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Research Article MATHEMATICS

Koenig's root-finding algorithms

Omar Ismael El Hasadi

Department of Mathematics, Faculty of Education,

University of Omar Al Mukhtar, Branch Derna, Libya.

0.1 Abstract In this paper, we first recall the definition of a family of Koenig's root-finding algorithms known as Koenig's algorithms ($K_{p,n}$) for polynomials. In the whole paper p has degree $d \geq 2$ with real coefficients and real (and simple) zeros x_k , $1 \leq k \leq d$.

Now we want to discuss Koenig's algorithms in details where $n = 4, (K_{P,4}(z))$.

Keywords: *Koenig's function,derivative of Koenig, immediate basins of Koenig.*

Definition 0.1.1.

Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d$$

be a polynomial with real coefficients and real (and simple) zeros x_k , $1 \leq k \leq d$, and $n \geq 2$ is an integer. Koenig's method of p of order n is defined by the formula

$$K_{p,n}(z) = z + (n-1)\frac{(\frac{1}{p})^{[n-2]}}{(\frac{1}{p})^{[n-1]}},$$

(0.1.1)

where $(\frac{1}{p})^{[n]}$ is the nth derivative of $\frac{1}{p}$.

For $n = 2$ the map $K_{p,n}$ is Newton's method of p, for $n = 3$ the map $K_{p,n}$ is Halley's method of p, and Householder's method

$$h_p(z) = K_{p,2}(z) - \frac{p}{2p'} K'_{p,2}$$

which we have discussed all of them in the previous papers

0.2 Koenig's root-finding algorithms of order four

Let p be a polynomial with real coefficients and real (and simple) zeros x_k , $1 \leq k \leq d$, then

$$K_{p,4} = z - 3\frac{p^2p'' - 2pp'^2}{6pp'p'' - 6p'^3 - p^2p'''} \quad ;$$

(0.2.1)

defined as Koenig's function of order four associated with p. The fixed points of $K_{p,4}$ are given by the zeros of $p^2p'' - 2pp'^2$. Since we have known $pp'' - 2p'^2 < 0$ on R, the fixed points of $K_{p,4}$ are the zeros of p together with ∞, and from proposition (0.5.1) the rational map $K_{p,4}$ has degree $3d - 2$.

Proposition 0.2.1. *Let $p: C \rightarrow C$ be a polynomial of degree d, then Koenig's method $K_{p,4}$ is a rational map, it has a repelling fixed point at ∞ with multiplier $(d+2)/(d-1)$.*

Proof. When $|z|$ tends to ∞, we have

$$p(z) \sim \lambda z^d \quad ,$$

we know

$$K_{p,4} = z + 3\frac{(\frac{1}{p})''}{(\frac{1}{p})'''} \ ,$$

where

$(\frac{1}{p})'' \sim \frac{d(d+1)}{\lambda z^{d+2}}$,
and
$(\frac{1}{p})''' \sim \frac{-d(d+1)(d+2)}{\lambda z^{d+3}}$,

Then
$K_{p,4} \sim z - 3\frac{z}{d+2}$,

$K'_{p,4}(z) \sim 1 - \frac{3}{d+2} \sim \frac{d-1}{d+2}$,

as we know that the multiplier λ , at ∞ is equal to

$$\lim_{n\to\infty} \frac{1}{K'_{p,4}(z)} = \frac{d+2}{d-1} .$$

0.3 Derivative of Koenig's method of order four

The derivative of Koenig's method $K_{p,4}$ is
$K'_{p,4} = \frac{p^3(4pp'''^2 - 24\ 'p''p''' + 6p'^2p^{(4)} + 18p''^3 - 4pp'\ \ ^{(4)})}{(6pp'p'' - 6p'^3 - p^2p''')^2}$,
(0.3.1)
from (0.3.1), we can see that the roots of $p(z)$ are superattracting fixed points of $K_{p,4}$, but of one degree higher order than for Halley's method. There are three critical points at each fixed point of $K_{p,4}$. The rational map $K_{p,4}$ has $2(3d-2)-2 = 6d-6$ critical points, and $3d-6$ of them are free critical points. Also from proposition (0.5.1), the local degree of $K_{p,4}$ at the roots of p is exactly equal to four.

Remark **0.3.1.** Let x be a simple zero of p, then $K_{p,4}(x) = x$ and from (0.5.1) $K'_{p,4}(x) = K''_{p,4}(x) = K'''_{p,4}(x) = 0$, while $K^{(4)}_{p,4} \neq 0$. Thus $K_{p,4}$ is of order four for simple roots.

