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STATISTICAL INFERENCE FOR M/M/1/N QUEUEING SYSTEM WITH BALKING, RENEGING AND REFLECTING BARRIER UNDER QUALITY CONTROL PROCESS

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KEY WORDS	ABSTRACT
balking, reneging, reflecting barrier, quality control, maximum likelihood estimate, confidence interval.	In this paper we develop the confidence intervals for estimating the expected value of the units in the system and the maximum likelihood estimates of the parameters are derived for truncated single - channel Markovian queue with balking, reneging and reflecting barrier by using quality control process under steady - state simulation. Some important queueing systems are developed and discussed as special cases of the model. A simulated numerical example is used to illustrate its application for estimating the parameters under quality control process.

1. Introduction

Statistical inferences in quality control process arise in several academic areas, which includes biomedical and life sciences, political sciences, computer science and communications, production systems, military sciences and biotechnology science. Point estimation and confidence intervals

estimate of the parameters had been studied by many researches. The pioneer work began by Clarke [3], who calculated the estimates of the parameters of the non - truncated single - channel Markovian queues using the maximum likelihood method. This work had been followed by Wolf [12], who studied

some Poissonian queues ignoring the initial probability. Lillifeors [7] treated confidence intervals of the Markovian queues *M/M/1* and *M/M/2* without any concepts. Kotb et al. [6] addressed and concerned statistical sampling design in quality control processes for queueing systems of units in different institutions adding the concept of balking via steady - state simulation. Tapiero and Hsu [10] integrated a quality control procedure into a job shop manufacturing process described by an *M/M/1* queue. Some papers can be found in the literature for the parameter estimation problems for many distributions. See, for example, Louzada et al. [8], Keeble [5], Acharya et al. [2] and Wang et al. [11]. Recently, Quinino and Cruz [9] studied the sample size planning problem by focusing on a Bayesian point estimation for *M/M/1* queue for the utilization factor.

The purpose of this paper is to study statistical inference in quality control procedure for truncated single-server Markovian queue with balking, reneging and reflecting barrier under steady-state simulation. As a result of this model, we treat both point estimation using modified likelihood function and confidence intervals for estimating the mean and control charts. Some important queueing systems are derived as special cases. Finally, the queueing system is then investigated to obtain greater insights regarding the mutual and join effects of quality control and model design.

2. BASIC NOTATIONS AND ASSUMPTIONS

The following notations are adopted for developing the model:

$1/\lambda =$ Mean inter-arrival time.

$1/\mu =$ Mean service time.

$\rho = \lambda/\mu =$ Utilization factor.

$k =$ Balking limit.

$n =$ Number of units in the system.

$N =$ System Capacity.

$P_0 =$ Probability that no units are in the system.

$P_n =$ Steady-state probability that there are n units in the system.

$P_n(t) =$ Transient state probability that there are n units in the system.

$\beta =$ Probability that an arriving unit joins the queue when it finds n units in the system.

$1 - \beta =$ Probability that a unit balks, $0 \le \beta < 1$ *for* $k \le n \le N-1$ *and* $\beta = 1$ elsewhere.

$g =$ Reneging rate of a certain length of time, which a unit will wait for service.

$r =$ Probability that a barrier reflects a unit at state $n = N$.

$1 - r =$ Probability that a barrier absorbs a unit at state $n = N$, $0 < r \le 1$.

$E(n) =$ Expected number of units in the system.

CL = Control limit (average of subgroup ranges).

$UCL = E_U(n)$ = Upper control limit.

$LCL = E_L(n)$ = Lower control limit.

$E_U(n) - E_L(n)$ = Subgroup ranges.

In addition, the following assumptions are made for developing the model:

(1) Inter-arrival times of the units follow the exponential distribution with mean $1/\lambda$.

(2) Service times are also an exponential distribution with mean $1/\mu$. The units are served according to FIFO discipline.

(3) An arriving unit joins the queue when it finds n units in the system with probability β and balks with probability $(1-\beta)$

(4) After joining the queue each unit will wait a certain length of time for his service to begin. If it does not begin by then, he will get reneged and may leave the queue without getting service with probability *(n-1)g* for $n \geq 2$.

(5) For truncated queue with capacity *N*, there is a barrier reflects or absorbs a unit at the state

$n = N$ with probability *r* or *(1-r)* respectively.

