DJS Vol. 40 (2019) 30-35

Delta Journal of Science

Available online at
https://djs.journals.ekb.eg/

Research Article MATHEMATICS

Perfect folding of graphs

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KEY WORDS

Clique number, chromatic number, perfect graphs, graph folding

ABSTRACT

In this paper we introduced the definition of perfect folding of graphs and we proved that cycle graphs of even number of edges can be perfectly folded while that of odd number of edges can be perfectly folded to C_3 . Also we proved that wheel graphs of odd number of vertices can be perfectly folded to C_3. Finally we proved that if G is a graph of n vertices such that 2 < clique number =chromatic number=$k < n$, then the graph can be perfectly folded to a clique of order k.

1. Introduction

Let $G = (V, E)$ be a graph , where V is the set of its vertices and E is the set of its edges. Two distinct vertices $u ,v \in V$ are called independent if $\{u ,v\}$ is not an edge in G . Two vertices u ,v are called neighbors (adjacent) if $\{u,v\}$ is an edge in G. The degree (valency) of a vertex is the number of edges with the vertex as an end point. A graph with no loops or multiple edges is called a simple graph . A graph is said to be connected if every pair of vertices has a path connecting them otherwise the graph is disconnect-ted . A graph $H=(V,E')$ is called induced subgraph of $G=(V,E)$ if $V' \subseteq V$ and $\{u,v\}$ is an edge in H wherever u and v are distinct vertices in V$'$ and $\{ u,v \}$ is an edge in G , H is called proper if $H \neq G$. A cycle graph is a graph that consists of a single cycle, or in other words , some number of distinct vertices connected in a closed chain. The cycle graph with n vertices is denoted by C_n . The number of vertices in C_n equals the number of edges , and every vertex has degree 2. The wheel graph W_n or n-wheel is a graph that contains a cycle of order $n-1$, and for which every graph vertex in the cycle is connected to one other graph vertex which is called the hub. A bipartite graph is a graph whose vertex set can be split into two sets A and B in such a way that each edge of the graph joins a vertex in A to a vertex in B . A vertex coloring of a graph $G=(V,E)$ is a way of coloring the vertices of the graph such that no two adjacent vertices share the same color. A clique of a graph G is a maximal complete subgraph. In this case each pair of vertices of the clique are adjacent. The clique number $W(G)$ of a graph is the number of graph vertices in the largest clique of G , [8]. The clique number of a cycle graph C_n , n odd is 3 and 2 otherwise. For a wheel graph W_n , n is even the clique number is 4 and is 3 otherwise. The chromatic number of a graph G is the smallest number of colors needed to color the vertices of a graph G so that no two adjacent vertices share the same color, and is often denoted by χ(G). A graph G is called perfect if for every induced subgraph H of G, $\chi(H) = W(H)$. Note that if G is a perfect graph, then every induced subgraph of G is also perfect,[2].

(2)Perfect folding

Definition (2-1)

Let G_1 and G_2 be two simple graphs and $f : G_1 \to G_2$ be continuous map. Then f is called a graph map, if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $dim(f(e)) \leq dim(e)$, [3] .

Definition (2-2)

A graph map $f : G_1 \to G_2$ is called a graph folding if and only if f maps vertices to vertices and edges to edges ,i.e., if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $f(e)$ is an edge in $E(G_2)$,[4] .

Note that if the vertices of an edge $e=(u,v) \in E(G_1)$ are mapped to the same vertex , then the edge e will collapse to this vertex and hence we cannot get a graph folding. In other words, any graph folding cannot maps edges to loops but it may maps loops, if there is any , to loops.

Definition (2-3)

Let G and H be simple connected graphs. We call a graph folding $f: G \to H$ perfect folding if its image $f(G)$ is a perfect subgraph of H.

