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THE MIKHAILOV STABILITY CRITERION REVISITED

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It is shown that the principle of the argument is the basis for the Mikhailov's stability criterion for linear continuous systems. Mikhailov's criterion states that a real Hurwitz polynomial $\delta(s)$ of degree n satisfies the monotonic phase increase, that is to say the argument of $\delta(jw)$ goes through n quadrants as w runs from zero to infinity. In this paper, the generalized Mikhailov criterion where a real polynomial of degree n with no restriction on the roots location is considered. A method based on the argument is used to determine the number of roots in each half of the s-plane as well as on the imaginary axis if any.

INTRODUCTION

The Mikhailov stability criterion [1, 2, 3] states that a real polynomial $\delta(s)$ of degree *n* is Hurwitz stable if and only if the argument $\theta(w)$ of $\delta(jw)$ changes monotonically increasing from 0 to ∞.

Consider a real polynomial of degree *n*

$$\delta(s) = a_0 s^n + a_1 s^{n-1} + \quad + a_{n-1}s + a_n \tag{1}$$

$$\delta(s) = a_0 \prod_{i=1}^{n} (s - s_i)$$

$$\delta(jw) = a_0 \prod_{i=1}^{n} (jw - s_i)$$

The roots of this polynomial can be real and complex conjugate as shown in Fig.(1).

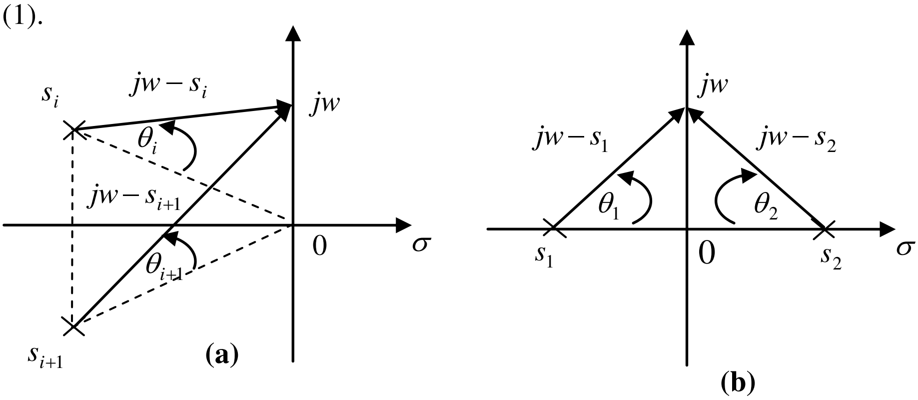

Fig. (1) Root Location

For a Hurwitz polynomial with all its roots $s_i \; (i=1 \quad n)$ real negative $(s_i = a_i)$ or some complex conjugate roots having negative real values, i.e. $s_i, s_{i+1} = a_i \pm jb_i$, $a_i < 0$, the argument of $\delta(jw)$ is

$$\theta(w) = Arg[\delta(jw)] = \left\{ \sum_{i=1}^{n_1} \left(\tan^{-1} \frac{w}{-a_i} \right) + \sum_{n_1+1}^{n} \left(\tan^{-1} \frac{w - b_i}{-a_i} \right) \right\} = \sum_{i=1}^{n} \theta_i(w) \qquad (2)$$

Let $\Delta_0^\infty \, \theta(w)$ denotes the net change in the argument $\theta(w)$ as w increases from zero to infinity. For a Hurwitz polynomial this monotonic change is $n\pi/2$ i.e.

$$\Delta_0^\infty \, \theta(w) = \frac{n\pi}{2} \qquad (3)$$

The plot of $\delta(jw)$ known as the (Mikhailov vector) with w increases from zero to infinity turns counterclockwise for positive argument and goes through n quadrants in turn as shown in Fig.(2).

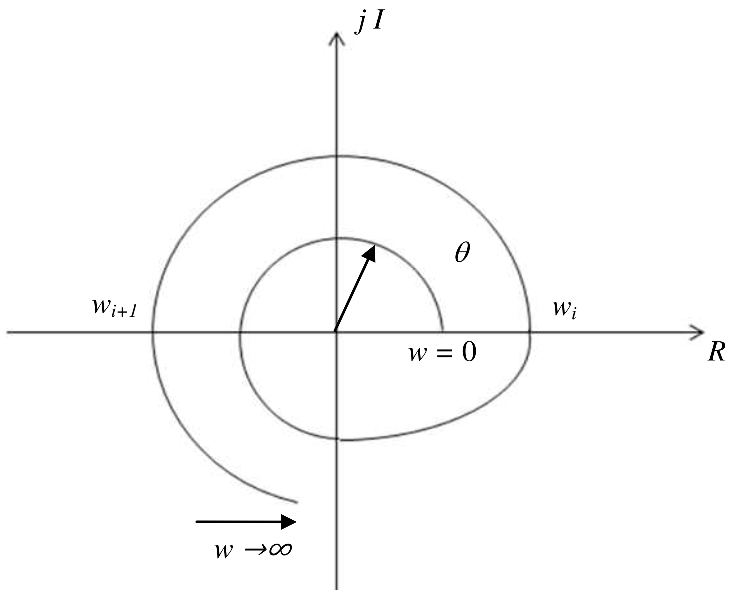

Fig. (2) Mikhailov plot

THE GENERALIZED MIKHAILOV CRITERION:

Assume that the polynomial $\delta(s)$ of degree n has l roots in the open left half plane (LHP), r roots in the open right half plane (RHP), y pairs of roots on the imaginary axis and k roots at the origin. In [1], it was shown that $\Delta_0^\infty \, \theta(w) = (n-2r)\dfrac{\pi}{2}$ for a polynomial of degree n which has r roots in RHP and no roots on the imaginary axis.

Taking into consideration that when w changes from zero to infinity the vector $(jw - s_1)$ in Fig.(1-b) will rotate counterclockwise an angle $\theta_1 = \pi/2$, while the vector $(jw - s_2)$ will rotate clockwise an angle $\theta_2 = -\pi/2$.

In [4], it was shown that the Mikhailov plot gives information about the roots distribution of $\delta(s)$, when it is not Hurwitz, as follows:

The even-odd decomposition

$$\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$$

Then

$$\begin{aligned}\delta(jw) &= \delta_e(-w^2) + jw\delta_o(-w^2)\\ &= R(w) + jI(w)\end{aligned} \tag{4}$$

with $R(w), I(w)$ are real polynomials in w.

Let the non-negative real zeros of $I(w)$ be

$$0 = w_0 < w_1 < w_2 < \quad < w_t$$

with respective multiplicities k_i, $i = 0, 1, \quad , t$ and let $w_{t+1} = \infty$. Then in [4], it was given that:

A. degree $[\delta(s)]$ is even, the polynomial signature $\sigma = l - r$ can be calculated as follows:

$$\begin{aligned}\sigma = l - r = Sign\left[I^{(k_0)}(w_0)\right]\left(Sign\left[R^{(k_0-1)}(w_0)\right] - Sign[R(w_1)]\right)\\ + \sum_{i=1}^{t} Sign\left[I^{(k_i)}(w_i)\right]\left(Sign[R(w_i)] - Sign[R(w_{i+1})]\right)\end{aligned} \tag{5}$$

B. degree $[\delta(s)]$ is odd, then

$$\begin{aligned}\sigma = l - r = Sign\left[I^{(k_0)}(w_0)\right]\left(Sign\left[R^{(k_0-1)}(w_0)\right] - Sign[R(w_1)]\right)\\ + \sum_{i=1}^{t-1} Sign\left[I^{(k_i)}(w_i)\right]\left(Sign[R(w_i)] - Sign[R(w_{i+1})]\right) + Sign\left[I^{(k_t)}(w_t)\right] * Sign[R(w_t)]\end{aligned} \tag{6}$$

where $I^{k_i}(w_o) = \dfrac{d^{k_i}}{dw^{k_i}}[I(w)]_{w=w_o}$, and

$$R^{k_i}(w_o) = \frac{d^{k_i}}{dw^{k_i}}[R(w)]_{w=w_o}$$

In this paper we will use the change in argument $\theta(w)$ of Eq.(2) to determine the number of roots in the LHP, RHP and on the imaginary axis. Consider the following cases:

1. No imaginary axis roots (with l roots in the LHP and r roots in the RHP) ($n=l+r$)

   then $$\Delta_0^{\infty}\,\theta(w)=\frac{\pi}{2}l-\frac{\pi}{2}r$$
   $$=\frac{\pi}{2}(n-2r)\neq\frac{n\pi}{2} \tag{7}$$

   The argument is monotonic but $\Delta_0^{\infty}\,\theta(w)$ is less than $\dfrac{n\pi}{2}$.

2. Roots allowed at the origin, i.e. $\theta(0^-)\neq\theta(0^+)$ or $\theta_{0^-}\neq\theta_{0^+}$. For a single root at the origin, $\Delta_{0^-}^{0^+}\,\theta(w)=Arg[\delta(j0^+)]-Arg[\delta(j0^-)]=\frac{\pi}{2}-(-\frac{\pi}{2})=\pi$, and if this root is of multiplicity k then,

   $$\Delta_{0^-}^{0^+}\,\theta(w)=2k\frac{\pi}{2}=k\pi \tag{8}$$

   The 0^- is used in the case of roots allowed at the origin to take into consideration the discontinuity in the argument $\theta(w)$ at the origin.