3. STEADY-STATE SIMULATION

Using the above notations and assumptions and applying Markov conditions, we obtain the following system of steady-state probability difference equations:

$$-\lambda p_0 + \mu p_1 = 0, \qquad n = 0 \qquad (1)$$

$$-(\lambda + \mu) p_1 + \lambda p_0 + (\mu + g) p_2 = 0, \qquad n = 1 \quad (2)$$

$$-[\lambda + \mu + (n-1)g] p_n + \lambda p_{n-1} + (\mu + ng) p_{n+1} = 0, \qquad 2 \leq n \leq k\text{-}1 \qquad (3)$$

$$-[\beta\lambda + \mu + (k-1)g] p_k + \lambda p_{k-1} + (\mu + kg) p_{k+1} = 0, \qquad n = k \qquad (4)$$

$$-[\beta\lambda + \mu + (n-1)g] p_n + \beta\lambda p_{n-1} + (\mu + ng) p_{n+1} = 0, \qquad k+1 \leq n \leq N-2 \qquad (5)$$

$$-[\beta\lambda + \mu + (N-2)g] p_{N-1} + \beta\lambda p_{N-2} + [\mu + (N-1)g] r p_N = 0, \qquad n = N\text{-}1 \qquad (6)$$

$$-[\mu + (N-1)g] r p_N + \beta\lambda p_{N-1} = 0, \quad n = N \quad (7)$$

Solving this system of equations iteratively, the probability that here are *n* units in the system is easily obtained as:

$$p_n = \begin{cases} \dfrac{\delta^n}{(\gamma)_n} p_0, & 0 \leq n \leq k-1 \\[3mm] \dfrac{(\beta\delta)^n}{\beta^k (\gamma)_n} p_0, & k \leq n \leq N-1 \\[3mm] \dfrac{(\beta\delta)^N}{r\beta^k (\gamma)_N} p_0, & n = N \end{cases} \qquad (8)$$

Where

$$\delta = \frac{\lambda}{g}, \gamma = \frac{\mu}{g}$$

And

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\ldots[\gamma+(n-1)],$$
$$n \geq 1, (\gamma)_0 = 1.$$

To find p_0, use the boundary condition $\sum_{n=0}^{N} p_n = 1$, then we get:

$$p_0^{-1} = {}_1F_1\left(\begin{matrix} 1-k \\ \gamma \end{matrix}; -\delta\right) + \beta^k \, {}_1F_1\left(\begin{matrix} 1+k-N \\ \gamma \end{matrix}; -\beta\delta\right) + \frac{(\beta\delta)^N}{r\beta^k(\gamma)_N} \quad (9)$$

Where

${}_qF_p(\ )$ is the general hypergeometric function.

Using Abou-El-Ata's theorem [1] of moments, the expected number of units in the system is:

$$E(n) = -\lambda \frac{\partial \ln p_0}{\partial \lambda}$$

$$= p_0 \left\{ (k-1)\delta \, {}_1F_1\left(\begin{matrix} 2-k \\ \gamma+1 \end{matrix}; -\delta\right) + \beta^{1-k}\,\delta(N-k-1) \right.$$

$$\left. \cdot {}_1F_1\left(\begin{matrix} 2+k-N \\ \gamma+1 \end{matrix}; -\beta\delta\right) + \frac{N(\beta\delta)^N}{r\beta^k(\gamma)_N} \right\} \quad (10)$$

4. ESTIMATE

To estimate the parameters λ, μ, g and ρ of this system, consider n units enter the system during the interval $[0, T]$, m units are served during the interval $[0, \tau]$, h units are reneged during the interval $[0, t^*]$ and v is the initial number of units at $t = 0$. Thus the modified likelihood function is:

$$\ell(\lambda, \mu, g) = \prod_{i=1}^{n} a(t_i) . \prod_{i=1}^{m} b(\tau_i) . \prod_{i=1}^{h} c(t_i^*) . p_v$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda t_i} . \prod_{i=1}^{m} \mu e^{-\mu \tau_i} . \prod_{i=1}^{h} g e^{-g t_i^*} . p_v$$

