

DETERMINATION OF CONTROLLERS GAINS LIMIT USING THE MIKHAILOV STABILITY CRITERION

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Mikhailov's criterion states that a real Hurwitz polynomial $\delta(s)$ of degree n satisfies the monotonic phase increase, that is to say the argument of $\delta(jw)$ goes through n quadrants as w runs from zero to infinity. In this paper, we utilize the generalized Mikhailov criterion presented in [1] to give a solution to the problem of finding the stabilizing feedback gains for a given linear-time invariant plant having a P controller, a PI controller and a PID controller.

INTRODUCTION

Consider a real polynomial $\delta(s)$ of degree n . We say that $\delta(s)$ is Hurwitz if all roots lie in the open left half plane ($\text{Re}[s] < 0$). We utilize the generalized Mikhailov criterion [1, 2, 3] to give a solution to the problem of feedback stabilization of a given linear-time invariant plant using a P controller, a PI controller and a PID controller. Even though this problem can be solved using classical approaches such as the Nyquist stability criterion and the Routh-Hurwitz criterion, it is not clear how to extend these methods to the more complicated cases where PI or PID controllers are involved.

A CHARACTERIZATION OF ALL STABILIZING P, PI AND PID CONTROLLERS

Consider the feedback control system shown in Fig. (1),

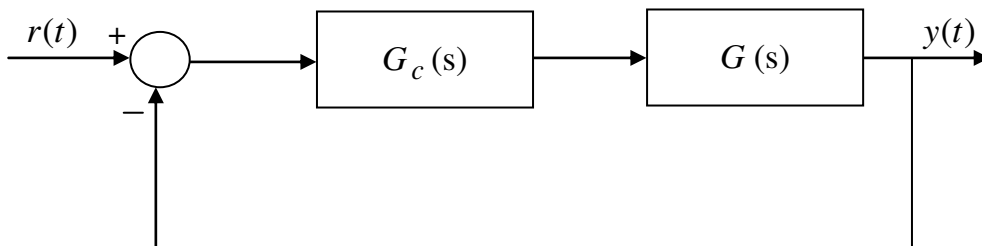


Fig. (1) Feedback control system

where $G(s) = \frac{N(s)}{D(s)}$

with $N(s), D(s)$ being coprime polynomials and $\text{degree}[N(s)] \leq \text{degree}[D(s)]$.

The closed-loop characteristic polynomial $\delta(s)$ can be expressed as:

• **For a stabilizing P controller**

$$G_c(s) = K_p \quad ,$$

then
$$\delta(s) = D(s) + K_p N(s) \quad (1)$$

We can write

$$N(s) = N_e(s^2) + sN_o(s^2) \quad , \text{ and}$$

$$D(s) = D_e(s^2) + sD_o(s^2)$$

so that

$$\delta(s) = [K_p N_e(s^2) + D_e(s^2)] + s[K_p N_o(s^2) + D_o(s^2)]$$

It is clear that from this expression that both the even as well as the odd parts of $\delta(s)$ depend on K_p , which causes difficulties when trying to find the range of K_p that makes the system stable. To overcome this problem, we will now construct a polynomial for which only the even part of this polynomial depends on K_p [2, 3].

Define

$$N^*(s) = N(-s) = N_e(s^2) - sN_o(s^2)$$

then

$$\delta(s)N^*(s) = h_1(s^2) + K_p h_2(s^2) + s g_1(s^2)$$

where

$$h_1(s^2) = D_e(s^2)N_e(s^2) - s^2 D_o(s^2)N_o(s^2)$$

$$h_2(s^2) = N_e(s^2)N_e(s^2) - s^2 N_o(s^2)N_o(s^2)$$

$$g_1(s^2) = N_e(s^2)D_o(s^2) - D_e(s^2)N_o(s^2)$$

Substituting $s = jw$, we obtain

$$\delta(jw)N^*(jw) = (R(w) + jI(w))(N_R^*(w) + jN_I^*(w))$$

where

$R(w)$ and $N_R^*(w)$ are real parts of $\delta(jw)$ and $N^*(jw)$ respectively.