3. Roots allowed on the imaginary axis except at the origin

   $s_i=jw_i$, $s_{i+1}=-jw_i$

   $\theta(w_i{}^-)\neq\theta(w_i{}^+)$ i.e. $\theta_{w_i^-}\neq\theta_{w_i^+}$, and $\Delta_{w_i^-}^{w_i^+}\,\theta(w)=\pi$

   If this pair of imaginary root is of multiplicity y then,

   $$\Delta_{w_i^-}^{w_i^+}\,\theta(w)=y\pi \tag{9}$$

In general, the order of the characteristic equation n will be equal to sum of the number of all the previous different kinds of roots as follows:

$$n=l+k+2y+r \tag{10}$$

$$\Delta_{0^-}^{\infty}\,\theta(w)=\frac{\pi}{2}\left(n+k-2r\right) \tag{11}$$

or $$\Delta_{0^+}^{\infty}\,\theta(w)=\frac{\pi}{2}\left(n-k-2r\right) \tag{12}$$

If there are no roots at the origin, the change can be written from 0 as ($0^-=0^+$).

In the following sections, we will illustrate the suggested method by some examples with no restrictions on the roots location.

EXAMPLE (1)

Given, $\delta(s)=(s^3+s^2+2s+2)=(s^2+2)(s+1)$

The Mikhailov vector $\delta(jw)$ is

$$\delta(w) = (2 - w^2) + j\,w(2 - w^2) = R(w) + jI(w)$$

The Mikhailov plot is shown in Fig.(3), where

$R(w) = (2 - w^2)$,

$I(w) = w(2 - w^2)$, and

$$\theta(w) = \tan^{-1}\left[\frac{I(w)}{R(w)}\right]$$

$R(w) = 0$ at $w = \pm\sqrt{2}$

$I(w) = 0$ at $w = 0, \pm\sqrt{2}$

Since $I(w) = R(w) = 0$ at $w = \sqrt{2}$ (positive real root), this indicates that

1- The characteristic equation have roots at $\pm j\sqrt{2}$
2- The argument will have a discontinuity of 180 at $w = \sqrt{2}$

From the phase plot in Fig. (4), we found that,

$$\Delta_0^{\infty}\theta(w) = 3\pi/2 = 270$$

1. $\Delta_{0^-}^{\infty}\,\theta = 3\dfrac{\pi}{2} = \dfrac{\pi}{2}(n - 2r) = \dfrac{\pi}{2}(3 - 2r)$
   $\therefore\ r = 0$ (no roots in the right half of the s-plane).

2. $\theta_{0^+} = \theta_{0^-}$ $\qquad\therefore\ k = 0$

3. $\theta_{w_i^+} \neq \theta_{w_i^-}$ where $w_i = \sqrt{2}$

$$y = \left(\frac{\Delta_{w_i^-}^{w_i^+}\theta(w)}{\pi}\right) = \frac{\pi}{\pi} = 1 \quad \text{(one pair of roots at } \pm j\sqrt{2}\text{)}$$

We can justify this information directly without drawing the phase plot as follows:

$$\theta(w) = \tan^{-1}\left[\frac{w(2 - w^2)}{(2 - w^2)}\right] \quad \text{and} \quad \theta(w_i) = \tan^{-1}(\sqrt{2})$$

at $w_i^- = \sqrt{2}^-$, $\theta(w_i^-) = \tan^{-1}(w_i^-)$ and

at $w_i^+ = \sqrt{2}^+$, $\theta(w_i^+) = \tan^{-1}\left[\dfrac{w_i^+(2 - (w_i^+)^2)}{(2 - (w_i^+)^2)}\right] = \tan^{-1}\dfrac{w_i^+(-\varepsilon)}{(-\varepsilon)}$, where $\varepsilon = (w_i^{+})^2 - 2$,

i.e. $\theta(w_i^{+}) = \tan^{-1}(w_i^{+})$ but it is in the third quadrant , i.e. $\theta(w_i^{+}) = \theta(w_i^{-}) + \pi$ (This is clear from Fig. (4)). But $n = l + k + 2y + r$, then $3 = l + 0 + 2 + 0 \quad \therefore \underline{l = 1}$
Then the signature $\sigma(\delta)$ is $\sigma(\delta) = l - r = 1 - 0 = 1$.
However due to the discontinuity of 180° at $w = \sqrt{2}$, i.e. due to the roots at $\pm j\sqrt{2}$, *the system is on the margin of the stability.*
Using the formula of Eq.(6), we can get the same result as follows:

$$l - r = Sign\left[I^{(1)}(0)\right]\left(Sign[R(0)] - Sign\left[R(\sqrt{2})\right]\right) + Sign\left[I^{(1)}(\sqrt{2})\right] \bullet Sign\left[R\left(\sqrt{2}\right)\right]$$
$$= 1(1-0) + (-1) \times 0 = 1$$