Since $p(x) = 0$, it follows that $N_p(x) = H_p(x) = K_{p,4}(x) = x$, and this fixed point is superattracting fixed point for the three methods because $N'_p(x) = H'_p(x) = K'_{p,4}(x) = 0$. And since the third derivative of $K_{p,4}$ vanishes, whereas the third derivative of H_p does not, the graph of $K_{p,4}$ is flatter than that of H_p near the fixed point. Thus $K_{p,4}$
is faster convergence to the fixed point than H_p. From figures (1,2), Koenig's function ($K_{p,4}$) looks like Newton's function but ($K_{p,4}$), where $p(z) = z^3 - z$, has non-real critical points wherea Newton's function does not.

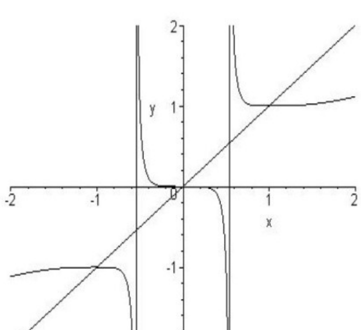


Figure 1: Koenig's function for the polynomial $p(x) = x^3 - x$.

Proposition 0.3.1. *Let $p: C \to C$ be a polynomial of degree d with real coefficients and real (and simple) zeros. Then the rational map$K_{p,4}$ has $2d-2$ repelling fixed points in* C *and their multipliers are all equal to four.*

And $pp'' - 2p'^2 < 0$ on R, it follows that, if $p > 0$ in (c_1, x_k), then $p' < 0$ and $p''' < 0$,
and if $p < 0$ in(c_1, x_k) , then $p' > 0$ and $p''' > 0$. Thus

$$\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''} < 0 \quad in \quad (c_1, x_k),$$

it follows that $K_{p,4}(x) > x$ in (c_1, x_k), thus

$$\lim_{x\to c_1^+} K'_{p,4}(x) = +\infty .$$

Similarly, we have

$$\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''} > 0 \quad in \quad (x_k, c_2),$$

So $K_{p,4}(x) < x \quad in \quad (x_k, c_2)$, thus

$$\min_{x\to c_2^-} K'_{p,4}(x) = -\infty.$$

and at the repelling fixed points of $K_{p,4}$, $g = 0$. Thus $K'_{p,4} = 4$ at each repelling fixed point.

Definition 0.3.1. If $c_1 < c_2$ are consecutive real poles of $K_{p,4}$, then the interval (c_1, c_2) is called a band for $K_{p,4}$.

Proposition 0.3.2. *If (c_1, c_2) is a band for $K'_{p,4}$ that contains a root of $p(x)$, then*

$$\lim_{x\to c_1^+} K'_{p,4}(x) = +\infty, \qquad \min_{x\to c_2^-} K'_{p,4}(x) = -\infty$$

Proof. From

$$K_{p,4} = z - 3\frac{p(pp'' - 2p'^2)}{6p'(pp'' - p'^2) - p^2p'''} ,$$

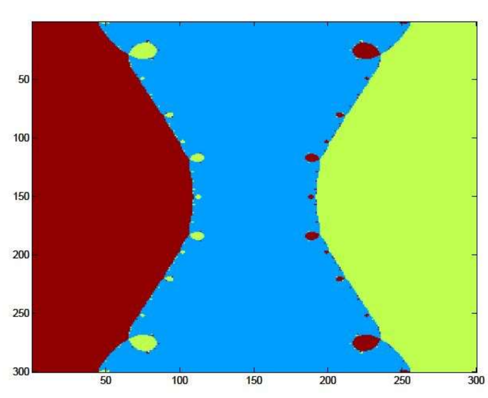


Figure 2: Iteration of Koenig's function for the polynomial $p(z) = z^3 - z$.

Proof. Let

$$K_{p,4}(z) = z + 3\frac{g(z)}{g'(z)} ,$$

Where

$$g = \left(\frac{1}{p}\right)'' = \frac{2p'^2 - pp''}{p^3} ,$$

a rational map $\mathcal{R}: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Assume that A is not simply connected. Then there exist in $\hat{\mathbb{C}}$ two disjoint domains U_0 and U_1 intersecting A, such that $V = \mathcal{R}(U_0) = \mathcal{R}(U_1) \supset \overline{U_0} \cup \overline{U_1}$, $\mathcal{R}(\partial U_i) = \partial V \subset A$ for $i = 0,1$, $V \cup A = \hat{\mathbb{C}}$ and V is homeomorphic to a disc.

$$g' = \left(\frac{1}{p}\right)''' = \frac{6pp'p'' - 6p'^3 - p^2p'''}{p^4} .$$

Let x_k , $1 \le x_k \le d$, be the zeros of p which are real and simple. The fixed points of $K_{p,4}(z)$ are ∞, the points x_k and the zeros of the rational map $g = \left(\frac{1}{p}\right)''$.

