

Research Article

MATHEMATICS

Picard and Homotopy Perturbation Methods for Solving the Time-Fractional Schrödinger Equations

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ABSTRACT

The time-dependent Schrödinger wave equation is the basic partial differential equation of quantum field theory. The study of this equation and its applications play an exceptionally important function in modern physics. From a mathematical point of view, the time-dependent Schrödinger equation is a commutable as mathematics itself. The newest analytical methods to solve linear and nonlinear differential equation is the Homotopy Perturbation Method (HPM) developed to the time-fraction Schrödinger wave equation, which is a combination of homotopy transformation and perturbation. Furthermore, Picard Method (PM) is applied to formulate an approximate iterative solution of the time-fraction Schrödinger equation.

Introduction

Let's consider the following linear time-fractional Schrödinger equation:

$$D_t^\alpha u + i u_{xx} = 0, \quad u(x, 0) = f(x), \quad (1)$$

where $i^2 = -1$, $0 < \alpha \leq 1$, and the non-linear time-fractional Schrödinger equation

$$i D_t^\alpha u + \beta u_{xx} + \nu(x)u + \gamma |u|^2 u = 0,$$

$$u(x, 0) = f(x), \quad i^2 = -1,$$

$$|u^2| = u\bar{u}, \quad 0 < \alpha \leq 1,$$

where $u(x, t)$ is a complex function, β, γ are constants, and $\nu(x)$ is a function in term of x .

The linear and non-linear Schrödinger equations are produced in study of the time evolution of the wave function [1, 6, 7, 8], in many of science and engineering sciences and quantum mechanics [2, 17, 24].

A large amount of this research was done in previous studies of the linear and nonlinear (time-fractional) Schrödinger equation [2, 6, 7, 11].

PM and HPM play an important large role in solving the fractional order differential equations as well as the ordinary order differential equations [9, 13, 15, 16, 18, 22, 23].

Scientists have noted that these are a large number of real-time problems formulated by fractional linear and nonlinear differential equations that are very difficult to process [10, 12, 14, 20]. New iterative method has been presented to solve this equation [3, 4, 5, 19].

In this chapter, PM and HPM will be used to approach the linear and nonlinear time-fraction Schrödinger equations, it is well known that these methods can be applied directly to both linear and nonlinear time-fractional Schrödinger equations [2, 6, 21].

We will circulate the general formula on the linear time-fractional Schrödinger equation, and apply some examples.

The analytical solutions along with the graphs reveal the reliability and full efficiency of the proposed algorithms.

Definitions

Definition A real function $g(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p (> \mu)$, such that $g(x) = x^p g_1(x)$, where $g_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^∞ if and only if $g^m \in C_\mu$, $\mu \geq 1$, $m \in \mathbb{N}$.

Definition The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $g \in C_\mu$, $\mu \geq -1$, is defined as:

$$I^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t) dt, \alpha > 0, x > 0 \quad (2)$$

$$I^0 g(x) = g(x).$$

Properties of the operator I^α can be found in [2], we mention only the following for $g \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

$$1- I^\alpha I^\beta g(t) = I^{\alpha+\beta} g(t),$$

$$2- I^\alpha I^\beta g(t) = I^\beta I^\alpha g(t),$$

$$3- I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 1.1.3. The fractional derivative of $g(x)$ in the Caputo sense is defined as

$$D_t^\alpha g(x) = I^{m-\alpha} D^m g(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} g^m(t) dt, \quad (3)$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1.1. If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $g \in C_\mu^m$, $\mu \geq -1$, then

$$D^\alpha I^\alpha g(x) = g(x),$$

and

$$J^\alpha D^\alpha g(x) = g(x) - \sum_{k=0}^{m-1} \frac{g^{(k)}(0^+) x^k}{k!}, \quad x > 0. \quad (4)$$

Picard Method (PM).

To illustrate the basic idea of this method, we consider the following general fractional differential equation of arbitrary order $\alpha > 0$

$$D_t^\alpha(t) = F(t, u^{(k)}(t)), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \quad (5)$$

$$\frac{d^k}{dt^k} u(0) = h_k, \quad k = 0, 1, 2, \dots, m-1, \quad (6)$$

where D_t^α is the fractional differential operator of order $\alpha > 0$. In view of the fractional integral

operators, the initial value fractional problem (5) and (6) is equivalent to the fractional integral equation:

$$u(t) = \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^\alpha [F(t, u_t^{(k)}(t))] = f + N(u), \tag{7}$$

where $\sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!}$, $N(u) = I_t^\alpha [F(t, u_t^{(k)}(t))]$,

and I_t^α is the inverse of D_t^α . The required solution $u(t)$ for (7) which is also the solution for (5) and (6) can be obtained as the limit of sequence of function $u_{r+1}(t)$ generated by the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_{r+1} = u_0 + N(u_r), \end{cases} \quad r = 0, 1, 2, \dots, \tag{8}$$

Homotopy Perturbation Method (HMP).