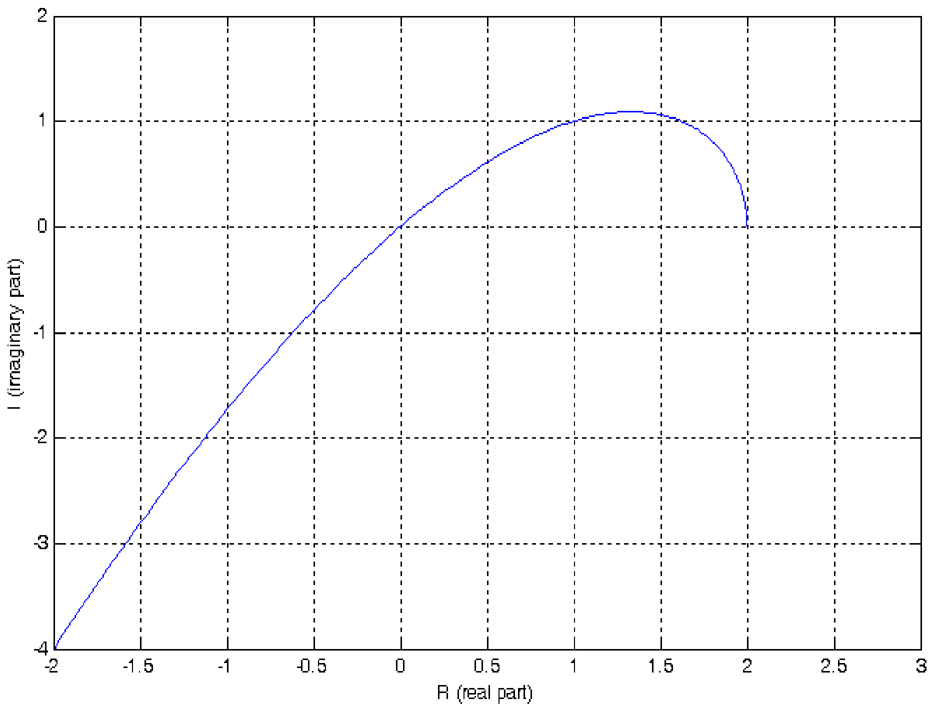

Fig. (3) The Mikhailov plot

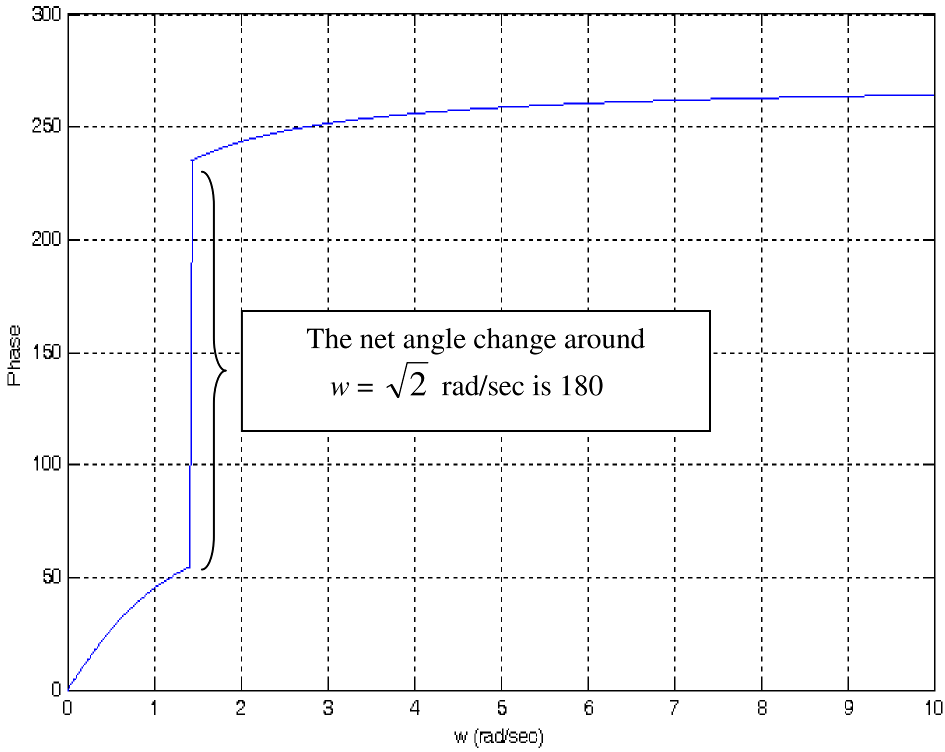

Fig. (4) The phase plot

EXAMPLE (2)

Given, $\delta(s) = (s^7 + s^6 + 6s^5 + 6s^4 + 12s^3 + 12s^2 + 8s + 8) = (s^2 + 2)^3 (s+1)$