$I(w)$ and $N_I^*(w)$ are imaginary parts of $\delta(jw)$ and $N^*(jw)$ respectively,

$$N_R^*(w) = N_e(-w^2) \quad , \text{ and}$$

$$N_I^*(w) = -w N_o(-w^2)$$

Then

$$\begin{aligned} \delta(jw)N^*(jw) &= (R(w)N_R^*(w) - I(w)N_I^*(w)) + j(R(w)N_I^*(w) + I(w)N_R^*(w)) \\ &= p(w) + jq(w) \end{aligned} \quad (2)$$

This last equation is a general form for any controller type.

If $\delta(s)N^*(s)$ has roots on the imaginary axis [1], then

$$\delta(jw)N^*(jw) = 0$$

and as a result $p(w) = 0$ and $q(w) = 0$

So,

$$R(w)N_R^*(w) - I(w)N_I^*(w) = 0 \quad , \quad (3)$$

$$\text{and} \quad R(w)N_I^*(w) + I(w)N_R^*(w) = 0 \quad (4)$$

Equations (3), (4) will equal zero either if

- i) $R(w)$ and $I(w)$ have common real roots, i.e. $\delta(s)$ has roots on the imaginary axis.,
- ii) Or $N_R^*(w)$ and $N_I^*(w)$ have common real roots, i.e. $N^*(s)$ has roots on the imaginary axis.

Since we test the characteristic polynomial $\delta(s)$, the case (i) is only considered.

For a P controller :

$$p(w) = D_e(-w^2)N_e(-w^2) + w^2 D_o(-w^2)N_o(-w^2) + K_p [N_e^2(-w^2) + w^2 N_o^2(-w^2)] \quad (5)$$

$$q(w) = w[D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2)] \quad (6)$$

In [2] and [3], the authors gave a computational method for the stabilizing range of feedback gain K_p . They used a set of strings that are generated to satisfy a certain requirements using the generalized Hermite-Biehler theorem and the generalized Mikhailov criterion respectively. In [1], it was shown that, the number of roots on the imaginary axis of the characteristic polynomial $\delta(s)$ equals the number of the common zeros of the real polynomials $R(w)$ and $I(w)$. The system can be marginally stable or unstable depending on the multiplicity of these imaginary roots taking into consideration that no roots are in the RHP. This last concept will be used in our proposed algorithm, where we write a MATLAB program to run this algorithm.

The Proposed Algorithm:

- 1- Obtain the nonnegative real roots of $q(w)$ of multiplicity one.
- 2- Eliminate the roots of $N^*(jw)$ of multiplicity one if any, from the roots obtained in step (1).
- 3- Equate $p(w)$ to zero using the above obtained nonnegative real roots in step (1) and obtain the unknown parameters K_p , i.e. $[K_{p0}, K_{p1}, \dots, K_{pt}]$.
- 4- At these values of K_p , the system is marginally stable or unstable depending on the presence of the roots of $\delta(s)$ in the right half plane (RHP). This can be checked by substituting these values of K_p in $\delta(s)$ and obtain the roots or

obtaining the argument of $\delta(s)$ i.e. $\theta(w)$ as w approaches infinity. If $\delta(s)$ has no roots in the RHP, then $\theta(w \rightarrow \infty) = n\pi/2$.

- 5- Sort these values of K_p that make the system on the margin of stability in ascending order ($[K_{p0}, K_{p1}, \dots, K_{pt}]$, where $K_{p0} < K_{p1} < \dots < K_{pt}$) and check the stability of $\delta(s)$ at $(K_{p0} - \varepsilon)$ and $(K_{pt} + \varepsilon)$ (where ε is a small positive number) to determine if $(-\infty, K_{p0})$ and (K_{pt}, ∞) are in the stability ranges or not.

• **For a stabilizing PI controller**

$$G_c(s) = \frac{K_i + K_p s}{s} \quad ,$$

then
$$\delta(s) = s D(s) + (K_i + K_p s)N(s) \tag{7}$$

As in the case of P controller,

$$\delta(jw)N^*(jw) = p(w) + jq(w)$$

where

$$p(w) = p_1(w) + K_i p_2(w) \quad , \tag{8}$$

$$q(w) = q_1(w) + K_p q_2(w) \quad , \tag{9}$$

and

$$p_1(w) = -w^2 (D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2))$$

$$p_2(w) = N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2)$$

$$q_1(w) = w(D_e(-w^2)N_e(-w^2) + w^2 D_o(-w^2)N_o(-w^2))$$

$$q_2(w) = w(N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2))$$

From these expressions [3], we note that K_i, K_p appear in $p(w)$ and $q(w)$ respectively. Moreover, for every fixed K_p , the roots of $q(w)$ do not depend on K_i , and we use the same algorithm presented in the case of P controller (using K_i instead of K_p). Thus, by sweeping over all real K_p and solving a constant gain stabilization problem (stabilizing P controller problem) for each value, we can determine the set of all stabilizing values (K_i, K_p) for a given plant.