To illustrate the basic concept of this method, we consider the following differential equation:

$$A(u) - g(r) = 0, \quad r \in \Omega, \tag{9}$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{10}$$

where A is a general differential operator, B is boundary operator, $g(r)$ is a known analytical function and Γ is the boundary of the domain Ω . The operator A can be generally divided into two operator L and N , where L is linear and N is a nonlinear. Then equation (9) can be written as follows:

$$L(u) + N(u) - g(r) = 0, \quad r \in \Omega. \tag{11}$$

Using the Homotopy perturbation technique, we can construct a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(v, p) = (1-p)[(L(v) - L(u_0)) + p[L(v) + N(v) - g(r)] = 0, \tag{12}$$

or

$$H(v, p) = L(v) - L(u_0) + p[L(u_0) + p[N(v) - g(r)]], \tag{13}$$

where $r \in \Omega$, $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation that satisfies the boundary conditions. Now the solution of (12) and (13) can be written as a power series in p , as follows:

$$v = v_0 + pv_1 + p^2v_2 + \dots, \tag{14}$$

and the best approximation solution is:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \tag{15}$$

Now for the general fractional differential equation:

$$D_t^\alpha(x, t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + g(x, t), \quad t > 0, \tag{16}$$

where L is a liner operator, N is a nonlinear operator, $g(x, t)$ is a known analytic function and D_t^α , is the Caputo fractional derivative of order α , where $m - 1 < \alpha \leq m$, subject to the initial conditions:

$$u^{(k)}(x, 0) = h_k(x), \quad k = 0, 1, 2, 3, \dots, m - 1. \tag{17}$$

In view of the homotopy technique, we can construct the following homotopy:

$$D_t^\alpha u - L(u, u_x, u_{xx}) - g(x, t) = p[N(u, u_x, u_{xx})], \tag{18}$$

or

$$D_t^\alpha u - g(x, t) = p[(u, u_x, u_{xx}) + N(u, u_x, u_{xx})]. \tag{19}$$

The basic assumption is that the solution of (19) can be written as a power series in p :

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{20}$$

Finally, the n -term approximate solution is

$$u(x) = \sum_{i=0}^{n-1} u_i. \quad (21)$$

Application

Linear Time-Fractional Schrödinger Equation.

Let's consider the linear time-fractional Schrödinger equation of the following form

$$D_t^\alpha u \pm i u_{xx} = 0, \quad u(x, 0) = f(x), \quad 0 < \alpha \leq 1 \quad (22)$$

To solve the linear time-fractional Schrödinger equation (22) by using:

1- PM: We proceed as follows:

$$u_{j+1} = u_0 \mp I_t^\alpha i [(u_j)_{xx}], \quad j \geq 0, \quad 0 < \alpha \leq 1, \quad (23)$$

$$u_0 = f(x),$$

$$u_1 = f(x) \mp I_t^\alpha i \left[\frac{\partial^2 f}{\partial x^2} \right] = f(x) \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = f(x) \mp I_t^\alpha i \left[f(x) \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} \right]_{xx} \\ = f(x) \pm i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + i^2 \frac{\partial^4 f}{\partial x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3 = f(x) \mp I_t^\alpha i \left[f(x) \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} \right. \\ \left. + i^2 \frac{\partial^4 f}{\partial x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] \\ = f(x) \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + i^2 \frac{\partial^4 f}{\partial x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \mp i^3 \frac{\partial^6 f}{\partial x^6} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (24)$$

For higher order terms, we can obtain:

$$u(x, t) = f(x) + \sum_{n=1}^{\infty} (\mp 1)^n (i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \frac{\partial^{2n} f}{\partial x^{2n}}. \quad (25)$$

For the special case $\alpha = 1$, we obtain from equation (25)

$$u(x, t) = f(x) + \sum_{n=1}^{\infty} (\mp 1)^n \frac{(it)^n}{n!} \frac{\partial^{2n} f(x)}{\partial x^{2n}}. \quad (26)$$

2-HPM: we suppose that the solution of the problem can be written as a power series in p :

$$u = u_0 + p u_1 + p^2 u_2 + \dots, \quad (27)$$

and let us construct the homotopy:

$$D_t^\alpha = \mp p [i u_{xx}], \quad (28)$$

Substitute (27) in (28), the homotopy for (22) becomes:

$$D_t^\alpha [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] = \mp p i [u_0 + p u_1 + p^2 u_2 \\ + p^3 u_3 + \dots]_{xx}, \quad (29)$$

Equating the coefficients of equal power of p in both sides of (29), we obtain:

$$\left. \begin{aligned} p^0 : D_t^\alpha u_0 = 0, \quad u_0(x, 0) = f(x), \\ p^1 : D_t^\alpha u_1 = \mp i u_{0xx}, \quad u_1(x, 0) = 0, \\ p^2 : D_t^\alpha u_2 = \mp i u_{1xx}, \quad u_2(x, 0) = 0, \\ p^3 : D_t^\alpha u_3 = \mp i u_{2xx}, \quad u_3(x, 0) = 0, \\ \vdots \end{aligned} \right\} \quad (30)$$

Solving the above set of equations (30), we get the first terms of the homotopy perturbation solution for (22):

$$\left. \begin{aligned} u_0 = f(x), \\ u_1 = \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 = +i^2 \frac{\partial^4 f}{\partial x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 = \mp i^3 \frac{\partial^6 f}{\partial x^6} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ \vdots \end{aligned} \right\} \quad (31)$$

So the solution will be

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \\ = f(x) \mp i \frac{\partial^2 f}{\partial x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + i^2 \frac{\partial^4 f}{\partial x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \mp i^3 \frac{\partial^6 f}{\partial x^6} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \quad (32)$$

For higher terms we get

$$u(x, t) = f(x) + \sum_{n=1}^{\infty} (\mp 1)^n (i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \frac{\partial^{2n} f(x)}{\partial x^{2n}}. \tag{33}$$

For special case $\alpha = 1$, we obtain from equation (33)

$$u(x, t) = f(x) + \sum_{n=1}^{\infty} (\mp 1)^n \frac{(it)^n}{n!} \frac{\partial^{2n} f(x)}{\partial x^{2n}}.$$

Example 3.1. Consider the linear time-fractional Schrödinger equation

$$D_t^\alpha u_x + i u_{xx} = 0, \quad 0 < \alpha \leq 1, \tag{34}$$

with the initial condition $u(x, 0) = k e^{imx}$.