$$= \lambda^n . \mu^m . g^h . e^{-(\lambda T + \mu \tau + g T^*)}$$

$$\cdot \begin{cases} \dfrac{\delta^v}{(\gamma)_v} p_{0,} & 0 \le v \le k-1 \\[2ex] \dfrac{\beta^{v-k}\delta^v}{(\gamma)_v} p_0, & k \le v \le N-1 \\[2ex] \dfrac{\beta^{N-K}\delta^N}{r(\gamma)_N} p_0, & v = N \end{cases} \quad (11)$$

where

$$T = \sum_{i=1}^{n} t_i \,, \ \tau = \sum_{i=1}^{m} \tau_i \,, \ T^* = \sum_{i=1}^{h} t_i^* \,,$$

t_i is the entrance time of the ith unit,

τ_i is the service time of the ith unit,

t_i^* is the reneging time of the ith unit

and

$a(t_i), b(\tau_i), c(t_i^*), p_v$ are independently distributed.

The maximum likelihood estimates (MLEs) of the parameters λ, μ, g and ρ are obtained by the direct maximization of the log-likelihood function given by:

$$L(\lambda, \mu, g) = \ln \ell = n \ln \lambda + m \ln \mu + h \ln g$$

$$- (\lambda T + \mu \tau + g T^*) + \ln p_0 \quad (12)$$

$$+ \begin{cases} v \ln \delta - \ln(\gamma)_v, & 0 \le v \le k-1 \\ v \ln \delta + (v-k)\ln \beta - \ln(\gamma)_v, & k \le v \le N-1 \\ v \ln \delta + (v-k)\ln \beta - \ln(\gamma)_v - \ln r, & v = N \end{cases}$$

Differentiating $L(\lambda, \mu, g)$ with respect to λ, μ, g respectively and using

Abou-El-Ata's theorem [1], we get for $0 \le v \le N$:

$$\hat{\lambda} = \frac{n + v - \hat{E}(n)}{T}, \qquad \hat{\mu} = \frac{m + \hat{E}(n)}{\tau + (\gamma)_v'/(\gamma)_v}$$

$$\hat{g} = \frac{h - \mu + \hat{E}(n)}{T^* + (\gamma)_v'/(\gamma)_v} \quad (13)$$

And

$$\hat{\rho} = \frac{\hat{\lambda}}{\hat{\mu}} = \frac{n + v - \hat{E}(n)}{T} \cdot \frac{\tau + (\gamma)_v'/(\gamma)_v}{m + \hat{E}(n)} \quad (14)$$

thus

$$f(\hat{\lambda}, \hat{\mu}, \hat{g}) = \hat{\rho} T m + [\hat{\rho} T + \tau + (\gamma)_v'/(\gamma)_v]\hat{E}(n) \quad (15)$$
$$- (n+v)[\tau + (\gamma)_v'/(\gamma)_v] = 0$$

where

$\hat{E}(n)$ is given in relation (10).

Which is an algebraic equation of degree $N+1$ in $\hat{\rho}$ .and it is impossible to solve mathematically except for some special cases, but in fact could be numerically oriented.

5. SPECIAL CASES

Some queueing models can be obtained as special cases of this system:

Case (1):

Let $\beta = 1$, this is the queue: *M/M/1/N* with reneging and reflecting barrier. In view (8), (9), (10) and (15):

The probability that there are *n* units in the system is:

$$p_n = \begin{cases} \dfrac{\delta^n}{(\gamma)_n} p_0, & 0 \le n \le N-1 \\ \dfrac{\delta^N}{r(\gamma)_N} p_0, & n = N \end{cases} \quad (16)$$

The delay probability is:

$$p_0^{-1} = {}_1F_1\begin{pmatrix} 1-N \\ \gamma \end{pmatrix};-\delta\end{pmatrix} + \frac{N\delta}{r(\gamma)_N} \quad (17)$$

The expected number of units in the system is:

$$E(n) = p_0 \left\{ (N-1)\delta\, {}_1F_1\begin{pmatrix} 2-N \\ \gamma \end{pmatrix};-\delta\end{pmatrix} + \frac{N\delta^N}{r(\gamma)_N} \right\} (18)$$

And

$$f(\hat{\lambda},\hat{\mu},\hat{g}) = \hat{\rho}Tm + [\hat{\rho}T + \tau + (\gamma)'_v/(\gamma)_v]\hat{E}(n) \\ - (n+v)[\tau + (\gamma)'_v/(\gamma)_v] = 0 \quad (19)$$