In general the image of a graph folding $f: G \to H$ is not a perfect graph e.g., if G_1 is the imperfect graph shown in Fig.(1-a), where $V(G_1)=\{v_1,v_2,v_3, v_4, v_5, v_6, v_7\}$ and $E(G_1) = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Then the graph folding $f: G_1 \to G_1$ defined by $f\{v_6 ,v_7\}=\{v_5 ,v_4\}$and $f\{ e_6 ,e_7\}=\{ e_2,e_4\}$ is not a perfect folding. While if we consider the imperfect graph G_2 shown in Fig.(1-b), where $V(G_2)=\{ u_1,\dots,u_7 \}$ and $E(G_2)=\{ e_1,\dots, e_7 \}$. Then the graph folding $g: G_2 \to G_2$ defined by $g\{u_1, u_4\} = \{u_6, u_6\}$ and $g\{e_4, e_7 \} =\{ e_5, e_6\}$ is a perfect folding. The omitted vertices and edges in this example and through the paper will be mapped to themselves.

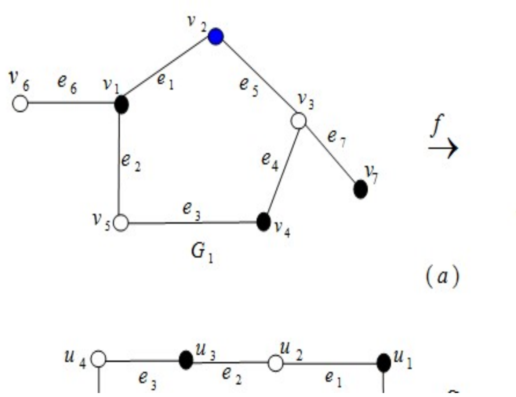

Fig.(1)

Theorem (2-1)

Let G be a simple connected graph such that the number of $E(G) \geq 2$. If the chromatic number $\chi(G)$ is equal to two, then G can be perfectly folded.

Proof

From [5], any simple connected graph G such that $E(G) \geq 2$ and $\chi(G)=2$ can be folded to an edge. In this case $\chi(f(G)) = W(f(G)) = 2$, and thus the graph G can be perfectly folded to an edge.

Example (2-4)

The cubic graph G with $\chi(G) = W(G) = 2$, shown in Fig. (2) can be folded to an edge by the graph folding $f(v_1, \ldots, v_8) = (v_1, v_2, v_1, v_2, v_1, v_2, v_1, v_2)$. This folding can be done by the composition of a sequence of foldings f_1, f_2, f_3 and f_4, see Fig.(2) . And hence the graph folding is a perfect.

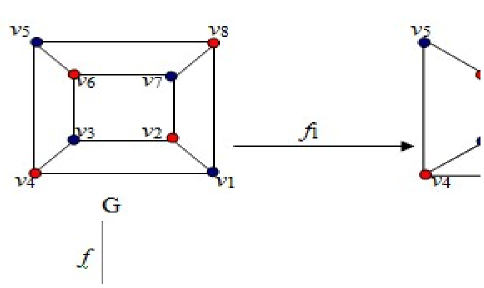

Fig.(2)

Lemma (2-2)

Any folding of a bipartite graph (complete) is a perfect folding.

Proof

This follows from the fact that the chromatic number of a bipartite graph is equal to two, and thus it can be perfectly folded.

Example (2-5)

Consider the bipartite graph G shown in Fig.(3). A graph folding $f: G \to G$ defined by $f\{v_1, v_3\} = \{v_2\}$ and $f\{e_1, e_4\} = \{e_2, e_3\}$ is a perfect folding.

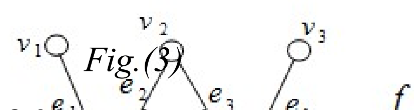

Fig.(3)

(3) Perfect folding of cycle graphs

The chromatic number of a cycle graph C_n , $n > 2$ where n is odd is 3 while that for n even is 2 , [1].

Theorem (3-1)

Any folding of a cycle graph C_n of an even number of edges is a perfect folding.

Proof

This follows from the fact that $\chi(C_n)$, n is an even number is equal to two. Thus C_n can be perfectly folded.