1- By PM:

We construct the following iteration formula for Picard:

$$u_{j+1} = u_0 - I_t^\alpha i [(u_j)_{xx}], \quad j \geq 0. \tag{35}$$

Therefore, we can obtain the following first few terms of the required solution.

$$u_0 = k e^{imx},$$

$$u_1 = k e^{imx} + i k m^2 \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2 = k e^{imx} + i k m^2 e^{imx} \frac{t^\alpha}{\Gamma(\alpha + 1)} + i^2 k m^4 e^{imx} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$u_3 = k e^{imx} + i k m^2 e^{imx} \frac{t^\alpha}{\Gamma(\alpha + 1)} + i^2 k m^4 e^{imx} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + i^3 k m^6 e^{imx} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)},$$

then

$$u = k e^{imx} \left(1 + i m^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + i^2 m^4 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + i^3 m^6 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \right).$$

In closed form, we obtain the solution

$$u = k e^{imx} \left(\sum_{n=0}^{\infty} (i m^2)^n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \right). \tag{36}$$

For the special case $\alpha = 1$, we obtain that:

$$u = k e^{imx} \left(1 + i m^2 t + \frac{(i m^2 t)^2}{2!} + \frac{(i m^2 t)^3}{3!} + \dots \right) = k e^{im(x+mt)},$$

which is an exact solution for (34).

The same result can be obtained by applying the general formula (25) for solving (4.34) in the form:

$$u = k e^{imx} + i k m^2 e^{imx} \frac{t^\alpha}{\Gamma(\alpha + 1)} + i^2 k m^4 e^{imx} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + i^3 k m^6 e^{imx} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots, \tag{37}$$

then

$$u = k e^{imx} \left(1 + i m^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + i^2 m^4 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + i^3 m^6 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \right).$$

In closed form, we obtain:

$$u = k e^{imx} \left(\sum_{n=0}^{\infty} (i m^2)^n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \right).$$

For the special case $\alpha = 1$, we obtain that

$$u = k e^{imx} \left(1 + i m^2 t + \frac{(i m^2 t)^2}{2!} + \frac{(i m^2 t)^3}{3!} + \dots \right) = k e^{im(x+mt)}, \tag{38}$$

which is the exact solution for (34).

2- By HPM: The homotopy for (34) becomes

$$D_t^\alpha u = -i p [u_{xx}], \tag{39}$$

we suppose that the solution of the problem (34) is in the form:

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots \tag{40}$$

Substituting (40) into (39), we obtain:

$$D_t^\alpha [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] = -i p [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]_{x x} \quad (41)$$

Equating the terms of equal powers of p in both sides of the above equation, we obtain:

$$\left. \begin{aligned} p^0 : D_t^\alpha u_0 &= 0, & u_0(x, 0) &= k e^{i m x}, \\ p^1 : D_t^\alpha u_1 &= -i u_{0_{xx}}, & u_1(x, 0) &= 0, \\ p^2 : D_t^\alpha u_2 &= -i u_{1_{xx}}, & u_2(x, 0) &= 0, \\ p^3 : D_t^\alpha u_3 &= -i u_{2_{xx}}, & u_3(x, 0) &= 0, \\ & \vdots & & \end{aligned} \right\} (42)$$

Solving the above set equations in (42), we obtain:

$$\left. \begin{aligned} u_0 &= k e^{i m x}, \\ u_1 &= i k m^2 e^{i m x} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= i^2 k m^4 e^{i m x} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= i^3 k m^6 e^{i m x} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ & \vdots \end{aligned} \right\} (43)$$

So the solution will be

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= k e^{i m x} + i k m^2 e^{i m x} \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + i^2 k m^4 e^{i m x} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + i^3 k m^6 e^{i m x} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots, \end{aligned} \quad (44)$$

then

$$u = \sum_{i=0}^3 u_i = k e^{i m x} \left(1 + i m^2 \frac{t^\alpha}{\Gamma(\alpha+1)} + i^2 m^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + i^3 m^6 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right).$$

In closed form, we obtain the solution

$$u = k e^{i m x} \left(\sum_{n=0}^{\infty} (i m^2)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \quad (45)$$

For the special cases $\alpha = 1$, we obtain

$$u(x, t) = k e^{i m x} + i k m^2 e^{i m x} t + i^2 k m^4 e^{i m x} \frac{t^2}{2!} + i^3 k m^6 e^{i m x} \frac{t^3}{3!} + \dots = k e^{i m(x + m t)}, \quad (46)$$

which is the same exact solution as obtained by PM.

It is clear that the result obtained by HPM in (45) is the same result as obtained by PM (36).