Case (2):

Let $g = 0$, this is the queue: *M/M/1/N* with balking and reflecting barrier. In view (8), (9), (10) and (15):

The probability that there are *n* units in the system is:

$$p_n = \begin{cases} \rho^n p_0, & 0 \le n \le k-1 \\ \dfrac{(\beta\rho)^n}{\beta^k} p_0, & k \le n \le N-1 \\ \dfrac{(\beta\rho)^N}{r\beta^k} p_0, & n = N \end{cases} \quad (20)$$

The delay probability is:

$$p_0^{-1} = \frac{1-\rho^k}{1-\rho} + \rho^k \frac{1-(\beta\rho)^{N-k}}{1-\beta\rho} + \frac{\beta^{N-k}\rho^N}{r} \quad (21)$$

The expected number of units in the system is:

$$E(n) = p_0 \left\{ \frac{-k\rho^k(1-\rho) + \rho - \rho^{k+1}}{(1-\rho)^2} + \frac{N\beta^{N-k}\rho^N}{r} \right. \\ \left. + \frac{k\rho^k - N\beta^{N-k}\rho^N(1-\beta\rho) + \beta\rho^{k+1} - \beta^{N-k+1}\rho^{N+1}}{(1-\beta\rho)^2} \right\} \quad (22)$$

And

$$f(\hat{\lambda},\hat{\mu},\hat{g}) = \hat{\rho}Tm + [\hat{\rho}T + \tau + (\gamma)'_v/(\gamma)_v]\hat{E}(n) \\ - (n+v)[\tau + (\gamma)'_v/(\gamma)_v] = 0 \quad (23)$$

Case (3):

Let $N \to \infty$, this is the queue: *M/M/1* with balking and reneging. In view (8), (9), (10) and (15):

The probability that there are *n* units in the system is:

$$p_n = \begin{cases} \dfrac{\delta^n}{(\gamma)_n} p_0, & 1 \le n \le k-1 \\ \dfrac{(\beta\delta)^n}{\beta^k(\gamma)_n} p_0, & n \ge k \end{cases} \quad (24)$$

The delay probability is:

$$p_0^{-1} = {}_1F_1\begin{pmatrix} 1-k \\ \gamma \end{pmatrix};-\delta\end{pmatrix} + \frac{\delta^k}{(\gamma)_k}{}_1F_1\begin{pmatrix} 1 \\ \gamma+k \end{pmatrix};\beta\delta\end{pmatrix} \quad (25)$$

The expected number of units in the system is:

$$E(n) = p_0 \left\{ (k-1)\delta \, {}_1F_1 \left(\begin{matrix} 2-k \\ \\ \gamma \end{matrix} ; -\delta \right) \right.$$

$$\left. + \frac{\delta^{k+1}}{(\gamma)_k} {}_1F_1 \left(\begin{matrix} 2 \\ \\ \gamma+k \end{matrix} ; \beta\delta \right) \right\} \quad (26)$$

And

$$f(\hat{\lambda}, \hat{\mu}, \hat{g}) = \hat{\rho}Tm + [\hat{\rho}T + \tau + (\gamma)'_v/(\gamma)_v]\hat{E}(n) \\ -(n+v)[\tau + (\gamma)'_v/(\gamma)_v] = 0 \quad (27)$$

Case (4):

Let $r = 1$, this is the queue: *M/M/1/N* with balking and reneging. In view (8), (9), (10) and (15):

The probability that there are n units in the system is:

$$p_n = \begin{cases} \dfrac{\delta^n}{(\gamma)_n} p_0, & 0 \le n \le k-1 \\ \\ \dfrac{(\beta\delta)^n}{\beta^k (\gamma)_N} p_0, & k \le n \le N \end{cases} \quad (28)$$

The delay probability is:

$$p_0^{-1} = {}_1F_1 \left(\begin{matrix} 1-k \\ \\ \gamma \end{matrix} ; -\delta \right) + \beta^{-k} {}_1F_1 \left(\begin{matrix} k-N \\ \\ \gamma \end{matrix} ; -\beta\delta \right) \quad (29)$$

