = d(F,K)$.

Proof . Let $\chi = qt$ which is continuous. From theorem 2, then H, G have a unique best proximity point $\xi \in F$ whereas $d(\xi,H\xi) = d(\xi,G\xi) = d(F,K)$.

Let's look at the common best proximity point for two self mappings for α-conjoint β-quasi contraction as follows:

Definition 7. *For a nonempty subset F of a metric space (E,d) the two self-mappings $H,G : F \to F$ are called α-generalized conjoint β-quasi contraction if occurring $\chi \in \Xi_\beta$ and $\alpha_i \geq 0$ for $i = 0,1,2,3,4,$ such that:*

$$d(H\xi, G\eta) \leq \chi(\mu_{H,G}(\xi,\eta)), \quad \forall \xi,\eta \in F, \tag{24}$$

where

$$\begin{aligned}\mu_{H,G}(\xi,\eta) = max\{&\alpha_0 d(\xi,\eta), \alpha_1 d(\xi,H\xi),\\ &\alpha_2 d(\eta,G\eta), \alpha_3 d(\eta,H\xi), \alpha_4 d(\xi,G\eta)\}.\end{aligned}$$

whenever $\beta \in (0,+\infty)$.

Corollary 3. *For a α-generalized conjoint β-quasi contraction between H and G on a complete metric space (E,d). Moreover, for any one of:*

–*χ is continuous;*
–*$\beta > max\{\alpha_2, \alpha_4\}$*

Then H and G have a unique common fixed point in E.

Proof . If we put $F = K$ in the definition of *α-conjoint β-quasi contraction mappings* then $\mu_{H,G}(\xi,\eta) = M_{H,G}(\xi,\eta)$ and $d(F,K) = 0$. So, since all conditions of theorem 2 are satisfied, then H,G have a unique common fixed point in E.

Now we will discuss a common fixed point for two self mappings for generalized conjoint β-quasi contraction as:

Definition 8. *Two self-mappings $H,G : F \to F$ over a nonempty subset F of a metric space (E,d) are called generalized conjoint β-quasi contraction if it occurs $\chi \in \Xi_\beta$ whereas*

$$d(H\xi, G\eta) \leq \chi(\mu^*_{H,G}(\xi,\eta)), \quad \forall \xi,\eta \in F,$$

where

$$\begin{aligned}\mu^*_{H,G}(\xi,\eta) = max\{&d(\xi,\eta), d(\xi,H\xi),\\ &d(\eta,G\eta), d(\eta,H\xi), d(\xi,G\eta)\}.\end{aligned}$$

whenever $\beta \in (0,+\infty)$.

Corollary 4. *Assume that, the two mappings H and G are generalized conjoint β-quasi contraction on a complete metric space (E,d) and χ is continuous. So, H and G have a unique common fixed point in E.*

Proof . Since $F = K = E$ and any set is approximately compact with itself, this is an obvious consequence of our fundamental theorem 2 where $\mu^*_{H,G}(\xi,\eta) = M_{H,G}(\xi,\eta)$ in theorem 2. Furthermore, on the self-mapping situation, the idea of generalized conjoint β-quasi contraction is a β-quasi contraction one. Then H,G have a unique common fixed point in E.

If for $H = G$ in the last definition we recall generalized conjoint β-quasi contraction by generalized β-quasi contraction then we will find the next corollary:

Corollary 5. *Let (E,d) be a complete metric space. Suppose that H is a generalized β-quasi contraction and χ is continuous. Then H has a unique fixed point in E.*

Proof . If we put $H = G$ in corollary 4. Then, we get that H has a unique fixed point in E.

5 Application

Around this part, we discuss an application of Theorem 2 in Volterra type integral equations. Consider the following integrals

$$r(m) = \int_0^m \Lambda_1(m,s,r(s))ds + f(m), \tag{25}$$

$$\sigma(m) = \int_0^m \Lambda_2(m,s,\sigma(s))ds + g(m), \tag{26}$$

for all $m \in [0,a]$. Let $\mathfrak{C}[0,a]$ be the set of all continuous functions defined on $[0,a]$. For $r \in \mathfrak{C}[0,a]$ with the norm

$$\|r\|_\infty = \max_{m\in[0,1]} |r(m)|$$

with metric

$$d(r,\sigma) = \|(r-\sigma)\|_\infty = \max_{m\in[0,1]} |r(m)-\sigma(m)|$$

for all $r,\sigma \in \mathfrak{C}[0,a]$. With these setting $(\mathfrak{C}[0,1], \|.\|_\infty)$ becomes Banach space. Now we prove the following theorem to ensure the existence of solution of the system of integral equations (25), (26).

Now for $F = \mathfrak{C}[0,a], K = \mathfrak{C}[0,b]$ and $d(F,K) = \inf\{d(x,k) \quad s.t. \quad x \in \mathfrak{C}[0,a] \quad and \quad k \in \mathfrak{C}[0,b]\}$.