In Fig. (3), we have Plotted the approximate solution for the linear Schrödinger equation (34) for the closed a) $\alpha = 0.25$, b) $\alpha = 0.5$, c) $\alpha = 0.75$, d) $\alpha = 1$, and e) the corresponding exact solution at $m = k = 1$.

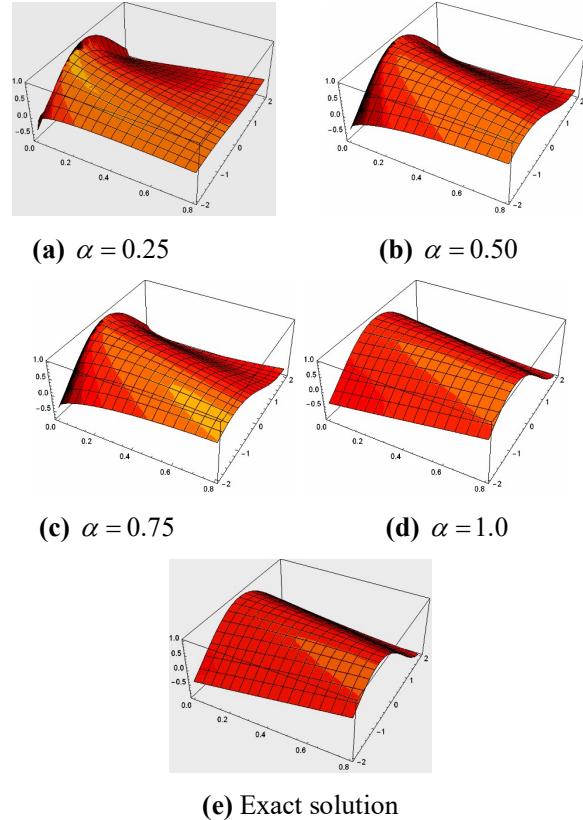


Fig. 3. The surfaces show the approximate solution for (34) for various values of α with the corresponding exact solution at $m = k = 1$.

Example 3.2. Consider the linear time-fractional Schrödinger equation.

$$D_t^\alpha u + i u_{xx} = 0, \tag{47}$$

with the initial condition

$$u(x, 0) = 1 + \cosh 2x.$$

1- By PM: using formula (25) lead to the following result:

$$\begin{aligned} u(x, t) &= 1 + \cosh 2x - 4i \cosh 2x \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + 16i^2 \cosh 2x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - 64i^3 \cosh 2x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &= 1 + \cosh 2x \left(1 - 4i \frac{t^\alpha}{\Gamma(\alpha+1)} + 16i^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\ &\quad \left. - 64i^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right), \end{aligned} \tag{48}$$

then

$$u(x, t) = 1 + \cosh 2x \left(\sum_{n=0}^{\infty} (-4i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \tag{49}$$

For the special case $\alpha = 1$, we obtain

$$\begin{aligned} u(x, t) &= 1 + \cosh 2x - 4i \cosh 2x t + 16i^2 \cosh 2x \frac{t^2}{2!} \\ &\quad - 64i^3 \cosh 2x \frac{t^3}{3!} + \dots = 1 + e^{-4it} \cosh 2x, \end{aligned}$$

which is the exact solution for (47).

2- By HPM: by using formula (33), we obtain that:

$$\left. \begin{aligned} u_0 &= 1 + \cos 2x, \\ u_1 &= -4i \cosh 2x \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= 16i^2 \cosh 2x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= -64i^3 \cosh 2x \frac{t^{3\alpha}}{\Gamma(2\alpha+1)}, \\ &\vdots \end{aligned} \right\} \tag{50}$$

So the solution will be:

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= 1 + \cosh 2x - 4i \cosh 2x \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + 16i^2 \cosh 2x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - 64i^3 \cosh 2x \frac{t^{3\alpha}}{\Gamma(2\alpha+1)} + \dots \\ &= 1 + \cosh 2x \left(1 - 4i \frac{t^\alpha}{\Gamma(\alpha+1)} + 16i^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\ &\quad \left. - 64i^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right) \end{aligned} \tag{51}$$

then

$$u(x, t) = 1 + \cosh 2x \left(\sum_{n=0}^{\infty} (-4i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right) \tag{52}$$

For the special case $\alpha = 1$, we obtain that

$$\begin{aligned} u(x, t) &= 1 + \cosh 2x - 4i \cosh 2x t + 16i^2 \cosh 2x \frac{t^2}{2!} \\ &\quad - 64i^3 \cosh 2x \frac{t^3}{3!} + \dots \\ &= 1 + e^{-4it} \cosh 2x. \end{aligned}$$

which is an exact solution for (47).

It is clear that the result obtained by HPM in (52) is the same result as obtained by PM (49).

In Fig. (2), we have Plotted the approximate solution for the linear Schrödinger equation (47) for the case a) $\alpha = 0.25$, b) $\alpha = 0.5$, c) $\alpha = 0.75$, d) $\alpha = 1$, and e) the corresponding exact solution.