The expected number of units in the system is:

$$E(n) = p_0 \left\{ (k-1)\delta \, {}_1F_1 \left(\begin{matrix} 2-k \\ \\ \gamma+1 \end{matrix} ; -\delta \right) \right.$$

$$\left. + \beta^{1-k} \delta(N-k) \, {}_1F_1 \left(\begin{matrix} 1+k-N \\ \\ \gamma+1 \end{matrix} ; -\beta\delta \right) \right\} \quad (30)$$

And

$$f(\hat{\lambda}, \hat{\mu}, \hat{g}) = \hat{\rho}Tm + [\hat{\rho}T + \tau + (\gamma)'_v/(\gamma)_v]\hat{E}(n) \\ -(n+v)[\tau + (\gamma)'_v/(\gamma)_v] = 0 \quad (31)$$

Case (5):

Let $r = 1, \beta = 1$ and $g = 0$, this is the queue:

M/M/1/N without any concepts.

Which is the same work as in Gross and Harris [4].

Case (6):

Let $k = 1$, $g = 0$ and $N \to \infty$, this is the queue:

M/M/1 with balking. which is the same work as in Kotb [6].

Case (7):

Let $k = 1, \beta = 1, g = 0$ and $N \to \infty$, this is the simple queue: *M/M/1* which is the same work as in Clarke [3].

6. SAMPLING DESIGN IN QUALITY CONTROL

In this section, we discuss the confidence intervals of *E(n)* and design in quality control for *M/M/1/N* system with balking, reneging and reflecting barrier. The $(1-\alpha)$ upper and lower limits of *E(n)* are given by:

$$E_U(n) = p_{0U} \left\{ (k-1)\delta_U \, {}_1F_1 \left(\begin{matrix} 2-k \\ \\ \gamma_U+1 \end{matrix} ; -\delta_U \right) + \beta^{-k} \delta_U(N-k-1) \right.$$

$$\left. \cdot {}_1F_1 \left(\begin{matrix} 2+k-N \\ \\ \gamma_U+1 \end{matrix} ; -\beta\delta_U \right) + \frac{N(\beta\delta_U)^N}{r\beta^k(\gamma_U)_N} \right\} \quad (32)$$

And

$$E_L(n) = p_{0L} \left\{ (k-1)\delta_L \, {}_1F_1 \left(\begin{matrix} 2-k \\ \\ \gamma_L+1 \end{matrix} ; -\delta_L \right) + \beta^{-k} \delta_L(N-k-1) \right.$$

$$\left. \cdot {}_1F_1 \left(\begin{matrix} 2+k-N \\ \\ \gamma_L+1 \end{matrix} ; -\beta\delta_L \right) + \frac{N(\beta\delta_L)^N}{r\beta^k(\gamma_L)_N} \right\} \quad (33)$$

Let

$$g = 10, \; k = 3, \; \rho_{est} = 0.70, \; r = 0.50, \; \lambda = 10, \; N =$$

And

$\beta = 0.25, \; 0.50, \; 0.75 \; and \; 1.00$, in relations

(32) and (33) then we get **TABLE 6.1** for each

values of n:

The results of $E(n)_U$, $E(n)_L$ and $(E(n)_U - E(n)_L)$

are given in **TABLE 6.1** for each values of n:

      Solution of the system may be

determined more readily by plotting

$[E(n)_U, \; E(n)_L]$ and $[(E(n)_U - E(n)_L)]$ again

st n for each values of β as given in

FIGUERE 6.1 and FIGUERE 6. 2

respectively.

