Mahmoud R. Mahmoud,*

*Fatma A. Khalil**,*

*Abeer A. El-helbawy**,*

*Ghada A. El-kelany***

** Institute of Statistical Studies and Research.*

Department of Statistics.

Cairo University.

*** Department of Statistics, Faculty of Commerce.*

Al-Azhar University (Girls' Branch).

CORRESPONDENCE:

Ghada A. El-kelany

Department of Statistics - Faculty of Commerce.

Al-Azhar University (Girls' Branch).

g_elkelany@hotmail.com

01061533900 – 01143519543

Competing Risks inComplete, Incomplete and *Type-II* CensoredDatainSome Weibull Models

Abstract:This paper presents estimators of the parameters included in independent competing risks in the presence of complete, incomplete and *Type-II*censored data.A procedure is established for analyzing data from some of Weibull models with consideration mainly that there are two independent causes of failures. We consider the case when the competing risks (two causes) have Weibull distribution and Rayleigh distribution, respectively. The maximum likelihood estimators, MLEs, of the different parameters are obtained.Properties of the estimated values have been studied through a simulation study.

Keywords:*Survival analysis; Competing Risks; Complete data; Incomplete data; Censored data; Weibull distribution; Rayleigh distribution; Maximum likelihood estimators and Relative risk.*

1- Introduction:

Survival analysis pertains to a statistical approach which deals with collection, modeling and statistical analysis of data on lifetimes. The use of survival analysis today has been of considerable interest in many branches of statistical applications such as medical and biological sciences, actuarial science, and business studies. Like most branches of statistics, modeling has been carried out using parametric and non-parametric setups.The parametric setup, in survival analysis,is performed assuming that the lifetime followssome distributions such as exponential, gamma, Weibull, and generalized exponential distributions. The non-parametric setup does not consider a specific lifetime distribution.

In the statistical analysis of lifetime data, the exponential distribution is widely used in statistics and reliability analysis as a popular model. Thisis because it providessimple and closed form solutions. Cox (1959) stated that: *"there are two situations in which an exponential distribution would be expected on general grounds. First, failure may be due to an external point occurrence, for example an accident, arising randomly in age-independent way. Secondly, there may be many more or less independent causes of failure, when the observed failure time is the smallest of a number of independent random variables; under some rather special conditions the resulting distribution will be exponential."* It is well known that, the exponential distribution can have only constant failure rate, so it might not be very practical to assume that the lifetime distribution is exponential; as in the case where there is a single process of wear going on at a fairly steady rate. In this case, there are many distributions that have been suggested to model lifetime data based on the behavior of their failure rates. Most popular among these alternative distributions is the two parameters Weibull distribution. It is commonly used to analyze lifetime data because of monotonic failure rates.

In many applications in survival analysis it is quite common to have several possible risks (causes) of failure present at the same time. The actual cause of failure of an item may be one, and only one, of these risks. Hence these risks are said to compete for the life of the item. Modelsforlifetime in the presence of such competing risks in statistical literature are known as the competing risks models.

Jayant and Sudha (2001) considered competing risks model as follows: The item is exposed to k (≥ 2) risks of failure.One and only one of these risks actually claim the life of the item and is called the cause of failure. It is presumed that $T_1, T_2, \ldots, T_k$ are positive-valued continuous random variables denoting the lifetime (time to failure) under the k risks, respectively. In other words, T_i called the ith latent lifetime of the item, represents the random lifetime of the item when the item is exposed to the ith risk alone. However, all the k risks act simultaneously and the actual life time of the ith item is another positive-valued continuous random variable $T = \min(T_1, T_2, \ldots\ldots, T_k)$. They also assumed that upon failure the cause of death or the risk which actually claimed the life becomes known. It is denoted by δ and is defined as:

$$\delta = j \text{ if } T = T_j,\ j = 1, 2, \ldots\ldots, k.$$

Thus data available from n independent copies of the unit are:

$$(t_i, \delta_i),\ i = 1, 2, \ldots\ldots, n.$$

Competing risks models have been studied by several authors using parametric and non-parametric setup. The parametric setup is performed assuming that the competing risks follow different lifetime distributions such as Exponential, Gamma, Weibull, and Generalized exponentialdistributions, see for example Berkson and Elveback (1960), Cox (1959), David and Moeschberger (1978), Park (2005), Kundu and Sarhan (2006) and Sarhan (2007). The non-parametric setup does not consider a specific lifetime distribution. The analysis of the non-parametric version of this model has been investigated by several authors such as Kaplan and Meier (1958). Nonparametric models will not be considered in this article. Most works discuss the case of two competing risks presuming that the results would be easily generalized to the case of more than two risks. Cox (1959) considered some general models involving arbitrary distributions for general independent risks. In one of these models, he assumed that the random variables T_1 and T_2 are independently distributed with continuous distribution functions $F_1(t)$ and $F_2(t)$. Then the probability that the failure occurs between $(t, t+\delta t)$ for the first cause is given by