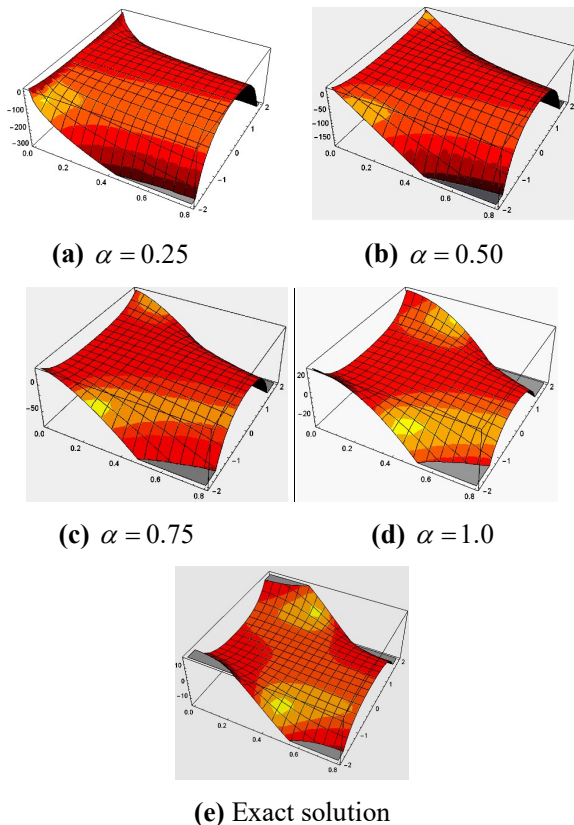


Fig. 3. The approximate solution for (47) for various values at α with corresponding exact solution.

Non Linear Time-Fractional Schrödinger Equation.

Consider the following nonlinear Schrödinger equation in the following form

$$i D_t^\alpha u + \beta u_{xx} + v(x)u + \gamma |u|^2 u = 0, \quad 0 < \alpha \leq 1, \quad (53)$$

which the initial conditions:

$$u(x, 0) = f(x).$$

We can rewrite the equation (53) in the form

$$D_t^\alpha u - \beta i u_{xx} - i v(x)u - \gamma i u^2 \bar{u} = 0 \quad \text{where } \bar{u}$$

is the complex conjugate of u , $|u|^2 = u\bar{u}$ and β, γ are constants, $u(x, t)$ is a complex function,

$v(x)$ is a function with x .

1- By PM: we proceed as follows

$$u_{j+1} = u_0 + I_t^\alpha [i \beta u_{jxx} + i v(x)u_j + i \gamma u_j^2 \bar{u}_j], \quad j \geq 0, \quad 0 < \alpha \leq 1, \quad (54)$$

$$u_0 = f(x),$$

$$u_1 = f(x) + [i \gamma \bar{u}_0 u_0^2 + i u_0 v(x) + i \beta u_{0xx}] \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = f(x) + [i \gamma \bar{u}_0 u_0^2 + i u_0 v(x) + i \beta u_{0xx}] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$+ [-2\gamma^2 u_0^2 u_0^3 - 3\gamma \bar{u}_0 u_0^2 v(x) - u_0 v^2(x)$$

$$- 2\beta \gamma \bar{u}_0 u_0^2 - 4\beta \gamma u_0 \bar{u}_0 u_{0x}^2 - \beta \gamma u_0^2 u_{0x}^2 \bar{u}_{0xx}$$

$$- 4\beta \gamma \bar{u}_0 u_0 u_{0xx} - 2\beta v(x) u_{0xx} - \beta \gamma u_0^2 \bar{u}_{0x} u_{0xx}$$

$$- \beta^2 u_{0xxxx} - 2\beta u_{0x} v'(x) - \beta u_0 v''(x)]$$

$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots,$$

$$u_3 = f(x) + [i \gamma \bar{u}_0^2 u_0 + i u_0 v(x) + i \beta u_{0xx}] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$+ [-2\gamma^2 u_0^2 u_0^3 - 3\gamma u_0^2 \bar{u}_0 v(x) - u_0 v^2(x)$$

$$- 2\beta \gamma \bar{u}_0 u_0^2 - 4\beta \gamma u_0 \bar{u}_0 u_{0x}^2 - \beta \gamma u_0^2 u_{0x}^2 \bar{u}_{0xx}$$

$$- 4\beta \gamma \bar{u}_0 u_0 u_{0xx} - 2\beta v(x) u_{0xx} - \beta \gamma u_0^2 \bar{u}_{0x} u_{0xx}$$

$$- \beta^2 u_{0xxxx} - 2\beta u_{0x} v'(x) - \beta u_0 v''(x)] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ [i \gamma^3 \bar{u}_0^3 u_0^4 - 2i \gamma^2 \bar{u}_0^2 u_0^3 v(x) - i \gamma \bar{u}_0 u_0^2 v^2(x)$$

$$- 2i \beta \gamma^2 \bar{u}_0^2 u_0^2 u_{0xx} - 2i \beta \gamma \bar{u}_0 u_0 v(x) u_{0xx}$$

$$- i \beta^2 \gamma \bar{u}_0 u_{0xx}^2] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots$$

2- By HPM: we suppose that the solution of the problem can be written as a power series in p

$$D_t^\alpha u = i p [\beta u_{xx} + v(x)u + \gamma u^2 \bar{u}], \quad (55)$$

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots \quad (56)$$

Substituting (56) in (53), the homotopy for (53) becomes

$$D_t^\alpha [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] = i p [\beta [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]_{xx} + v(x) [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] + \gamma [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]^2 [\bar{u}_0 + p \bar{u}_1 + p^2 \bar{u}_2 + p^3 \bar{u}_3]] \quad (57)$$