FIGUERE 6.1
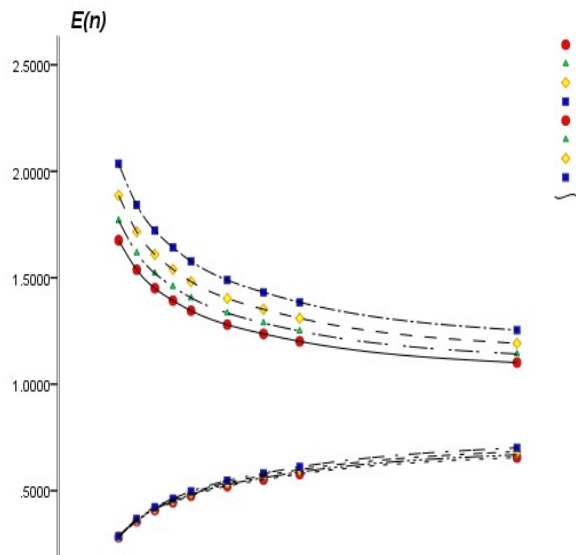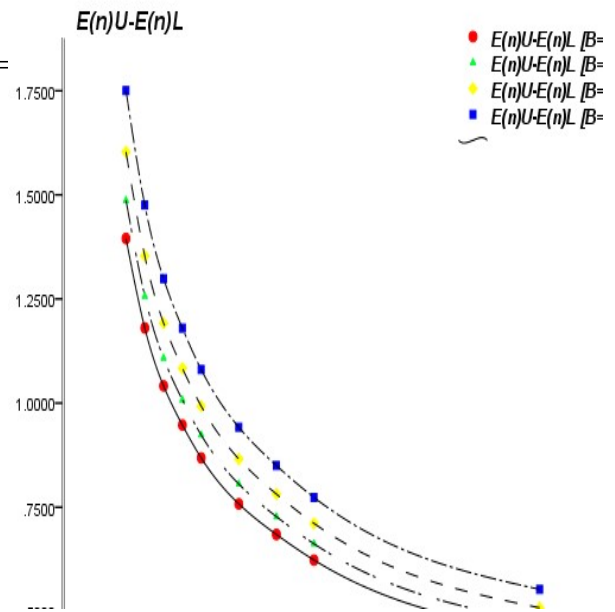
FIGUERE 6.2

7. QUALITY CONTROL FOR ALL DIFFERENT β VALUES:

      In order to study quality control for the system, applying control limit (CL) approach when σ is unknown. The control charts for upper control limit $UCL = E_U(n)$ and lower control limit $LCL = E_L(n)$.

      The average of the subgroup ranges (CL) for all values of β is:

$$\overline{E}(n) = \frac{1}{9} \sum_{j=1}^{9} E_j$$
$$= 0.8827, \; 0.9391, \; 1.0103 \; and \; 1.1004$$

For

$\beta = 0.25, 0.50, \; 0.75 \; and \; 1.00$ respectively

Where

$$E_j = (E_U - E_L)_j$$

      The standard deviation of the subgroup ranges is:

$$S_E = \sqrt{\frac{1}{9}\sum_{j=1}^{9}(E_j - \overline{E})^2}$$

$$= 0.278 \,,\; 0.296 \,,\; 0.32 \;\; and \;\; 0.35$$

For

$\beta = 0.25, 0.50, 0.75 \; and \; 1.00$ respectively

The control charts for upper control limit $UCL = E_U(n) = \overline{E} + 3S_E$, lower control limit $LCL = E_L(n) = \overline{E} - 3S_E$ and the averages of the subgroup ranges $\overline{E}$ are shown in **FIGUERE 7.1, FIGUERE 7.2, FIGUERE 7.3 and FIGUERE 7.4**
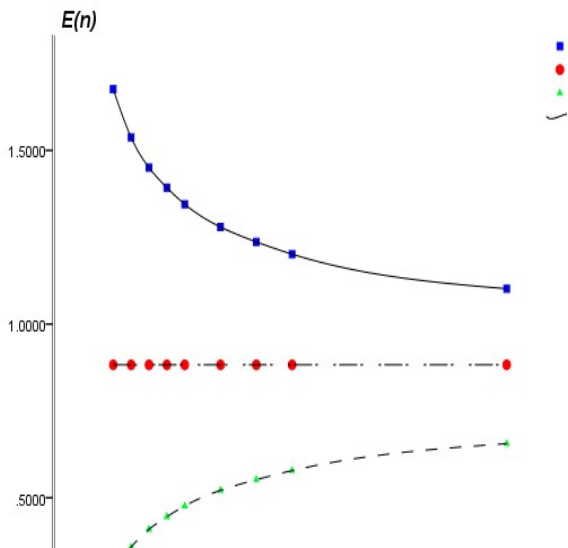
respectively for all values of β.