From (0.3.2), we can see that g has $3d$ poles. When $z \to \infty$, then $p(z) \sim \lambda z^d$ and it follows that g has a zero of order $d + 2$ at ∞. Since the number of zeros for any rational map is equal to the number of poles, then has $3d - (d + 2) = 2d$ finite zeros. Since we have proved that $2p'^2 - pp'' > 0$ on R, $2d - 2$ zeros of g are non-real repelling fixed points of $K_{p,4}$. Now we have

$$K'_{p,4} = 4 - \frac{3gg''}{g'^2}$$

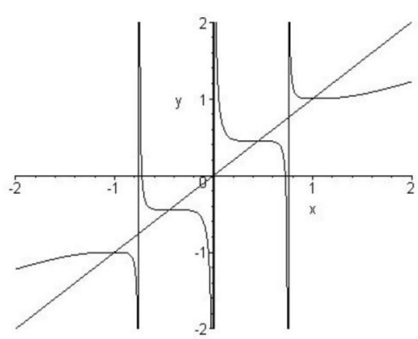


Figure 3: Koenig's function for the polynomial $p(x) = (x^2 - 1)(x^2 - 1/5)$.

0.4 Immediate basins of Koenig's method of order four

In this section, we want to prove that each component of Fatou set of Koenig's method $K_{p,4}$ is simple connected.

Lemma 0.4.1. *([6]) Let A be the immediate basin of attraction to a fixed point for*

Theorem 0.4.2. *The immediate basins of attraction to the roots of any polynomial with real coefficients and only real (and simply) zeros x_k, $1 \le k \le d$ for Halley's method, are simply connected.*

Proof. In [6] Feliks Przytycki has proved that the immediate basins of attraction for N_p is simply connected. We can apply the same proof, so we can assume that A is a multiply connected immediate basin of attraction for $\mathcal{R} = H_p$ to a root $x \in \mathbb{R}$ of a polynomial p. Choose $z \in V \cap A$, V given by Lemma (0.4.1), and branches $\mathcal{R}^{-1}$,

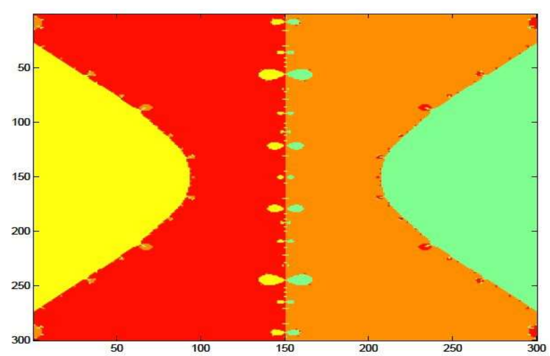


Figure 4: Iteration of Koenig's function for the polynomial $p(z) = (z^2 - 1)(z^2 - 1/5)$.

so that $w_i = R^{-1}(z) \in U_i \cap A$. Join z with w_i by a curve $\gamma_i^0 \subset V \cap A$. Take care additionally to have $\gamma_i^0 \cap cl\left(\cup_{n>0} R^n(critR)\right) = \emptyset$. Define by induction $\gamma_i^n = R^{-1}(\gamma_i^{n-1})$, where R^{-1} is the extension of the preliminary branch along the curve $\cup_{j=0}^{n-1} \gamma_i^j$. Define $\gamma_i = \cup_{n=0}^{\infty} \gamma_i^n$. The curve γ_i converges to a fixed point $\zeta_i \in U_i$ of R. The reason is that $R_{v_i}^{-1} \circ \ldots \circ R_{v_i}^{-1}$, n times, $n = 0,1, \ldots,$ is a normal family of functions on a neighborhood of γ_i with the set of limit functions on boundary of A which is nowhere dense. So all limit functions are constant, hence $\lim_{n\to\infty} diam(\gamma_i^n) = 0$. Therefore all limit points of the sequence of curves γ_i^n are fixed points for R. On the other hand they must be isolated from each other. So we actually have only one limit

point. The conclusion is that the boundary of A contains two different fixed points ζ_0, ζ_1 belonging to two
different components of the boundary of A. But the only fixed points for H_p are the roots of p (real), the roots of p' (real), and ∞. Since we have proved that H_p is continuous on $\mathbb{R}$. Thus $A \cap \mathbb{R}$ is an interval. We arrived at a contradiction.

Theorem 0.4.3. *Immediate basins of attraction of Koenig' function $K_{p,4}$ are simply connected, whenever p is a complex polynomial with real coefficients and only real and simple zeros $x_k, 1 \leq k \leq d$.*

Proof. We follow the same steps of proof of theorem (0.4.2) with some changes. In this case we work on the interval (a_1, a_2), where a_1, a_2 are two consecutive poles of $K_{p,4}$ instead of the interval (r_1, r_2), where r_1, r_2 are two repelling fixed points of H_p.