The Mikhailov vector $\delta(jw)$ is

$$\delta(w) = 8 - 12w^2 + 6w^4 - w^6 + j\,w(8 - 12w^2 + 6w^4 - w^6) = R(w) + jI(w)$$

The Mikhailov plot is shown in Fig.(5), where

$R(w) = 8 - 12w^2 + 6w^4 - w^6$,

$I(w) = w(8 - 12w^2 + 6w^4 - w^6)$, and

$$\theta(w) = \tan^{-1}\left[\frac{I(w)}{R(w)}\right]$$

$I(w) = R(w) = 0$ at $w = \pm\sqrt{2}\,,\ \pm\sqrt{2}\,,\ \pm\sqrt{2}$

Since $I(w) = R(w) = 0$ at six common zeros, these roots $\pm\, j\sqrt{2}$ are imaginary roots of multiplicity three.

From the phase plot in Fig. (6) and let $w_i = \sqrt{2}$, we found that ,

$\theta(w)$ at $w = w_i^- = 54.737$, and at $w = w_i^+$ the Mikhailov vector will rotate counterclockwise an angle of $3 \times 180 = 540$ due to the roots $\pm\sqrt{2}$ with multiplicity three.

$$\Delta_0^\infty \theta(w) = 7\frac{\pi}{2} = \frac{\pi}{2}(n-2r) = \frac{\pi}{2}(7-2r) \quad \therefore \underline{r=0} \text{ (no roots in the RHP).}$$

We can justify these information directly without drawing the phase plot as follows:

$$\theta(w) = \tan^{-1}\left[\frac{w(2-w^2)^3}{(2-w^2)^3}\right] \quad \text{and} \quad \theta(w_i) = \tan^{-1}(\sqrt{2})$$

at $w_i^- = \sqrt{2}^-$, $\theta(w_i^-) = \tan^{-1}(w_i^-)$ and at $w_i^+ = \sqrt{2}^+$, $\theta(w_i^+) = \tan^{-1}\frac{w_i^+(-\varepsilon)^3}{(-\varepsilon)^3}$

i.e. $\theta(w_i^+) = \tan^{-1}(w_i^+)$ but it is in the third quadrant , i.e. $\theta(w_i^+) = \theta(w_i^-) + 3\pi$ (This is clear from Fig. (6)).

This is also clear since $I(w) = R(w) = 0$ at the common zero $\pm\sqrt{2}$ with multiplicity three.

$\Delta_0^\infty \theta(w) = 7\pi/2 = 630$

From the previous information, $y = 3, k = 0, l = 1$ $\quad \therefore l - r = 1 - 0 = 1$

Using the formula of Eq.(6), we can get the same result as follows:

$$l - r = Sign\left[I^{(1)}(0)\right]\left(Sign[R(0)] - Sign\left[R(\sqrt{2})\right]\right) + Sign\left[I^{(2)}(\sqrt{2})\right] \bullet Sign\left[R(\sqrt{2})\right]$$
$$= 1(1-0) + (-1)\times 0 = 1$$