Equating the coefficient of equal power of p in both sides of equation (57), we obtain

$$\left. \begin{aligned} p^0 : D_t^\alpha u_0 &= 0, & u_0(x, 0) &= f(x), \\ p^1 : D_t^\alpha u_1 &= \beta i u_{0,xx} + i v(x) u_{0,x} + \gamma i u_0^2 \bar{u}_0, & u_1(x, 0) &= 0, \\ p^2 : D_t^\alpha u_2 &= \beta i u_{1,xx} + i v(x) u_{1,x} + \gamma i u_0^2 \bar{u}_1 \\ &+ 2\gamma i u_0 u_1 \bar{u}_0, & u_2(x, 0) &= 0, \\ p^3 : D_t^\alpha u_3 &= \beta i u_{2,xx} + i v(x) u_{2,x} + \gamma i u_0^2 \bar{u}_2 \\ &+ \gamma i \bar{u}_0 (u_1^2 + 2u_0 u_2) + 2\gamma i u_0 u_1 \bar{u}_1, & u_3(x, 0) &= 0, \\ &\vdots & & \end{aligned} \right\} \quad (58)$$

Solving the above set of equation in (58), we get the first terms of the homotopy perturbation

$$\begin{aligned} u_0 &= f(x), \\ u_1 &= [\beta i u_{0,xx} + i v(x) u_{0,x} + \gamma i u_0^2 \bar{u}_0] \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= [\beta i u_{1,xx} + i v(x) u_{1,x} + \gamma i u_0^2 \bar{u}_1 + 2\gamma i u_0 u_1 \bar{u}_0] \\ &\quad \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= [\beta i u_{2,xx} + i v(x) u_{2,x} + \gamma i u_0^2 \bar{u}_2 + \gamma i \bar{u}_0 (u_1^2 \\ &\quad + 2u_0 u_2) + 2\gamma i u_0 u_1 \bar{u}_1] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\vdots \end{aligned}$$

So the solution will be

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$$

Example 3. Consider the following cubic nonlinear time-fractional Schrödinger equation

$$i D_t^\alpha u + \beta u_{xx} - 2|u|^2 u = 0, \quad (59)$$

with the initial condition

$$u(x, 0) = e^{ix}.$$

The exact solution is $u(x, t) = e^{i(x-3t)}$. We have $\beta = 1, v(x) = 0, \gamma = -2$, we can rewrite the equation (59) in the form

$$D_t^\alpha - i u_{xx} + 2i u^2 \bar{u} = 0. \quad (60)$$

1- By PM: we construct the following iteration formula for Picard:

$$u_{j+1} = u_0 + I_t^\alpha [i u_{j,xx} - 2i u_j^2 \bar{u}_j], \quad j \geq 0, \quad (61)$$

therefore, we can obtain the following first few terms of the required solution

$$\begin{aligned} u_0 &= e^{ix}, \\ u_1 &= e^{ix} - 3i e^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= e^{ix} - 3i e^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)} + 9i^2 e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= e^{ix} - 3i e^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)} + 3^2 e^{ix} i^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - 3^3 i^3 e^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\vdots \end{aligned}$$

then

$$u = e^{ix} \left(1 - 3i \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(3i)^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{(3i)^3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right). \quad (62)$$

In closed form, we obtain the solution

$$u = e^{ix} \left(\sum_{n=0}^{\infty} (-3i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \quad (63)$$

For the special case $\alpha = 1$, we obtain

$$u(x, t) = e^{ix} \left(1 - 3it + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \dots \right),$$

which is an exact solution for (59).

2- By HPM: the homotopy (59) becomes

$$D_t^\alpha u = i p [u_{xx} - 2u^2 \bar{u}], \quad (64)$$

we suppose that the solution of the problem (64) is in the form

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots \quad (65)$$

Substituting (65) into (64), we obtain:

$$\begin{aligned} D_t^\alpha [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] &= i p [[u_0 + p u_1 + p^2 u_2 \\ &\quad + p^3 u_3 + \dots]_{xx} - 2[u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]^2 \\ &\quad [\bar{u}_0 + p \bar{u}_1 + p^2 \bar{u}_2 + p^3 \bar{u}_3 + \dots]] \end{aligned} \quad (66)$$

Equating the terms of equal powers p in both sides of the above equation we obtain:

$$\left. \begin{aligned} p^0 : D_t^\alpha u_0 &= 0, & u_0(x, 0) &= e^{ix}, \\ p^1 : D_t^\alpha u_1 &= i u_{0_{xx}} - 2i u_0^2 \bar{u}_0, & u_1(x, 0) &= 0, \\ p^2 : D_t^\alpha u_2 &= i u_{1_{xx}} - 2i u_0^2 \bar{u}_1 - 4i u_0 u_1 \bar{u}_0, & u_2(x, 0) &= 0, \\ p^3 : D_t^\alpha u_3 &= i u_{2_{xx}} - 2i u_0^2 \bar{u}_2 - 2i \bar{u}_0 (u_1^2 + 2u_0 u_2) \\ &\quad - 4i u_0 u_1 \bar{u}_1, & u_3(x, 0) &= 0, \\ &\vdots & & \end{aligned} \right\} \quad (67)$$

Solving the above set of equation in (67), we obtain

$$\left. \begin{aligned} u_0 &= e^{ix}, \\ u_1 &= -3i e^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= 9i^2 e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= -63i^3 e^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 18i^3 e^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}. \end{aligned} \right\} \quad (68)$$