$$g_1(t) = f_1(t)[\,1 - F_2(t)]$$

and similarly, for the second cause, is given by:

$$g_2(t) = f_2(t)[\,1 - F_1(t)].$$

In this model, the probability that a failure is of the first cause, given that it occurs at t, is equal to $g_1(t)/[g_1(t) + g_2(t)] = \pi_1(t)$, say. The probability does not involve t if and only if:

$$\frac{f_1(t)}{1-F_1(t)} = \psi \frac{f_2(t)}{1-F_2(t)},$$

for some constant ψ; that is, if and only if:

$$1-F_1(t) = [1-F_2(t)]^{\psi}$$

In this case, the probability that a failure is of the first cause is $\psi/(1+\psi)$.

In many applications, in analyzing competing risk models, it is assumed that each observation consists of a failure time and an indicator denoting the cause of failure. It is usually assumed, in either parametric or non-parametric models, that both the failure times and the causes of failure are observed. Thissituation is referred to as the case of complete data. However, in certain situations, the determination of the cause of failure may be expensive and requires time, very difficult or impossible to observe. Thus it might happen that the failure time of that item is observed but the corresponding cause of failure is not observed [see Alwasel (2009)].

Balakrishnan and Han (2010) stated that; "a failure is associated with one of several fatal risk factors the test unit is exposed to. Since it is not usually possible to study the test units with an isolated risk factor, it becomes necessary to assess each risk factor in the presence of other risk factors. In order to analyze such a competing risks model, each failure observation must come in a bivariate form composed of a failure time and the cause of failure".

Kundu and Basu (2000) considered the following two types of data:

a) The item has failed due to a certain cause of failure, and both its time of failure and the cause are observed.

b)The item has failed, and its time of failure has been observed, but not the cause of failure.

They referred to the first type in *(a)* as a complete data, and referred to the second type in *(b)* as an incomplete data. Also, they assumed that every observation in the sample can be monitored until failure. That is, there is no censoring. But in most applications, some observations may be alive at the end of the project period; that is, the data are censored; see for example David and Moeschberger (1978).In addition to the above types of observations, Sarhan (2007) considered the following third type:

*c)*The item was still working at the end of the project period.

Naturally, he referred to it as a censored data.

Censoring is inevitable in live testing and survival studies because the experimenter is unable to obtain complete information on lifetime for all observations. For example, patients in a clinical trial may withdraw from the study, or the study may have to be terminated at a pre-fixed time point. In industrial experiments, units may break accidentally. The two most common censoring schemes are termed *Type-I* and *Type-II* censoring.

Type-I censoring: occurs if an experiment has a set number of observations or items and stops the experiment at a predetermined time.

Type-II censoring: occurs if an experiment has a set number of observationsor items and stops the experiment when a predetermined number are observed to have failed.

In this paper, we consider the competing risks model when the underlying lifetime distributions are some of Weibull models that have not been subjected to this type of analysis with consideration mainly that there are two independent causes of failures. We consider the case when the independent competing risks (two causes) have the two parameters Weibull distribution and Rayleigh distribution respectively, in the presence of complete, incomplete and *Type-II* censored data, since it is the mode of censoring most common in practice.The maximum likelihood estimatorsof different parameters with different sampling schemes and their properties are studied under theseassumptions. We also conduct a simulation study for studying the properties of the estimators for unknown parameters.

The rest of the paper is organized as follows: Notations which are needed for describing the model and some Weibull models are presented in Section *2.* Section 3 is concerned with model assumptions. The MLEs of the unknown parameters and the approximate asymptotic variance-covariance matrix for *Type-I* censoring are considered in Section *4.* The relative risk rates are obtained in Section *5.* A simulation study is analyzed in Section *6* and conclusions are presented.

2- Notations and Some Weibull models:

2-1 Notations:

In survival analysis, some basic concepts need to be reintroduced. Since we deal with lifetimes here, we further assume that the random variables are continuous positive-valued. Without loss of generality, we assume that there are only two causes of failure and we use the following notations:

n:	the sample size.
τ_m:	censored value in *Type-II* censoring.
f. (.):	denotes the probability density function, (p.d.f).
F.(.):	denotes the cumulative distribution function, (c.d.f).
S. (.):	denotes the survival function.
h. (.):	denotes the hazard function, (h.f).
$\delta_i = j$:	indicator variable means the observation i has failed at time T_i due to cause j, j=1,2 while δ_i =* means the cause of observation *i* to fail is unknown.
I(.):	indicator function of the observation (.)

