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ACCREDITED 2016

Delta Journal of Science

Available online at

https://djs.journals.ekb.eg/

Research Article

MATHEMATICS

Higher estimates for the solution of a heavy rigid body moving under the action of a Newtonian force field

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Abstract In this paper, we introduce a new solution of the Euler's dynamic equations for the rotational motion of a rigid body about a fixed point under the action of a Newtonian force field. The components of the angular velocity vector for this solution are differing from the most famous cases. We assumed that the center of mass of the rigid body coincides with the fixed point and a restriction on an initial condition is applied. The obtained solution is represented graphically using most recent computer codes to describe the motion at any time and is considered as a modification of Euler's case.

Key words: *Euler's equations, Rigid body motion, Newtonian field*

Introduction:

The rotational motion of a rigid body about a fixed point in a Newtonian force field is one of the important problems in theoretical classical mechanics. This problem attracted the interest of many researchers during the last five decades e. g. [1-7]. The great importance of this research subject is due to the wide range of its applications in mechanics. To solve these problems we need to deal with intricate techniques because they are governed by a system contains six non-linear differential equations besides with three first integrals [8]. The exact solutions of such systems require an additional fourth algebraic first integral. Many researches realized such integral for famous special cases, which have some restrictions on the body center of mass location and on the torques acting on the body [9].

The perturbed rotational motion of a heavy solid close to regular precession with constant restoring moment was treated in [2] and [3]. The authors assumed some initial conditions to achieve the analytical solutions of the equations of motion using averaging method [10] up to the first and second approximations. The rotatory motion of a symmetric gyrostat about a fixed point when one component of the gyrostatic torque is applied and in the presence of some torques was considered in [4] and generalized in [5]. The motion of an electromagnetic gyroscope is investigated in [6] when a Newtonian field, perturbed moments and restoring ones are applied. The averaging technique [10] is used to obtain the first order approximate analytical solutions. The graphical representations of these solutions are presented to describe the motion at any instant. The rotational motion of that body under the action of a Newtonian force field with the application of the third component of a gyrostatic moment is investigated in [7]. The approximate periodic solutions of the governing equations are obtained using the small parameter method of Poinacré [11]. This method and its modifications [12-13] are used in [14] to construct the periodic solutions of limiting case for the motion of a rigid body about a fixed point in a Newtonian force field.

The rotational motion of a heavy solid about a fixed point in the presence of a gyrostatic moment vector is presented in [15]. The authors supposed that the body has rapidly spinning about the major or the minor principal axis of the ellipsoid of inertia. Krylov-Bogoliubov-Mitropolski technique [10] is modified and used to achieve the periodic solutions of the equations of motion.

The perturbed self-excited rigid body problem with a fixed point is investigated in [16]. The averaging theory [17] is used to study the periodic orbits up to first order. In [18], the authors presented the possibility of constructing exact analytic solutions concerning the dynamic Euler equations of motion.

The spinning motion of the hovering magnetic top and its dynamic stability were analyzed in [19] and [20]. The numerical integration of a heavy magnetic top is investigated in [21].

Existence of periodic motions of a rigid body was investigated in [22]. The small parameter method was used to obtain the periodic solutions of the equations of motion. The center of mass of the body is slightly shifted from a dynamically symmetric axis. The generalization of this problem was treated in [23] when the body rotates under the action of a Newtonian field and in the presence of one component of the gyrostatic moment vector. A new exact solution of the equations of motion of a rigid body is investigated in [24] when the body moves under the action of a uniform force field. The author assumed that the center of mass of the body is located at meridional plane and the principal torques of inertia satisfied a simple algebraic condition.

In this work, we extend the previous studies when the rigid body moves under the action of a Newtonian force field arising from an attracting center located on the downward fixed axis. We assume that the center of mass of the body coincides with the fixed point (origin). The achieved solution is obtained after taking account some algebraic assumptions concerning on the moments of inertia. This solution is represented graphically, in the rest of this paper, to show the behavior of the body motion under the action of Newtonian force field. From this point of view, the current study may be regarded as a modification of Euler's case for the motion of a rigid body.