However, there is no need to sweep over all real values of K_p . From [3] a necessary condition for the existence of a stabilizing K_i value at a fixed value of K_p is that the number of real, non-negative, distinct finite roots of odd multiplicities of $q(w)$ be at least

$$\frac{|n - (l(N(s)) - r(N(s)))|}{2} \quad \text{if } m + n \text{ is even} \tag{10}$$

or

$$\frac{|n - (l(N(s)) - r(N(s)))| + 1}{2} \quad \text{if } m + n \text{ is odd} \quad (11)$$

where

$n =$ degree of $\delta(s)$

$m =$ degree of $N(s)$

$l(N(s)) =$ number of the roots of $N(s)$ in the LHP.

$r(N(s)) =$ number of the roots of $N(s)$ in the RHP.

These ranges of K_p that satisfy the condition can be determined by applying the root locus ideas which is presented in [3].

• **For a stabilizing PID controller**

$$G_c(s) = \frac{K_i + K_p s + K_d s^2}{s} \quad ,$$

$$\delta(s) = s D(s) + (K_i + K_p s + K_d s^2) N(s) \quad (12)$$

then,

$$\delta(jw)N^*(jw) = p(w) + jq(w)$$

where

$$p(w) = p_1(w) + (K_i - K_d w^2)p_2(w) \quad (13)$$

$$q(w) = q_1(w) + K_p q_2(w) \quad (14)$$

and

$$p_1(w) = -w^2(D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2))$$

$$p_2(w) = N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2)$$

$$q_1(w) = w(D_e(-w^2)N_e(-w^2) + w^2 D_o(-w^2)N_o(-w^2))$$

$$q_2(w) = w(N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2))$$

From these expressions [3], we note that K_i, K_d appear in $p(w)$ while K_p appears in $q(w)$. Moreover, for every fixed K_p , the roots of $p(w)$ will depend on both K_i and K_d . Since there are two unknown variables in $p(w)$, a linear programming problem has to be solved for each fixed gain K_p , where every two equations are solved and the stability of the solution points of all two equations are checked via the roots types of $\delta(s)$ or the argument $\theta(w)$ as w approaches infinity, so a modification is made on our proposed algorithm. The ranges of K_p can be obtained as discussed in the PI controller.

Modified Proposed Algorithm (at every fixed K_p):

- 1- Obtain the nonnegative real roots of $q(w)$ of multiplicity one.
- 2- Eliminate the roots of $N^*(jw)$ of multiplicity one if any from the roots obtained in step (1).

- 3- Equate $p(w)$ to zero using the above obtained nonnegative real roots in step (1). Since there are two unknown variables ((K_i, K_d)) in $p(w)$, a linear programming problem has to be solved for each fixed gain K_p .
- 4- At these set of (K_i, K_d) values, the system is marginally stable or unstable depending on the presence of the roots of $\delta(s)$ in the RHP. This can be checked by substituting these values in $\delta(s)$ and obtain the roots or obtaining the argument of $\delta(s)$ i.e. $\theta(w)$ as w approaches infinity. If $\delta(s)$ has no roots in the RHP, then $\theta(w \rightarrow \infty) = n\pi/2$.
- 5- Since the equations is linear, so the set of (K_i, K_d) values that make the system on the margin of stability will form boundaries of a convex polygon or an intersection of half planes.

In the following section, we will illustrate the suggested method by an example presented in [3].

EXAMPLE

Given the plant described by the rational transfer function $G(s) = \frac{N(s)}{D(s)}$, where

$$N(s) = s^3 - 4s^2 + s + 2$$

$$D(s) = s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17$$

We will consider the three types of controllers (P, PI and PID controller).