So the solution will be

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= e^{ix} - 3i e^{ix} \frac{t^\alpha}{\Gamma(\alpha+1)} + 3^2 i^2 e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - 63i^3 e^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 18i^3 e^{ix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + \dots \end{aligned} \quad (69)$$

then

$$\begin{aligned} u &= \sum_{i=0}^3 u_i = e^{ix} \left(1 - 3i \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (3i^2) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\ &\quad \left. - (3i)^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right) \end{aligned}$$

In closed form, we obtain the solution:

$$u = e^{ix} \left(\sum_{n=0}^{\infty} (-3i)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \quad (70)$$

The special case $\alpha = 1$, we obtain that

$$\begin{aligned} u(x, t) &= e^{ix} - 3i e^{ix} t + (3i)^2 e^{ix} \frac{t^2}{2!} \\ &\quad - (3i)^3 e^{ix} \frac{t^3}{3!} + \dots = e^{i(x-3t)}. \end{aligned}$$

Which is the same exact solution as obtained by PM.

It is clear that the result obtained by HPM in (70) is the same result as obtained by PM in (63).

In Fig. (3), we have plotted the approximate solution for nonlinear Schrödinger equation (55) for the cases a) $\alpha = 0.25$, b) $\alpha = 0.5$, c) $\alpha = 0.75$, d) $\alpha = 1$, and e) the corresponding exact solution.

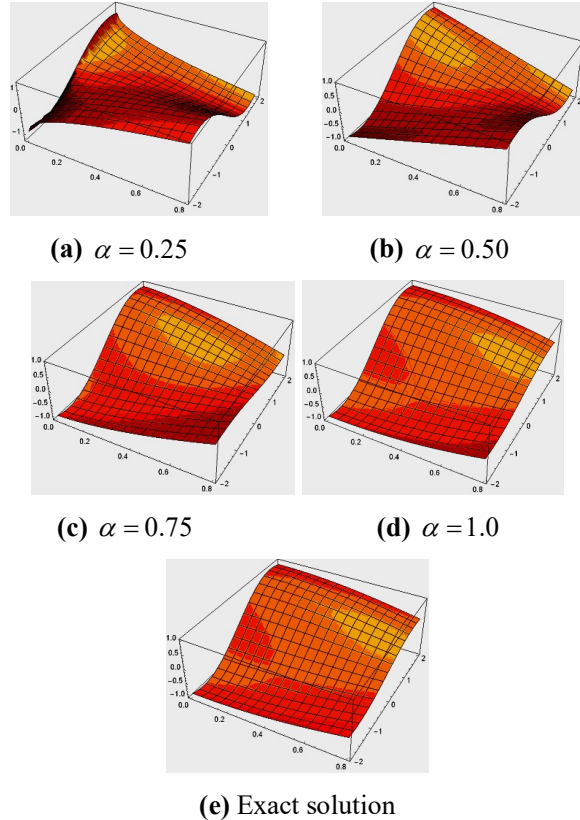


Fig. 3. The surface shows the approximate solution for (59) for various values at α with the corresponding exact solution.

Example 4. We consider the nonlinear time-fractional Schrödinger equation

$$i D_t^\alpha = -\frac{1}{2} u_{xx} + u \cos^2 x + |u|^2 u, \quad (71)$$

with the initial condition

$$u(x, 0) = \sin x .$$

The exact solution is

$$u(x, t) = e^{-\frac{3it}{2} \sin x} .$$

We rewrite the equation (71) in the form

$$D_t^\alpha u = \frac{1}{2} i u_{xx} - i u \cos^2 x - i u^2 \bar{u} . \quad (72)$$

In the equation we have $\beta = \frac{1}{2}$, $\gamma = -1$, and

$$v(x) = -\cos^2 x .$$

1- By PM: we construct the following iteration formula for Picard:

$$u_{j+1} = u_0 + I_t^\alpha \left[\frac{1}{2} i u_{jxx} - i u_j \cos^2 x - i u_j^2 \bar{u}_j \right], \quad j \geq 0, \quad (73)$$

Therefore, we can obtain the following first few terms of required solution

$$u_0 = \sin x ,$$

$$u_1 = \sin x - \frac{3}{2} i \sin x \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = \sin x - \frac{3}{2} i \sin \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4} i^2 \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots,$$

$$u_3 = \sin x - \frac{3}{2} i \sin \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4} i^2 \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{27}{8} i^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots, \quad (74)$$

then

$$u = \sin x \left(1 - \frac{3}{2} \frac{i t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4} i^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{27}{8} i^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right)$$

In closed form, we obtain the solution:

$$u = \sin x \left(\sum_{n=0}^{\infty} \left(-\frac{3i}{2} \right)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \quad (75)$$

For the special case $\alpha = 1$, we obtain that:

$$u(x, t) = \sin x \left(1 - \frac{3}{2} i t + \left(\frac{3it}{2} \right)^2 \frac{1}{2!} - \left(\frac{3it}{2} \right)^3 \frac{1}{3!} + \dots \right) = e^{-\frac{2it}{2} \sin x} ,$$

which is an exact solution for (72).