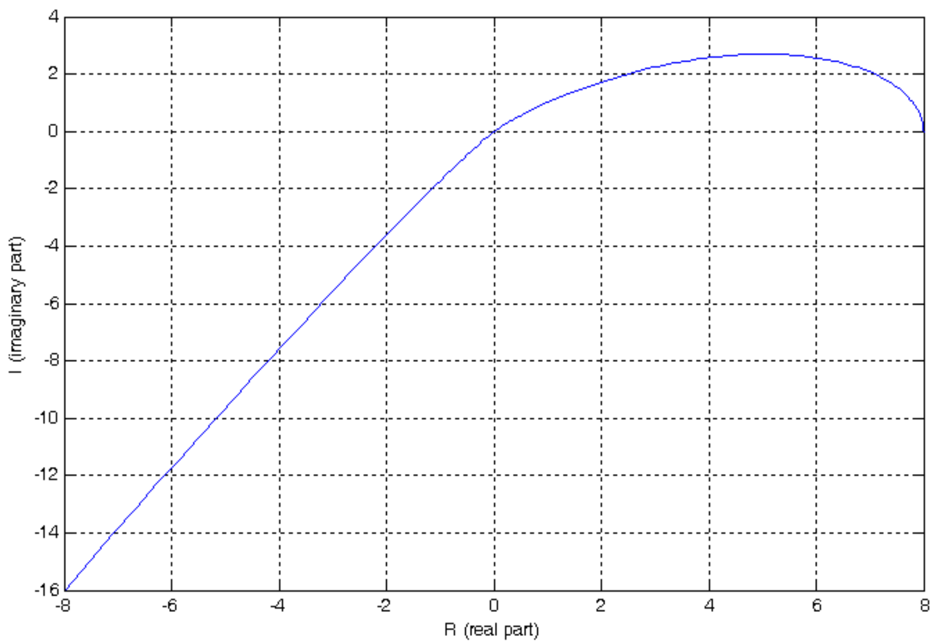

Fig. (5) The Mikhailov plot

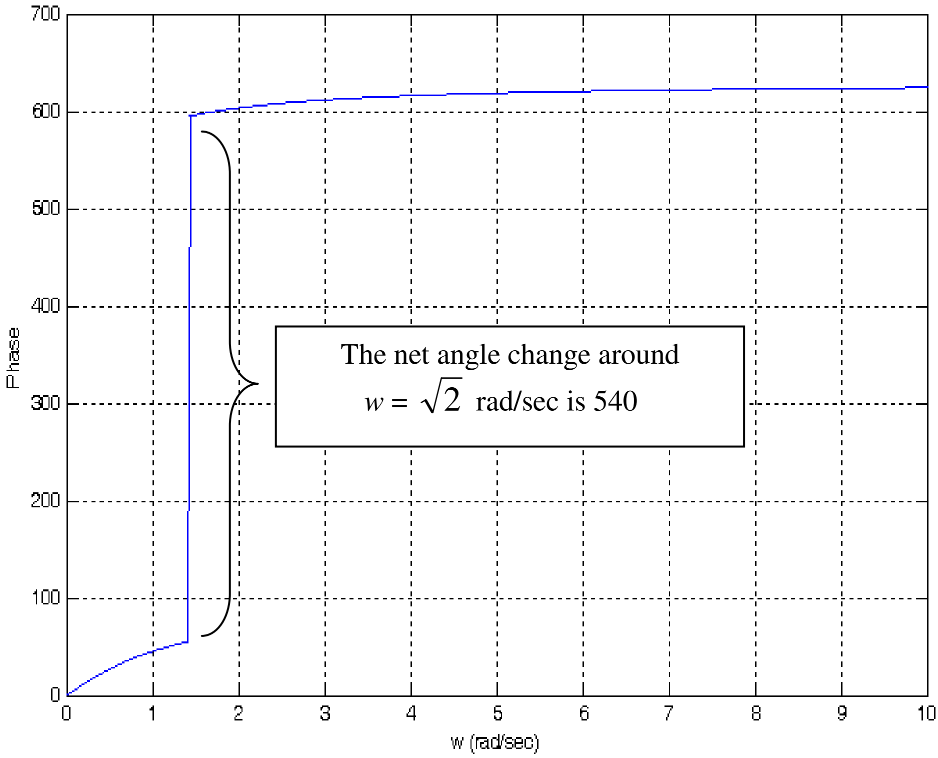

Fig. (6) The phase plot

<u>Notes</u>:

1- Comparing the Mikhailov vectors $\delta(jw)$ of both examples (1) and (2) in Fig.(3) and Fig.(5), although the two systems are of different orders, the two Mikhailov plots are similar. It is the argument rather than the Mikhailov vector that illustrates the stability as well as the roots location.

2- It is clear that, if some of the roots of the characteristic equation of order n are located in the LHP, and others on the imaginary axis (no roots in the RHP) then,

$$\Delta_0^{\infty}\, \theta(w) = \frac{n\pi}{2}$$

However, the change in argument $\theta(w)$ in this case is non-monotonic and we can conclude the following:

i. A discontinuity at the origin, $\Delta_{0^-}^{0^+}\, \theta(w) = k\pi$ indicates that there is a root at the origin with multiplicity k. The system is unstable for any value of k.

ii. A discontinuity at w_i, $\Delta_{w_i^-}^{w_i^+}\, \theta(w) = y\pi$ indicates that there is a pair of imaginary roots at w_i with multiplicity y. If y = 1, the system is marginally stable and if $y > 1$, the system is unstable.

EXAMPLE (3)

Given, $\delta(s) = s^2(s^4 - 2s^3 + 6s^2 - 8s + 8) = s^2(s^2 + 4)(s^2 - 2s + 2)$

The Mikhailov vector $\delta(jw)$ is

$$\delta(w) = w^2(-w^4 + 6w^2 - 8) + j\,w^3\,(8 - 2w^2) = R(w) + jI(w)$$

The Mikhailov plot is shown in Fig.(7), where

$R(w) = w^2(-w^4 + 6w^2 - 8)$,

$I(w) = w^3\,(8 - 2w^2)$, and

$\theta(w) = \tan^{-1}\left[\dfrac{I(w)}{R(w)}\right]$

$R(w) = 0$ at $w = 0, 0, \pm\sqrt{2}, \pm 2$

$I(w) = 0$ at $w = 0, 0, 0, \pm 2$

Since $I(w) = R(w) = 0$ at $w = 0, 0, 2$ (nonnegative real roots), this indicates that

1- The characteristic equation has multiple roots (two at the origin), and two pure imaginary roots at $\pm j2$

2- The argument will have a discontinuity of $2\pi/2$ at $w = 0$, and a discontinuity of π at $w = 2$

From the phase plot in Fig. (8), and let $w_i = 2$, we found that,

1. $\theta(0^-) = -\pi$, $\theta(0^+) = \pi$, *and* $\theta(\infty) = \pi$

   then $\Delta_{0^-}^{\infty}\theta(w) = \pi - (-\pi) = 2\pi$ and $\Delta_{0^+}^{\infty}\theta(w) = \pi - \pi = 0$.