1- For a P controller

$$G_c(s) = K_p$$

The closed-loop characteristic polynomial as in Eq.(1) is

$$\delta(s) = D(s) + K_p N(s)$$

Write

$$N(s) = N_e(s^2) + sN_o(s^2)$$

$$D(s) = D_e(s^2) + sD_o(s^2)$$

where

$$N_e(s^2) = -4s^2 + 2$$

$$N_o(s^2) = s^2 + 1$$

$$D_e(s^2) = 8s^4 + 46s^2 + 17$$

$$D_o(s^2) = s^4 + 32s^2 + 46$$

$$\text{and, } N^*(s) = N(-s) = N_e(s^2) - sN_o(s^2) = (-4s^2 + 2) - s(s^2 + 1)$$

Then we have

$$N_e(-w^2) = 4w^2 + 2$$

$$N_o(-w^2) = -w^2 + 1$$

$$D_e(-w^2) = 8w^4 - 46w^2 + 17$$

$$D_o(-w^2) = w^4 - 32w^2 + 46$$

Now we construct the polynomial,

$$\delta(jw)N^*(jw) = p(w) + jq(w)$$

where $p(w)$, $q(w)$ are obtained from Eq.(5) and Eq.(6)

$$p(w) = -w^8 + (65 + K_p)w^6 + (14K_p - 246)w^4 + (22 + 17K_p)w^2 + (34 + 4K_p)$$

$$q(w) = 12w^7 - 180w^5 + 183w^3 + 75w$$

We apply the proposed algorithm

- 1- The nonnegative real roots of $q(w) = 0$ of multiplicity one are :
 $w_0 = 0$, $w_1 = 1.2018$, $w_2 = 3.7240$.
- 2- There are no roots of $N^*(s)$ on the iw -axis.
- 3- For the above obtained roots in step (1), equate $p(w)$ to zero and obtain the unknown parameters K_p . The calculated gain constants K_p are:
 $[K_{p0} = -8.5, K_{p1} = 4.2109, K_{p2} = -15.9665]$.
- 4- At these values of K_p , we check the stability of the characteristic polynomial $\delta(s)$ of Eq.(1).
 - a. For $K_{p0} = -8.5$, $p(w) = q(w) = 0$ and $\delta(s)$ has a root at the origin and the system is unstable and $\theta(w \rightarrow \infty) = 5\pi/2$.
 - b. For $K_{p1} = 4.2109$, $p(w) = q(w) = 0$ and $\delta(s)$ has two pure imaginary roots $\pm j1.2018$ and the system is marginally stable as $\delta(s)$ has no roots in the RHP and $\theta(w \rightarrow \infty) = 5\pi/2$.
 - c. For $K_{p2} = -15.9665$, $p(w) = q(w) = 0$ and $\delta(s)$ has two pure imaginary roots $\pm j3.724$ and the system is unstable as $\delta(s)$ has a positive real root. It is found that, $\theta(w \rightarrow \infty) = 4\pi/2 \neq 5\pi/2$.
- 5- To find the range of K_p for a stable system, the system stability is checked at $K_p = -8.5 + \varepsilon$ where ε is a small positive number and it was stable.
 Thus the range of K_p is $-8.5 < K_p < 4.2109$.

2- For a PI controller

$$G_c(s) = \frac{K_i + K_p s}{s}$$

The closed-loop characteristic polynomial as in Eq.(7) is

$$\delta(s) = sD(s) + (K_i + K_p s)N(s)$$

Now we construct the polynomial,

$$\delta(jw)N^*(jw) = p(w) + jq(w)$$

where $p(w)$, $q(w)$ are obtained from Eq.(8) and Eq.(9) by determining the following polynomials as follows:

$$p_1(w) = -12w^8 + 180w^6 - 183w^4 - 75w^2$$

$$p_2(w) = w^6 + 14w^4 + 17w^2 + 4$$

$$q_1(w) = -w^9 + 65w^7 - 246w^5 + 22w^3 + 34w$$

$$q_2(w) = w^7 + 14w^5 + 17w^3 + 4w$$

then

$$p(w) = p_1(w) + K_i p_2(w)$$

$$q(w) = q_1(w) + K_p q_2(w)$$

Using the root locus ideas [3] to satisfy the condition of Eq.(11), the range of K_p values can be narrowed down to $K_p \in (-8.5, 4.2333)$.