Example (3-1)

Consider the cycle graph C_4 where $\chi(C_4) = W(C_4) = 2$, the graph folding

$f : C_4 \rightarrow C_4$ defined by $f\{v_1, v_4\} = \{v_3, v_2\}$ and $f\{e_i\} = \{e_3\}$, $i=1,2,4$ is a perfect folding , see Fig.(4).

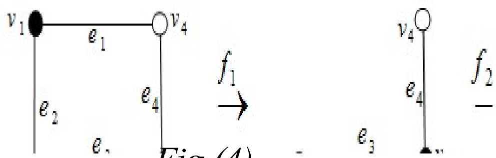

Fig.(4)

It should be noted that the cycle graph C_3 cannot be folded,[4].

Theorem (3-2)

Let $G = C_n$, $n > 3$ be a cycle graph of an odd number of edges (vertices). Then G can be perfectly folded to C_3.

Proof

Since $G = C_n$ has an odd number of edges (vertices). Thus the graph C_n has three color classes, say V_1, V_2 and V_3. We can color the vertices of C_n alternatively with the two colors of V_1 and V_2 except the last two edges one will join a vertex colored by the color of V_2 and a vertex colored by the color of V_3 and the other edge will join a vertex colored by the color of V_3 and a vertex colored by the color of V_1. Thus the number of vertices of color class V_1 = the number of vertices color class $V_2=(n-1)/2$, but V_3 has only one vertex w. We can define a graph folding $f : C_n \rightarrow C_n$, n is odd , by mapping vertices of V_1 to a vertex of V_1 , say u, and mapping the vertices of V_2 to a vertex of V_2 , say v, finally mapping w into itself. Thus we have three vertices u, v, w and hence three edges in the image i.e., we have C_3. But $\chi(C_3) = W(C_3) = 3$, i.e., the graph folding f is perfect .

Example (3-2)

Let $G=C_5$ and $h: G \rightarrow G$ be the graph folding defined by $h\{v_5, v_4\}=\{v_3, v_1\}$ and $h\{e_i\} = \{e_2\}$, $i=3,4$ is a perfect folding , see Fig.(5). This can be done by the composition of the two graph foldings $h_1 : C_5 \rightarrow C_5$ defined by $h_1\{v_5\}=\{v_3\}$, $h_1\{e_3\}=\{e_4\}$ and $h_2: h_1(C_5) \rightarrow h_1(C_5)$ defined by $h_2\{v_4\}= \{v_1\}$, $h_2\{e_4\} = \{e_2\}$.

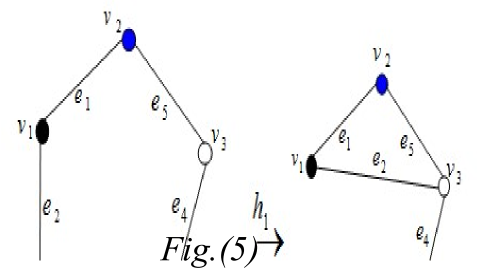

Fig.(5)

(4) Perfect folding of wheel graphs

The chromatic number of a wheel graph W_n is 3 ; if n is odd ; and 4; if n is even, [1].

Theorem (4-1)

Any wheel graph W_n of an odd number of vertices can be perfectly folded to C_3.

Proof:

A wheel graph W_n of order n , n is an odd number, is a graph that contains a cycle of even order $n-1$, and each vertex in the cycle is connected to the hub. In this case the chromatic number $\chi(W_n) = 3$, thus the graph W_n can be colored by using three colors A, B and C. One color for the hub, say A, and the vertices of the even cycle C_{n-1} can be colored alternatively with two colors B and C , i.e., if the set of vertices of the cycle C_{n-1} is $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$, then the colors B and C have the following vertices, $B=\{v_1, v_3, \ldots, v_{n-2}\}$ and $C=\{v_2, v_4, \ldots, v_{n-1}\}$. Now we can define a graph folding by mapping the vertices of B to a vertex of B , the vertices of C to a vertex of C and the hub onto itself. The image of this map will contains three vertices, three edges and thus we have C_3, i.e., the graph folding is perfect.