2. $k = \left(\dfrac{\Delta_{0^-}^{0^+}\theta(w)}{\pi}\right) = \dfrac{2\pi}{\pi} = 2$

3. $\Delta_{0^-}^{\infty}\,\theta(w) = \dfrac{\pi}{2}(n + k - 2r) = \dfrac{\pi}{2}(8 - 2r) = 2\pi$

   $\therefore$ $\underline{r = 2}$ (Two roots in the RHP).

   The system is unstable.

   or if we use the equation $\Delta_{0^+}^{\infty}\,\theta(w) = \dfrac{\pi}{2}(n - k - 2r) = \dfrac{\pi}{2}(4 - 2r) = 0$,

   then $\underline{r = 2}$, the same as before.

4. $\theta_{w_i^+} \neq \theta_{w_i^-}$, $y = \left(\dfrac{\Delta_{w_i^-}^{w_i^+} \theta(w)}{\pi} \right) = \dfrac{\pi}{\pi} = 1$ (one pair of roots at $\pm j2$)

Since $n = l + k + 2y + r$

then $6 = l + 2 + 2 + 2 \quad \therefore \; \underline{l = 0}$

and the signature $\sigma(\delta)$ is

$\sigma(\delta) = l - r = 0 - 2 = -2$

Using the formula of Eq.(5), we can get the same results as follows:

$$l - r = Sign\left[I^{(3)}(0)\right]\left(Sign\left[R^{(2)}(0)\right] - Sign[R(2)]\right) + Sign\left[I^{(1)}(2)\right]\left(Sign[R(2)] - sign[R(\infty)]\right)$$
$$= 1(-1-0) + (-1)(0-(-1)) = -2$$