By sweeping over different K_p values in this interval and following the algorithm illustrated above as in P controller (but for K_i instead of K_p), we can generate the set of (K_p, K_i) values for which $\delta(s)$ is stable. This set is sketched in Fig.(2).

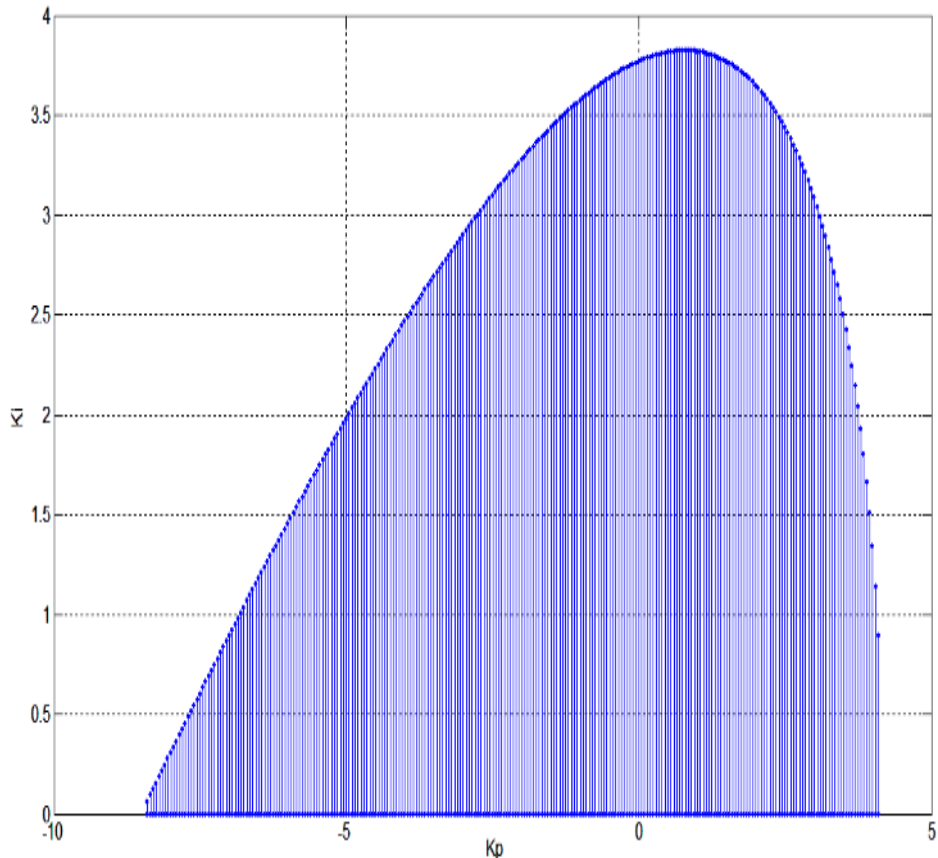


Fig. (2) The stabilizing set of (K_p, K_i) values

For a PID controller

$$G_c(s) = \frac{K_i + K_p s + K_d s^2}{s}$$

The closed-loop characteristic polynomial as in Eq.(12) is

$$\delta(s) = s D(s) + (K_i + K_p s + K_d s^2) N(s)$$

Now we construct the polynomial,

$$\delta(jw)N^*(jw) = p(w) + jq(w)$$

where $p(w)$, $q(w)$ are obtained from Eq.(13) and Eq.(14) by using the same polynomials that was determined in the PI controller, then

$$p(w) = p_1(w) + (K_i - K_d w^2)p_2(w)$$

$$q(w) = q_1(w) + K_p q_2(w)$$

Using the root locus ideas [3] to satisfy the condition of Eq.(8) as in the PI controller, the range of K_p values can be narrowed down to $K_p \in (-8.5, 4.2333)$.

By sweeping over different K_p values in this interval and following the modified algorithm illustrated before, we can generate the set of (K_p, K_i, K_d) values for which $\delta(s)$ is stable.