2- 2 Some Weibull models:

2-2-1 The Weibull distribution:

The Weibull distribution is often used in the field of life data analysis due to its flexibility. It is commonly used to model systems with monotone failure rates. In 1939, Weibull introduced a distribution to explain the variation in the strength of a specimen. Due to the effect WW //on communications among scientists the paper did not become known in the society for community. That was just the same article as the highly cited one of Weibull 1951, after which it became known as Weibull distribution. This paper is concerned with the two parameters Weibull distribution. The two parameters Weibull distribution can be used to analyze lifetime data because it can model a variety of life behaviors.The *p.d.f* of Weibull distribution is:

$$f(t;\alpha,\lambda)=\alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}} \quad ;t>0,\alpha>0,\lambda>0,$$

where α is a shape parameter, λ is a scale parameter. If $\alpha<1$, the failurerate decreases over time t. If $\alpha=1$, the failure rate is constant over time. If $\alpha>1$, the failure rate increases over time.The corresponding *c.d.f*, survival function and failure rate (or hazard rate) respectively are:

$$F(t)=1-e^{-\lambda t^{\alpha}}$$

$$S(t)=e^{-\lambda t^{\alpha}}$$

$$h(t)=\alpha\lambda t^{\alpha-1}$$

2- 2-2 The Rayleigh distribution:

The Rayleigh distribution is a special case of the Weibull distribution, that is, when $\alpha=2$ and $\lambda=\frac{1}{2\sigma^{2}}$. The Rayleigh distribution occurs in works on radar, properties of sine wave plus-noise, etc. It was introduced Sir Rayleigh in around 1880. He derived it from the amplitude of sound resulting from many important sources. The Rayleigh distribution is widely used in communication, engineering, reliability analysis and applied statistics. In this study, we are concerned with Rayleigh distribution because it has a linearly increasing rate, so, it is appropriate for components which might not have manufacturing defects, but age rapidly with time. The *p.d.f* of the Rayleigh distribution is:

$$f(t;\sigma)=\frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}} \quad ;\sigma>0,t>0$$

The corresponding *c.d.f*, survival function and hazard rate respectively are:

$$F(t)=1-e^{-\frac{t^2}{2\sigma^2}}$$

$$S(t)=e^{-\frac{t^2}{2\sigma^2}}$$

$$h(t)=\frac{t}{\sigma^2}$$

3- Model assumptions:

The following assumptions are needed throughout this study:

The random latent failure times (T_{i1}, T_{i2}); $i = 1,2,\ldots,n$ are independent random variables for all $i = 1, 2, \ldots, n$, hence $F(t) = F_1(t)\times F_2(t)$ and $T_i=\min[T_{i1},T_{i2}]$.

1- Let t_{i1} denotes the failure time according to the first cause, for t_{i1}'s, follows Weibull distribution, t_{i2} denote the failure time according to the second cause, for t_{i2}'s,follows Rayleigh distribution and τ denotes the censored time.

2- Probability that $[t_{i1}=, t_{i2}] = 0$

3- In the first n_1 observations we observe, without loss of generality, the failure times and also causes of failure. Whereas for the following $n_2 - n_1$ observations we only observe the failure times and not the causes of failure that is the cause of failure is unknown.

In the remaining $n-n_2$ observations, the system still alive at the end the project periods. Nameiy, we observe the foiiowing data:

$$(t_1,\delta_1),(t_2,\delta_2),\ldots\ldots,(t_{n_1},\delta_{n_1}),(t_{n_1+1},*),\ldots\ldots,(t_{n_2-n_1},*),\ and\ (t^*_{n_2-n_1+1},*),\ldots\ldots,(t^*_{n-n_2},*).$$

Here, (t,δ) means the experiment has failed at time t due to cause δ, $(t,*)$ means the experiment has failed at time t but the cause of failure is unknown and $(t^*,*)$ means the experiment has tested until time t without failing (censored data). We denote this set by Ω which can be categorized as a union of three disjoint classes Ω_1, $\Omega_2$and Ω_3. Where$\Omega_1$represents the set of data when the cause of experiment failure is known, while $\Omega_2$denotes the set of observations when the cause of experiment failure is unknown and $\Omega_3$denotes the set of censored observations. Further, the set $\Omega_1$can be divided into two disjoint subsets of observations $\Omega_{11}$and Ω_{12}, where Ω_{1j}represents the set of all observations when the failure of the experiment is due to the cause j, j= 1, 2. We also assume that $|\Omega_i|=r_i$, $|\Omega_{1j}|=r_{1j}$. Namely,$|\Omega_1|=r_1=(r_{11}+r_{12})$,$|\Omega_2|=r_2=(n_2-n_1)$ and $|\Omega_3|=r_3=(n-n_2)$.