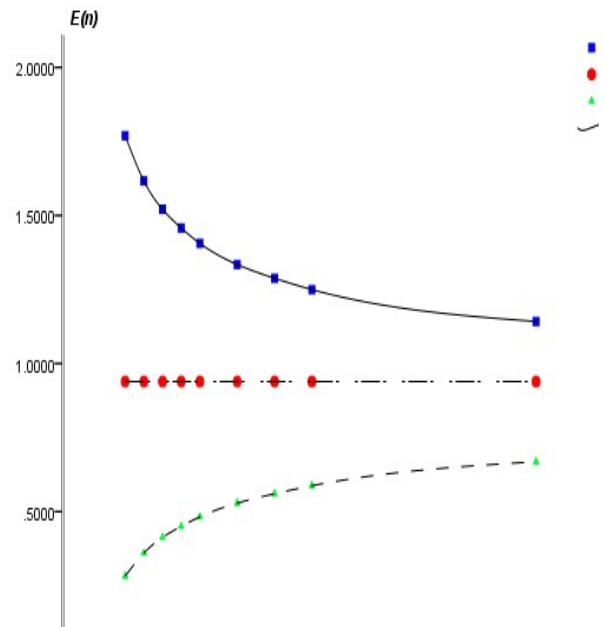**FIGUERE 7.2** ($\beta = 0.50$)

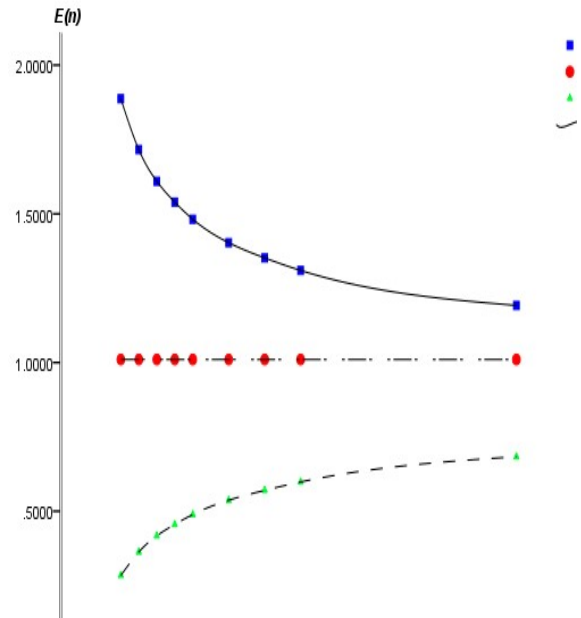**FIGUERE 7.1** ($\beta = 0.25$)

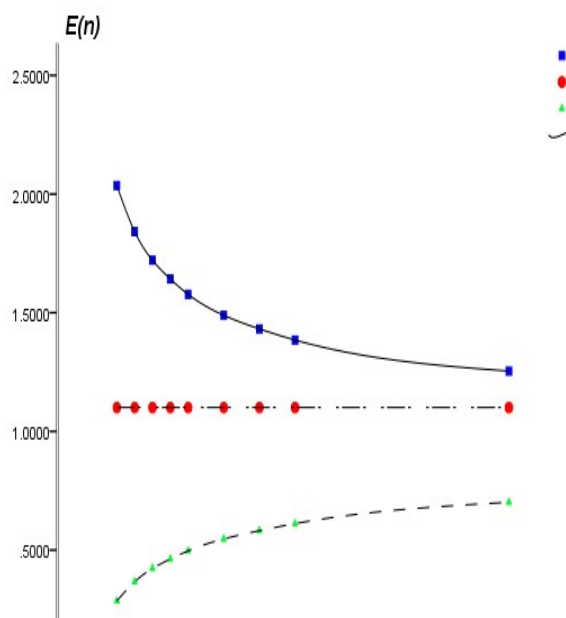
FIGUERE 7.3 ($\beta = 0.75$)

FIGUERE 7.4 ($\beta = 1.00$)

It is clear that all of values in the mean charts fall between the control limits, therefore the process is in control.

With the same manner, we can discuss the confidence intervals of *E(n)* and design in quality control for other concepts at *M/M/1/N* system.

8. CONCLUSION

this paper is has discussed statistical inference in quality control procedure truncated single-server Markovian queue with balking, reneging and reflecting barrier under steady-state simulation. We treated both point estimation using modified likelihood function and confidence intervals for estimating the mean, difference of means and control charts. Some important queueing systems are derived as special cases.

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