Example (4-1)

Consider the wheel graph W_7 and the graph folding $f: W_7 \to W_7$ defined by $f\{v_i\} =\{v_1\}$, $i=3,5$, $f\{v_j\}=\{v_2\}$, j=4,6 and $f\{e_k\}=\{ e_1,e_1,e_1,e_1, e_1,e_1, e_7,e_8, e_7,e_8, e_7,e_8\}$, $k=1,\ldots,12$. This graph folding is perfect, see Fig.(6).

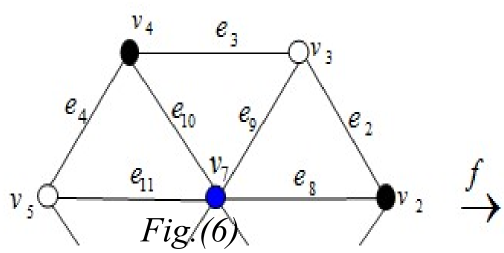

Fig.(6)

It should be noted that the wheel graph of an even number of vertices cannot be folded, [4], and hence cannot be perfectly folded.

(5) The clique number and perfect folding

The chromatic number of any graph is equal to or greater than its clique number, i.e., $\chi(G) \geq W(G)$. For connected graphs $2 \leq W(G) \leq \chi(G) \leq n$, where n is the number of vertices of the graph G , [7].

Theorem (5-1)

Let G be a simple connected graph, if the clique number $W(G)$ equal to the chromatic number $\chi(G)$ equal to 2 and $E(G) \geq 2$, then the graph G can be perfectly folded .

Proof

It is immediately follows from Theorem (2-4) and since $\chi(G)=2,$ then G can be perfectly folded.

Example (5-1)

Consider the cycle graph C_6 shown in Fig.(7). A graph folding $f: C_6 \to C_6$ defined by $f\{ v_2 ,v_3 ,v_4 ,v_5 \} = \{v_6 ,v_1 ,v_6 ,v_1\}$ and $f\{e_i\}=\{e_6\}$, $i=1,\ldots,5$ is a perfect folding.

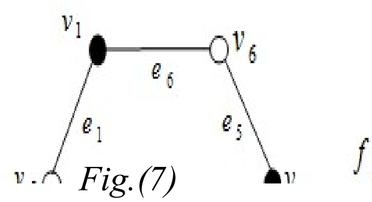

Fig.(7)

Theorem (5-2)

Let G be a simple connected graph such that $no.V(G) = n$. If $2 < W(G) = \chi(G) = k < n$,then the graph can be perfectly folded to a clique of order k.

Proof

Let $W(G) = \chi(G) = k$, then we have a maximal complete subgraph of k vertices. This complete graph cannot be folded, [3].These vertices must be colored by different colors $A_1, A_2, \ldots, A_k$. Now the other $(n-k)$ vertices of G, will be colored by the colors $A_1, \ldots, A_m$, $m \leq k$ in such a way that any edge will joins two vertices of different colors. So we can define a seque- nce of graph folding $f_i: G \to G_i$, where $G_i = f_i (G_{i-1})$, $i = 1,\ldots,m$, $G_0 = G$, by mapping the $(n-k)$ vertices to other vertices but of the same color, until we get the k-clique which cannot be folded any more . And hence $W(f_i(G_i)) = \chi(f_i(G_i)) = k$, i.e., the graph folding is a perfect.

Example (5-2)

Consider the house graph G *with 5 vertices and 6 edges* shown in Fig.(8), where $2 < W(G) = \chi(G) = 3 < n = 5$. This graph can be folded to a triangle by the graph folding $f : G \rightarrow G$ defined by $f\{ v_4 , v_5 \} = \{v_2 , v_3\}$ which is a perfect folding.

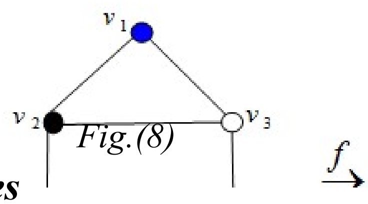

Fig.(8)

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