4- The Maximum likelihood estimators andthe approximateasymptotic variance-covariance matrix for *Type-I*censoring

4-1 The Maximum likelihood estimators:

According to our assumptions, for Type II censoring we assume that the experiment still working at time t, and the number of censoring observations is considered as the number of observations which has failed due to a certain cause of failure, and both its time of failure and the cause are observed (complete observations). Then the likelihood function of the observed datathen the likelihood function of the observed data $(t_1,\delta_1),(t_2,\delta_2),\ldots\ldots,(t_{n_1},\delta_{n_1}),(t_{n_1+1},*),\ldots\ldots,(t_{n_2-n_1},*),\ and\ (t^*_{n_2-n_1+1},*),\ldots\ldots,(t^*_{n-n_2},*)$, f or the general case, takes the following form:

$$L = \prod_{i=1}^{n_1} \left((f_1(t_i)\, S_2(t_i))^{I(\delta_i=1)} \quad (f_2(t_i)\, S_1(t_i))^{I(\delta_i=2)} \right)$$
$$\times \prod_{i=n_1+1}^{n_2} (d\, F(t_i)) \times \prod_{i=n_2+1}^{n} (S(t_i)),$$

where $d\, F(t)$ is the derivative of $F(t)$, [see Miyakawa (1984)]. Hence:

$$L = \prod_{i=1}^{n_1} \left((f_1(t_i)\, S_2(t_i))^{I(\delta_i=1)} \quad (f_2(t_i)\, S_1(t_i))^{I(\delta_i=2)} \right)$$
$$\times \prod_{i=n_1+1}^{n_2} \left(f_1(t_i).S_2(t_i) + f_2(t_i)\, S_1(t_i) \right)^{I(\delta_i=*)} . \times \prod_{i=n_2+1}^{n} (S(\tau_m))$$

Then,

$$L = \prod_{i=1}^{n_1} \left(\left(\alpha\, \lambda\, t_i^{\alpha-1}\, e^{-\lambda t_i^{\alpha}}\, e^{-\frac{t_i^2}{2\sigma^2}} \right)^{I(\delta_i=1)} \left(\frac{t_i}{\sigma^2}\, e^{-\frac{t_i^2}{2\sigma^2}}\, e^{-\lambda t_{i1}^{\alpha}} \right)^{I(\delta_i=2)} \right)$$
$$\times \prod_{i=n_1+1}^{n_2} \left(\alpha\, \lambda\, t_i^{\alpha-1}\, e^{-\lambda t_i^{\alpha}}\, e^{-\frac{t_i^2}{2\sigma^2}} + \frac{t_i}{\sigma^2}\, e^{-\frac{t_i^2}{2\sigma^2}}\, e^{-\lambda t_i^{\alpha}} \right)^{I(\delta_i=*)}$$
$$\times \prod_{i=n_2+1}^{n} \left(e^{-\lambda \tau_m^{\alpha}}\, e^{-\frac{\tau_m^2}{2\sigma^2}} \right)$$

Rearranging the terms we may write this as:

$$L = \prod_{t_i \in \Omega_{11}} \left(\alpha\, \lambda\, t_i^{\alpha-1}\, e^{-\lambda t_i^{\alpha}}\, e^{-\frac{t_i^2}{2\sigma^2}} \right) \times \prod_{t_i \in \Omega_{12}} \left(\frac{t_i}{\sigma^2}\, e^{-\frac{t_i^2}{2\sigma^2}}\, e^{-\lambda t_i^{\alpha}} \right)$$
$$\times \prod_{t_i \in \Omega_2} \left(\alpha\, \lambda\, t_i^{\alpha-1}\, e^{-\lambda t_i^{\alpha}}\, e^{-\frac{t_i^2}{2\sigma^2}} + \frac{t_i}{\sigma^2}\, e^{-\frac{t_i^2}{2\sigma^2}}\, e^{-\lambda t_i^{\alpha}} \right) \times \left(e^{-\lambda \tau^{\alpha}}\, e^{-\frac{\tau^2}{2\sigma^2}} \right)^{n-n_2}$$