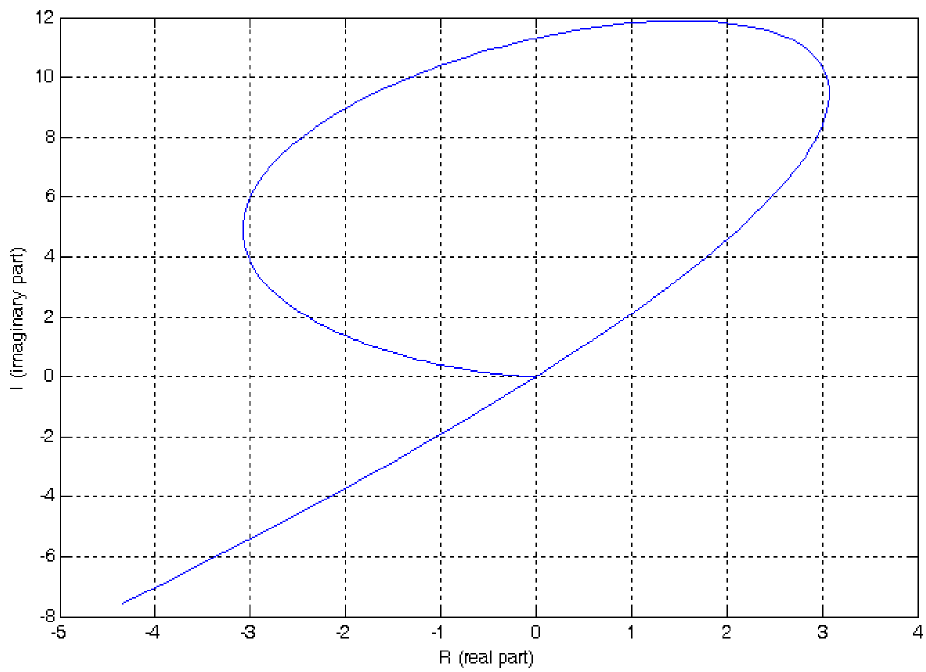

Fig. (7) The Mikhailov plot

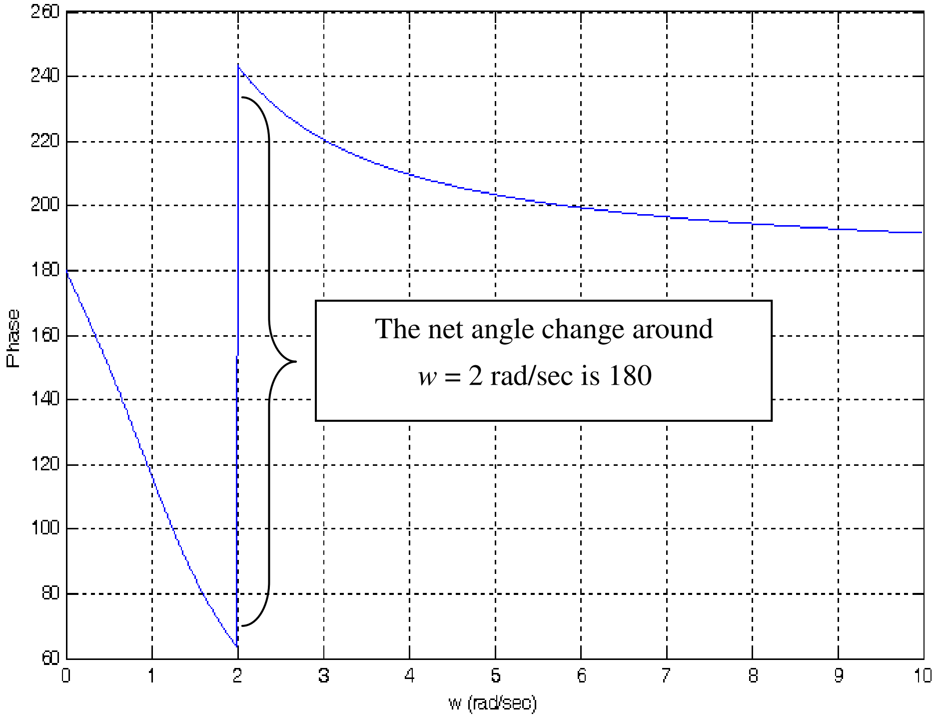

Fig. (8) The change in phase plot

CONCLUSION

In this paper the generalized Mikhailov Criterion is revisited, where a real polynomial of degree n with no restriction on root location is considered. It is the change in argument $\theta(w)$ that differentiate between stable, marginally and unstable systems. As stated in [1], [3], $\Delta_0^\infty \theta(w) = \frac{n\pi}{2}$ is a required condition for the system to be stable provided that the increase in $\theta(w)$ is monotonic. For systems with roots on the imaginary axis including the origin and other roots in the LHP, the argument $\theta(w)$ has a discontinuity at the origin and the system is unstable. If the argument has discontinuity at other frequencies, then the system can be marginally stable or unstable depending on the multiplicity of the imaginary roots. A method is given to determine the number of roots on the imaginary axis depending on this discontinuity besides the number of other roots in the LHP and the RHP. The number of roots on the imaginary axis equals the number of the common zeros of the real polynomials $R(w)$ and $I(w)$ after the even-odd decomposition of $\delta(w)$.

The proposed method is a simple one and determines the number of roots in each half of the s-plane as well as on the imaginary axis if any.

The discussions made above concerning the common zero between $R(w)$ and $I(w)$ will be used in another paper to determine the range of stabilizing values of the parameters of the different controllers.

REFERENCES

[1] E. P. Popov, "The Dynamics of automatic Control Systems", Reading, Massachusetts, U.S.A 1962.
[2] A. N. Michel, "Stability The Common Thread in the Evolution of Feedback Control", IEEE Control Systems, Vol. 16, No. 3,pp. 50-60, June 1996.
[3] H. Chapellat, M. Mansour and S. Bhattacharyya, "Elementary Proofs of Some Classical Stability Criteria", IEEE Trans. Education, Vol. 33, No. 3, pp.232-239, August 1990.
[4] L. H. Keel and S. P. Battacharyya, "A Generalization of Mikhailov's Criterion with Applications" Proceedings of the American Control Conference, pp. 4311-4315, Chicago, June 2000.

نظرة أخرى إلى نظرية ميخايلوف للاتزان

يتناول البحث نظرية ميخايلوف للاتزان فى مجال التردد والتى تفيد بأن دوران متجه ميخايلوف مع تغير التردد من صفر إلى مالانهاية يستخدم للحكم على الاتزان للنظم الخطية المتصلة.
وهذا البحث يتناول تغير زاوية الوجه لمتجه ميخايلوف بطريقة أخرى للحالة العامة عندما تكون المعادلة الواصفة لنظام التحكم فى الصورة العامة والتى لا يوجد قيود على وضع جذورها فى مجال المستوى المركب للجذور.
والطريقة المقدمة فى البحث تتناول معادلة واصفة من الدرجة ن وتوضح كيفية ايجاد عدد الجذور ومكانهم من تغير زاوية الوجه مع التردد وأعطيت أمثلة توضيحية تبين ذلك.