2. Equations of motion

Consider the motion of a heavy rigid body that rotates about a fixed point O, in the body, under the influence of a Newtonian force field arising from an attracting center O_1 being located on a downward fixed axis passing through the fixed point O. Let $OXYZ$ be a fixed coordinate system and another moving one $Oxyz$ which is fixed in the body and whose axes are directed along the principal axes of inertia of the body with origin O. The equations of motion are given below [14]

$$\begin{aligned} &A\,\dot{p}+(C-B)\,qr = M\,g\,(\gamma_2\, z_0-\gamma_3\, y_0)+N(C-B)\,\gamma_2\,\gamma_3,\\ &B\,\dot{q}+(A-C)\,rp = M\,g\,(\gamma_3\, x_0-\gamma_1\, z_0)+N(A-C)\,\gamma_3\,\gamma_1,\\ &C\,\dot{r}+(B-A)\,pq = Mg\,(\gamma_1\, y_0-\gamma_2\, x_0)+N(B-A)\,\gamma_1\,\gamma_2, \end{aligned} \tag{1}$$

with

$$\begin{aligned} &\dot{\gamma}_1 = r\,\gamma_2 - q\,\gamma_3, \quad \dot{\gamma}_2 = p\,\gamma_3 - r\,\gamma_1,\\ &\dot{\gamma}_3 = q\,\gamma_1 - p\,\gamma_2, \end{aligned} \tag{2}$$

where A, B and C are the principal moments of inertia of the body; p, q and r are the projections of the angular velocity $\underline{V}$ of the body on the principal axes of inertia; $\underline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector in the direction of the Z-axis; M is the mass of the body; g is the gravitational acceleration; x_0, y_0 and z_0 are the coordinates of the center of mass in the moving coordinate system $Oxyz$. The overdot here refers to differentiation with respect to the time t and $N = (3\lambda / R^3)$ where R is the distance from the fixed point O to the attracting center O_1 and λ is the coefficient of such center.

Equations (1) and (2) admit the following three first integrals

$$\begin{aligned} &\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1,\\ &A\,p\gamma_1 + B\,q\gamma_2 + C\,r\gamma_3 = C_0,\\ &(A\,p^2 + B\,q^2 + C\,r^2) + 2M\,g\,(x_0\,\gamma_1 + y_0\,\gamma_2\\ &+ z_0\,\gamma_3) + N(A\gamma_1^2 + B\,\gamma_2^2 + +C\,\gamma_3^2) = 2C_1, \end{aligned} \tag{3}$$

where C_0 and C_1 are constants.

3. Euler's case

As in Euler's case, we obtain the following first fourth integral according to the presence of Newtonian field

$$A^2 p^2 + B^2 q^2 + C^2 r^2 - N(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2) = C_0^2. \tag{4}$$

Making use of the first two integrals in (3) and the fourth integral (4), one obtains

$$\gamma_1 = \frac{Ap}{C_0}, \quad \gamma_2 = \frac{Bq}{C_0}, \quad \gamma_3 = \frac{Cr}{C_0}. \tag{5}$$

Substituting from systems (1) and (2) into the third equation of system (3), one gets

$$C_2[(N^* A^2+1)Ap^2 + (N^* B^2+1)Bq^2 + (N^* C^2+1)Cr^2] + x_0\,Ap + y_0\,Bq + z_0\,Cr = C_3, \tag{6}$$

where

$$C_2 = \frac{C_0}{2Mg}, \qquad C_3 = \frac{C_0 C_1}{Mg}, \qquad N^* = \frac{N}{C_o^2}.$$

Equation (6) represents a linear combination of the first integrals (3), the fourth integral (4) and (5). So, we seek for a solution that satisfies the previous equation (6).

4. The modified solution

For our scope, let us consider the following choice together with the assumptions of Euler's case

$$A > B > C.$$

This choice allows us to rewrite equation (6) in the form

$$p^2 = \frac{2C_1 - N_2^* Bq^2 - N_3^* Cr^2}{N_1^* A}, \tag{7}$$

where

$$N_1^* = N^* A^2 + 1, \quad N_2^* = N^* B^2 + 1, \quad N_3^* = N^* C^2 + 1.$$

Substituting from (7) into (4), we can obtain directly q^2 in the form

$$q^2 = C_4 - C_5 r^2. \tag{8}$$

Here,

$$C_4 = \frac{[C_0^2 - (2C_1 A/N_1^*) + (2C_1 N^* ABC/N_1^*)]}{[1 - N^* AC - (N_2^* A/N_1^* B) + (N^* N_2^* AC/N_1^*)]B^2},$$

$$C_5 = \frac{[1 - N^* AB - (N_3^* A/N_1^* C) + (N^* N_3^* AB/N_1^*)]C^2}{[1 - N^* AC - (N_2^* A/N_1^* B) + (N^* N_2^* AC/N_1^*)]B^2}.$$

The substitution from (8) into (7) gives

$$p^2 = C_6 + C_7 r^2;$$
$$C_6 = (2C_1 - N_2^* BC_4)/N_1^* A, \tag{9}$$
$$C_7 = (N_2^* BC_5 - N_3^* C)/N_1^* A.$$