Hence,

$$F_0 = \{x \in \mathfrak{C}[0,a] \quad s.t. \quad there \quad exists \quad k \in \mathfrak{C}[0,b] \quad s.t. \quad d(x,k) = d(\mathfrak{C}[0,a],\mathfrak{C}[0,b])\}$$

$$K_0 = \{k \in \mathfrak{C}[0,b] \quad s.t. \quad there \quad exists \quad x \in \mathfrak{C}[0,a] \quad s.t. \quad d(x,k) = d(\mathfrak{C}[0,a],\mathfrak{C}[0,b])\}$$

Theorem 4. *Consider $\Lambda_1,\Lambda_2 : [0,a]\times[0,a]\times\mathbb{R} \to \mathbb{R}$, $f,g : [0,a] \to \mathbb{R}$ are continuous and $H,G : \mathfrak{C}[0,a] \to \mathfrak{C}[0,b]$ as*

$$Hr(m) = \int_0^m \Lambda_1(m,s,r(s))ds + f(m), \tag{27}$$

$$G\sigma(m) = \int_0^m \Lambda_2(m,s,\sigma(s))ds + g(m), \tag{28}$$

1.$\mathfrak{C}[0,a]$ and $\mathfrak{C}[0,b]$ satisfies P-property in addition $H(F_0) \subset K_0, G(F_0) \subset K_0$,
2.$\mathfrak{C}[0,b]$ is approximately compact with respect to $\mathfrak{C}[0,a]$,
3.Occurring $\xi_1,\xi_2,\xi_3 \in \mathfrak{C}[0,a]$ such that

$$d(\xi_2,H\xi_1) = d(\xi_3,G\xi_2) = d(\mathfrak{C}[0,a],\mathfrak{C}[0,b]),$$

4.Occurs $\beta \geq max_{0\leq h\leq 3}\{\alpha_h, 2\alpha_4\}$ whereas H and G are α-conjoint proximal β-quasi contraction mappings.

In Addition, either χ is continuous, or $\beta > max\{\alpha_2,\alpha_4\}$
Also,If we have

$$|\Lambda_1(m,s,r(s)) - \Lambda_2(m,s,\sigma(s))| + |f(m)-g(m)| \leq \frac{\beta}{m}(|M_(H,G)(r(m),\sigma(m))|) \tag{29}$$

for some $m \in [0,a]$, and $r,\sigma \in \mathfrak{C}[0,a]$ then the system of integral equations (25) and (26) has a solution (a unique common best proximity point $\xi \in \mathfrak{C}[0,a]$).

Proof . Choosing x^* and k^* to be among the best approximations of $G\sigma$ and Hr, we have

$$\begin{aligned}
d(Hr, G\sigma) &= \max_{t\in[0,1]} |(Hr(m) - G\sigma(m)) \\
&= \max_{m\in[0,1]} (|\Lambda_1(m,s,r(s)) - \Lambda_2(m,s,\sigma(s))|ds + |f(m) - g(m)|) \\
&\leq \frac{\beta}{m}(\int_0^m \max_{m\in[0,1]} |M_(H,G)(r(s),\sigma(s))|ds) \\
&= \frac{\beta}{m}\int_0^m \|M_(H,G)(r,\sigma)\|_\infty ds \\
&= \frac{\beta}{m}\|M_(H,G)(r,\sigma)\|_\infty \int_0^m ds \\
&= \frac{\beta}{m}\|M_(H,G)(r,\sigma)\|_\infty m \\
&= \beta\|M_(H,G)(r,\sigma)\|_\infty
\end{aligned}$$

So, from theorem 2. Hence occurs $\theta \in \mathfrak{C}[0,a]$ whereas

$$\begin{aligned}
d(H\theta(m), \theta(m)) &= d(G\theta(m), \theta(m)) \\
&= d(\mathfrak{C}[0,a], \mathfrak{C}[0,b]) \\
&= \inf_{x\in\mathfrak{C}[0,a] k\in\mathfrak{C}[0,b]} d(x,k)
\end{aligned}$$

whereas $\theta(m)$ is a solution of equations (25) and (26).

On the other hand, There are several applications in nonlinear Volterra integral equation. For example, application on Abel-type nonlinear Volterra integral equation in Heat Radiation in a Semi-Infinite Solid, which is given by

$$u(r) = \frac{1}{\sqrt{\pi}}\int_0^r \frac{f(s) - u^n(s)}{\sqrt{r-s}} ds$$

where $u(x)$ gives the temperature at the surface for all time.

6 Main text

A non-self mapping $H: F \to K$, where F and K are subsets of a metric space E, does not have in general a fixed point. As a result, many people try to locate an element f that is closest to Hf in some way which is called *best proximity* point. Ayari achieved the best proximity point findings for a non-self mapping $H: F \to K$ known as special generalized proximal β-quasi contraction in 2019. The field continues to expand, with ongoing research exploring concepts such as conjoint proximal β-quasi contraction, α-generalized conjoint β-quasi contraction, and their applications to finite numbers of non-self mappings on metric spaces. This paper has led to a deeper understanding of fixed point theory and its extensions, providing powerful tools for analyzing diverse mathematical structures and their properties as follows:

6.1 Common best proximity point for two non-self mappings.

First, we defined the concept of α *conjoint proximal* β*-quasi contraction*. Also, we proved a unique common best proximity point between two mappings using this concept. Moreover, we enhanced our result by inserting an example.

6.2 Common best proximity point for several non-self mappings.

Second, we proved a unique common best proximity point results between several mappings.

6.3 Repercussions

Third, we inserted some new definitions *conjoint proximal* β*-quasi contraction mappings*, α*-generalized conjoint* β*-quasi contraction*, *generalized conjoint* β*-quasi contraction* and *generalized* β*-quasi contraction*. Also, we used these new concepts to derive some repercussions.

6.4 Application

Fourth, we gave an application in integral equations.

7 List of abbreviations

$d(F,K)$, F_0, K_0,β*-Quasi contraction*, α *conjoint proximal* β*-quasi contraction*, *conjoint proximal* β*-quasi contraction mappings*,α*-generalized conjoint* β*-quasi contraction*,*generalized conjoint* β*-quasi contraction* and *generalized* β*-quasi contraction*.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

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