We apply the modified algorithm (for every fixed K_p):

- 1- Obtain the nonnegative real roots of $q(w)$ of multiplicity one.
- 2- There is no roots of $N^*(s)$ on the iw -axis.
- 3- Equate $p(w)$ to zero using the above obtained nonnegative real roots in step (1). Since there are two unknown variables $((K_i, K_d))$ in $p(w)$, a linear programming problem has to be solved for each fixed gain K_p . These equations are:

$$p_1(w_0) + (K_i - K_d w_0^2)p_2(w_0) = 0$$

$$p_1(w_1) + (K_i - K_d w_1^2)p_2(w_1) = 0$$

$$p_1(w_2) + (K_i - K_d w_2^2)p_2(w_2) = 0$$

$$p_1(w_t) + (K_i - K_d w_t^2)p_2(w_t) = 0$$

- 4- At these set of (K_i, K_d) values, the system is marginally stable or unstable depending on the presence of the roots of $\delta(s)$ in RHP. This can be checked by substituting these values in $\delta(s)$ and obtain the roots or obtaining the argument

of $\delta(s)$ i.e. $\theta(w)$ as w approaches infinity. If $\delta(s)$ has no roots in the RHP, then $\theta(w \rightarrow \infty) = n\pi / 2$.

- 5- Since the equations are linear, so the set of (K_i, K_d) values that make the system on the margin of stability will form boundaries of a convex polygon. The set is shown in Fig.(3).

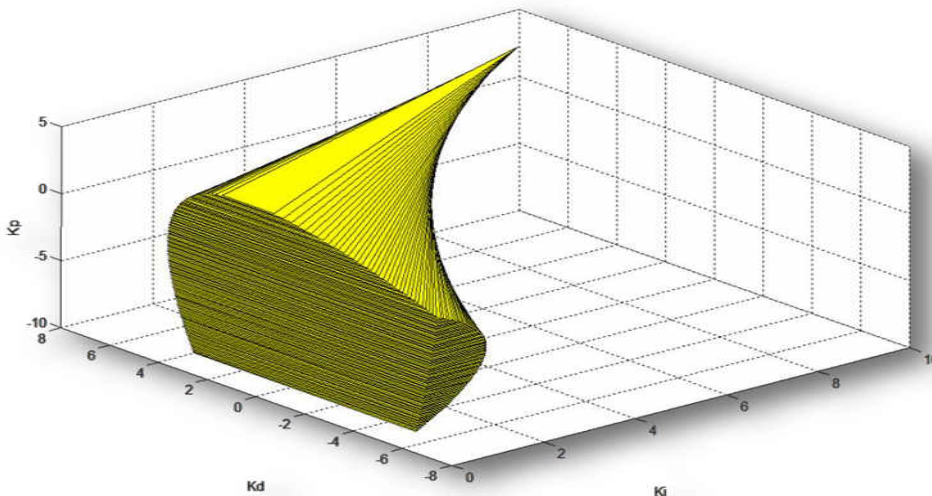


Fig. (3) The stabilizing set of (K_p, K_i, K_d) values

In this paper we use the generalized Mikhailov criterion to find a solution to the problem of feedback stabilization of a given linear-time invariant plant using a P controller, a PI controller and a PID controller. In [1], it was shown that, the number of roots on the imaginary axis of the characteristic polynomial $\delta(s)$ equals the number of the common zeros of the real polynomials $R(w)$ and $I(w)$. The system can be marginally stable or unstable depending on the multiplicity of these imaginary roots. We proposed an algorithm using MATLAB software using the last concept. An example with three types of controllers is given to illustrate the proposed algorithm to define the controllers gains limit.

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تحديد مدى معاملات المحكم باستخدام نظرية ميخايلوف للاتزان

تفيد نظرية ميخايلوف أن المعادلة الواصفة للنظام من الدرجة n لكي تكون "هوريفتزر" يجب أن تتغير زاوية الوجه لمتجه ميخايلوف انسيابيا مع الزيادة. و هذا البحث يتناول استخدام الحالة العامة لنظرية ميخايلوف لايجاد مدى معاملات المحكم لنظام خطى والتي تؤدي إلى اتزانه وهذا المحكم من نوع المحكم التناسبي والمحكم التناسبي التكاملي, والمحكم التناسبي التكاملي التفاضلي. ولقد أعطيت أمثلة توضيحية تبين الطريقة المقترحة لايجاد مدى تغير معاملات المحكم مع ضمان الاتزان.