Substituting from equalities (8) and (9) into the third one of the system of equations (1), we get

$$\int \frac{dr}{[(A-B)/C]\sqrt{(C_6 + C_7)(C_4 - C_5)}\ r^2} = \int dt.$$

Under the present circumstances, the solution of the previous integration can be obtained easily as

$$r = \{k - [(A-B)/C]\sqrt{(C_6 + C_7)(C_4 - C_5)}\ t\}^{-1}; k = const \tag{10}$$

An inspection of equations (8), (9) and (10), broadly speaking, provides the solution of the problem when the rigid body rotates under the action of a Newtonian force field. This elucidates that, we can separately determine the components of the angular velocity vector p, q and r as functions of time t from these equations. Consequently, we can obtain directly the scalar value of the angular velocity vector in the form

$$V = |\underline{V}| = \left|\sqrt{(C_4 + C_6) + [\frac{1 + C_7 - C_5}{(k - C_8\,t)^2}]}\right|, \tag{11}$$

where

$$C_8 = [(A-B)/C]\sqrt{(C_6 + C_7)(C_4 - C_5)}.$$

5. Discussion of results

In this section, our aim is to provide some numerical results using the computer programs. The following data are used to determine the motion in the considered problem

$$A = 7\,kg.m^2,\ B = 6\,kg.m^2, C = 4\,kg.m^2, M = 100\,kg,$$
$$g = 9.8\,m/s^2,\ N = (200, 400, 500)\,kg.m/s^2$$

Figures (1-4) show the variation of the angular velocity V versus time t in 2-D plane when $N = 200 kg.m/s^2$ and $N = 400 kg.m/s^2$. It is to be noted that, the value of the angular velocity of the body monotonically increases with the increase in time (see figures 1, 3) till it has attained its maximum value whenever $t = k/C_8$, i.e. when the dominator of the second bracket in equation (11) vanishes, at different values of Newtonian force field. The domain of equation (11) is $\Re^+ \cup \{0\} - \{k/C_8\}$ and its range is $\Re^+ \cup \{0\}$, where $\Re^+$ is the positive real numbers, $(C_4 + C_6) > 0$ and $(1 + C_7 - C_5) > 0$.

Above the value $t = k/C_8$, the numerical computations show that the angular velocity gradually decreases as the time goes on, in a similar manner to its increase, (see figures 2, 4). Further, we observe that the growth in the value of Newtonian force field leads to increase in t and V as well.

To make the results more favor, we proceed to illustrate the numerical results in 3-D space. Figures (5-7) and (8-10) represent the behavior of the angular velocity V and time t via $\xi = (V - t)$ when the Newtonian force field equals to 200 and 400, respectively. It should be noticed that figures (5, 8), (6, 9) and (7, 10) describe the behavior of the body below, near and above the maximum value of time t, respectively. The spatial figures for most values of Newtonian force field are presented; see figures (11-16).

It is clear from all previous figures that, the Newtonian force field has acquired a significant influence on the behavior of our model. Such results may be utilized in many industrial applications in various fields; like satellite, spacecraft and manipulators.