2- By HPM: the homotopy for (72) becomes

$$D_t^\alpha = i p \left[\frac{1}{2} u_{xx} - \cos^2 x u - u^2 \bar{u} \right], \quad (76)$$

we suppose that the solution of the problem (74) is in the form

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots, \quad (77)$$

substituting (76) into (77), we obtain:

$$D_t^\alpha [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] = i p \left[\frac{1}{2} [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]_{xx} - \cos^2 x [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots] - [u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots]^2 [\bar{u}_0 + p \bar{u}_1 + p^2 \bar{u}_2 + p^3 \bar{u}_3 + \dots] \right]. \quad (78)$$

Equating the terms of equal powers p in both sides of the above equation, we obtain:

$$\left. \begin{aligned} p^0 : D_t^\alpha u_0 &= 0, & u_0(x, 0) &= \sin x, \\ p^1 : D_t^\alpha u_1 &= \frac{1}{2} i u_{0xx} - i \cos^2 x u_0 - i u_0^2 \bar{u}_0, & u_1(x, 0) &= 0, \\ p^2 : D_t^\alpha u_2 &= \frac{1}{2} i u_{1xx} - i \cos^2 x u_1 - i u_0^2 \bar{u}_1 \\ &\quad - 2i u_0 u_1 \bar{u}_0, & u_2(x, 0) &= 0, \\ p^3 : D_t^\alpha u_3 &= \frac{1}{2} i u_{2xx} - i \cos^2 x u_2 - i u_0^2 \bar{u}_2 - i \bar{u}_0 (u_1^2 \\ &\quad + 2u_0 u_2) - 2i u_0 u_1 \bar{u}_1, & u_3(x, 0) &= 0, \\ &\vdots & & \end{aligned} \right\} \quad (79)$$

Solving the above set of equation in (79), we obtain

$$\left. \begin{aligned}
 u_0 &= \sin x, \\
 u_1 &= \frac{3}{2} i \sin x \frac{t^\alpha}{\Gamma(\alpha+1)}, \\
 u_2 &= \frac{9}{4} i^2 \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
 u_3 &= \frac{9}{8} i^3 \sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{9}{4} i^3 \sin^2 x \sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
 &\quad - \frac{27}{4} i^3 \sin^3 x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{9}{4} i^3 \sin^3 x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}
 \end{aligned} \right\} \quad (80)$$

So the solution will be

$$\begin{aligned}
 u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\
 &= \sin x - \frac{3}{2} i \sin x \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4} i^2 \sin x \\
 &\quad \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{9}{8} i^3 \sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
 &\quad - \frac{9}{4} i^3 \cos^2 x \sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{27}{4} i^3 \sin^3 x \\
 &\quad \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{9}{4} i^3 \sin^3 x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + \dots,
 \end{aligned} \quad (81)$$

then,

$$\begin{aligned}
 u = \sum_{i=0}^3 u_i &= \sin x \left(1 - \frac{3}{2} i \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4} i^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\
 &\quad \left. - \frac{27}{8} i^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right).
 \end{aligned}$$

In closed form, we obtain the solution

$$u = \sin x \left(\sum_{n=0}^{\infty} \left(-\frac{3}{2} i \right)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right). \quad (82)$$

For the special case $\alpha = 1$, we obtain that

$$\begin{aligned}
 u(x, t) &= \sin x - \frac{3it}{2} \sin x + \frac{(\frac{3}{2}it)^2}{2!} \sin x \\
 &\quad - \frac{(\frac{3}{2}it)^3}{3!} \sin x + \dots
 \end{aligned}$$

$$= e^{-\frac{3it}{2}} \sin x,$$

which is the same exact solution as obtained by PM.

It is clear that the result obtained by HPM in (82) is the same result as obtained by PM in (75)

In Fig. (4), we have plotted the approximate solution for the nonlinear Schrödinger equation (77) for cases a) $\alpha = 0.25$, b) $\alpha = 0.5$, c) $\alpha = 0.75$, d) $\alpha = 1$, and e) the corresponding exact solution.

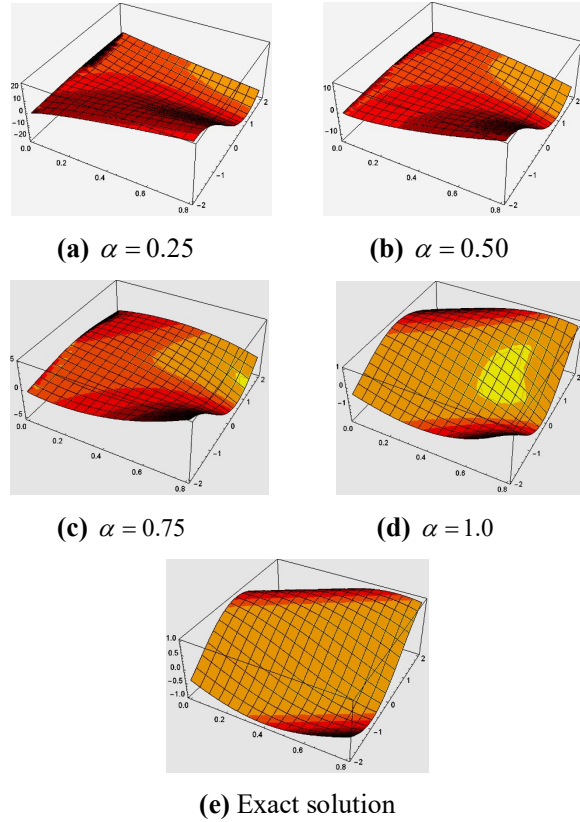


Fig. 4. The surface shows the approximate solution for (72) for various values of α with the corresponding exact solution.

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