Assume that A is a non simply connected immediate basin of attraction for $K_{p,4}$ to a root $x \in \mathbb{R}$ of a polynomial p. We follow the same proof of theorem (0.4.2) until we arrive to the conclusion that boundary A contains two different fixed points belonging to two different components of boundary of A. But the only fixed points for $K_{p,4}$ are the roots of p and ∞ . We arrived at a contradiction.

Since $K_{p,4}$, (where for simplicity $p(z) = z^3 - z$), has non real free critical points, then we are in the same situation of Halley's method.

0.5 General form of Koenig's method

The following rational map

$$K_{p,n}(z) = z + (n-1)\frac{(\frac{1}{p})^{[n-2]}}{(\frac{1}{p})^{[n-1]}},$$

is the general form of K¨oing's function. We end this chapter with some general remarks describe, without proof, the dynamics of the general form of Koenig's function $K_{p,4}$. We will consider p be a special polynomial of degree $d > 2$ which is a complex polynomial with real coefficients and real (and simple) zeros $x_k, 1 \leq k \leq d$, and $p'(x_k) = p''(x_k) = 0$.

Proposition 0.5.1. *Let $p: \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Then for any $n \geq 2$,*

(a) *The rational map $K_{p,n}$ has degree $(n-1)(d-1)+1$.*

(b) *If p has d distinct roots, then $K_{p,4}$ has*

(c) *The local degree of $K_{p,n}$ at the roots of p is exactly n.*

(d) *Koenig's method $K_{p,n}$ is a rational map, it has a repelling fixed point at ∞ with multiplier $1 + \frac{n-1}{d-1}$.*

Proof. For details proof see [9].

In general case of the map $K_{p,n}, n \geq 2$and p is special polynomial of degree $d \geq 2$ with real coefficients and real (and simple) zeros, we have two cases.

Case (1) If n is even, then the map $K_{p,n}$ has $nd - 2$ real critical points, and $(n-2)(d-2)$ non-real critical points which are distributed as follows; each basind of $x_k, 2 \leq k \leq d-1$, contains n real critical points and $n-2$ non-real critical points, symmetric to the real line; the two basins of x_1, x_d each contains $(n-1)$
real critical points. And there are $(d-1)$ real poles of $K_{p,n}$.

Case (2) If n is odd then the map $K_{p,n}$ has $(n-1)d$ real critical points and $(n-1)(d-2)$ non-real critical points, where each basin $x_k, 1 \leq k \leq d$, contains $(n-1)$ real critical points and each basin $x_k, 2 \leq k \leq d-1$, contains $(n-1)$ non-real critical points. And there are no real poles.

The following figures show how the critical points (c.p) distributed around the fixed points of the map $K_{p,n}$, where p is special polynomial.

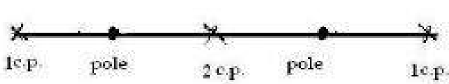
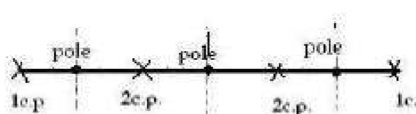


Figure 5: $n = 2, d = 3$ (Newton), number of critical points $2d - 2$.



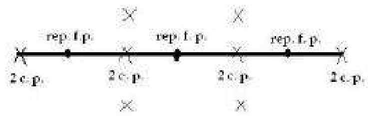


Figure 8: $n = 3, d = 4$ (Halley), number of critical points $4d - 4$.

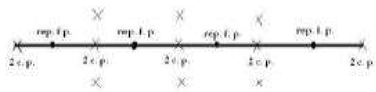


Figure 9: $n = 3, d = 5$ (Halley), number of critical points $4d - 4$.

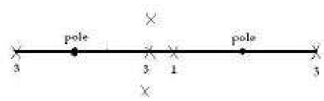


Figure 10: $n = 4, d = 3$ ($K_{p,4}$), number of critical points $6d - 6$.

Figure 12: $n = 5, d = 4$ ($K_{p,5}$), number of critical points $8d - 8$.

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الملخص

في هذه الورقه العلمية نعرف طريقة كونج لإيجاد الجذور لكثيرة الحدود

وسيتم مناقشة طريقة $d \geq 2$ في هذه الورقه $p(z)$ ونعتبر درجة $1 \leq k \leq d$ حيث x_k وكذلك ايجاد تفاضل دالة كونج للتعرف على حركة النقاط الحرجه $n = 4$ كونج بالتفصيل عندما تحت تكرار الداله وتم اثبات ان الاحواض الفورية للجذور تكون مرتبطه ارتباط بسيط وفي النهايه تعرضنا للصوره العامه لدالة كونج .