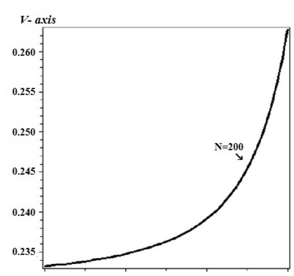

Fig. 1

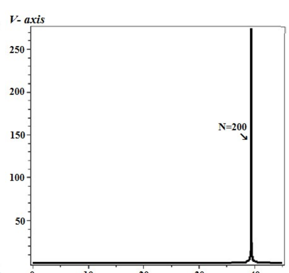

Fig. 2

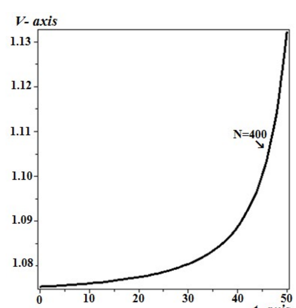

Fig. 3

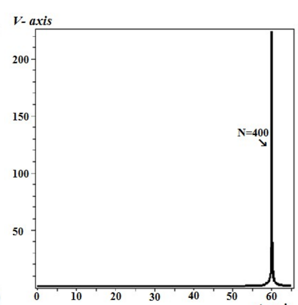

Fig. 4

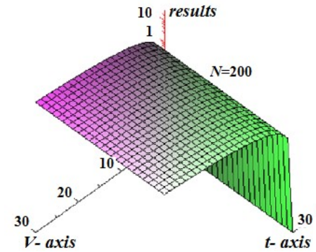

Fig. 5

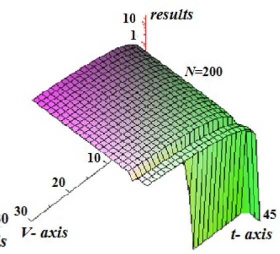

Fig. 6

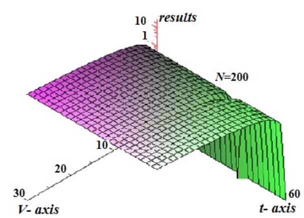

Fig. 7

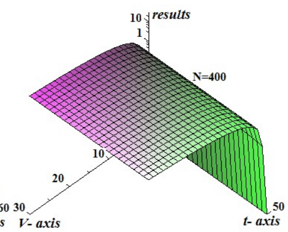

Fig. 8

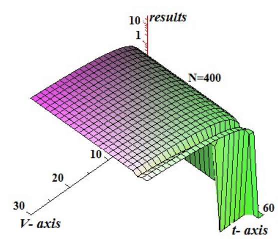

Fig. 9

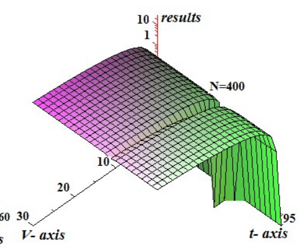

Fig. 10

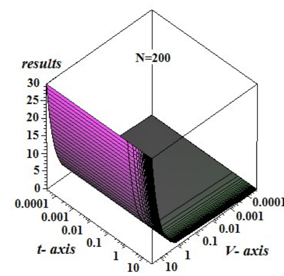

Fig. 11

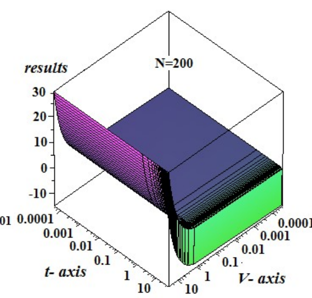

Fig. 12

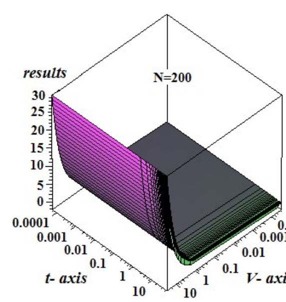

Fig. 13

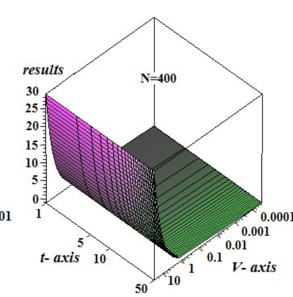

Fig. 14

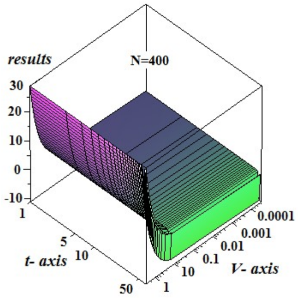

Fig. 15

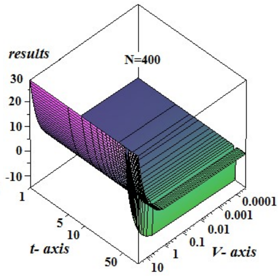

Fig. 16

6. Conclusion

In this work, we have developed a modified solution, represented by (8)-(10), for the Euler's dynamic equations (1) with the aid of Poinsot's equations (2), when the rigid body rotates under the action of a Newtonian force field. The obtained angular velocity components are different from Lagrange's case, Kovaleveskaya's case, Euler's case (when the body rotates without any applied torques) or from any special case. A restriction on the choosing of initial conditions of $p(0), q(0), r(0), \gamma_1(0), \gamma_2(0)$ or $\gamma_3(0)$ according to the meaning of C_0^2 is considered. The obtained solution is considered as a modification for both Euler's case and Ershkov [24] work when the Newtonian filed has no effect, i.e. vanishes. The graphical representations of the obtained angular velocity solution are presented through different figures. The numerical results have shown that the Newtonian force field value has an important effect on the rigid body motion. However, the analytical results of the rotational motion of a rigid body about a fixed point can be exploited in industrial applications, such as satellites, autopilots and aircrafts.

7. References

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تقديرات أعلى لحل جسم متماسك ثقيل يتحرك تحت تأثير مجال نيوتوني

في هذا البحث تم دراسة إيجاد حل جديد لمعادلات أويلر الديناميكية للحركة الدورانية للجسم المتماسك حول نقطة ثابتة تحت تأير مجال نيوتيوني، مع الأخذ في الاعتبار أن مركبات متجه السرعة الزاوية الخاصة بهذا الحل تختلف عن معظم الحالات الخاصة المشهورة، وبفرض أن مركز الكتلة للجسم المتماسك ينطبق على نقطة الأصل مع تطبيق شروط إبتدائية معينة. وتم تمثيل الحل الذي تم الوصول إليه هندسياً باستخدام برامج حديثة وذلك لوصف الحركة عند أية لحظة زمنية، ويعتبر هذا الحل تعديلاً لحالة أويلر.