Therefore the log-likelihood function can be derived as in the following form:

$$\log L = l = r_{11}\log\alpha + r_{11}\log\lambda + (\alpha-1)\sum_{t_i\in\Omega_{11}}\log(t_i) - \lambda\sum_{t_i\in\Omega_{11}} t_i^{\alpha} - \frac{1}{2\sigma^2}\sum_{t_i\in\Omega_{11}} t_i^2 +$$

$$+\sum_{t_i\in\Omega_{12}}\log(t_i) - r_{12}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{t_i\in\Omega_{12}} t_i^2 - \lambda\sum_{t_i\in\Omega_{12}} t_i^{\alpha} +$$

$$+\sum_{t_i\in\Omega_2}\log\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right) - \lambda\sum_{t_i\in\Omega_2} t_i^{\alpha} - \frac{1}{2\sigma^2}\sum_{t_i\in\Omega_2} t_i^2 +$$

$$- r_3\left[\lambda\tau_m^{\alpha} - \frac{\tau_m^2}{2\sigma^2}\right]$$

To estimate the unknown parameters, we need the first partial derivations of the log-likelihood function with respect to α, λ and σ^2 respectively as the following:

$$\frac{\partial l}{\partial\alpha} = \frac{r_{11}}{\alpha} + \sum_{t_i\in\Omega_{11}}\log(t_i) - \lambda\sum_{t_i\in\Omega_{11}} t_i^{\alpha}\log(t_i) - \lambda\sum_{t_i\in\Omega_{12}} t_i^{\alpha}\log(t_i) +$$

$$+\sum_{t_i\in\Omega_2}\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-1}\left(\alpha\lambda t_i^{\alpha-1}\log(t_i) + \lambda t_i^{\alpha-1}\right) - \lambda\sum_{t_i\in\Omega_2} t_i^{\alpha}\log(t_i) +$$

$$- r_3\lambda\tau_m^{\alpha}\log(\tau_m)$$

$$\frac{\partial l}{\partial\lambda} = \frac{r_{11}}{\lambda} - \sum_{t_i\in\Omega_{11}} t_i^{\alpha} - \sum_{t_i\in\Omega_{12}} t_i^{\alpha} + \sum_{t_i\in\Omega_2}\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-1}\left(\alpha t_i^{\alpha-1}\right) +$$

$$-\sum_{t_i\in\Omega_2} t_i^{\alpha} - r_3\tau_m^{\alpha}$$

$$\frac{\partial l}{\partial\sigma^2} = \frac{1}{\sigma^4}\sum_{t_i\in\Omega_{11}} t_i^2 - \frac{r_{12}}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{t_i\in\Omega_{12}} t_i^2 - \sum_{t_i\in\Omega_2}\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-1}\left(\frac{t_i^2}{\sigma^4}\right) +$$

$$+\frac{1}{2\sigma^4}\sum_{t_i\in\Omega_2} t_i^2 - r_3\left(\frac{\tau_m^2}{2\sigma^4}\right)$$

Setting $\frac{\partial l}{\partial\alpha}=0$, $\frac{\partial l}{\partial\lambda}=0$ and $\frac{\partial l}{\partial\sigma^2}=0$, we get the likelihood equations. These equations constitute a system of three nonlinear equations, that must be solved in α, λ

and σ^2 to get the *MLEs* of these parameters. It is obvious that the system of nonlinear equations has no closed form solutions. So, a numerical technique is required to get the estimates of the unknown parameters.

4-2 The approximateasymptotic variance-covariance matrix for *Type-II* censoring:

Since the *MLE* of the unknown parameters for *Type-II* censoring are not in closed form, it is not possible to derive the exact distributions of the *MLEs*. We obtained the approximate asymptotic variance-covariance matrix of the unknown parameters replacing $\theta$by the value $\hat{\theta}$. It is known that the asymptotic distribution of the *MLE* $\hat{\theta}$ is given by:

$$\left(\hat{\theta} - \theta\right) \rightarrow N_3\left(0, I^{-1}(\theta)\right),$$

where $I^{-1}(\theta)$ is theinverse of Fisher information matrix of the vector of unknown parameters θ. The elements of the 3×3matrix I^{-1}, $I_{iq}(\theta), i, q = 1, 2, 3,$ can be approximated by $I_{iq}\left(\hat{\theta}\right)$, given by:

$$I_{iq}\left(\hat{\theta}\right) = -\frac{\partial^2 l(\theta)}{\partial\theta_i\, \partial\theta_q}\Big|_{\theta=\hat{\theta}},$$

where:

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{r_{11}}{\alpha^2} - \lambda \sum_{t_i \in \Omega_{11}} t_i^{\alpha} \left(\log(t_i)\right)^2 - \lambda \sum_{t_i \in \Omega_{12}} t_i^{\alpha} \left(\log(t_i)\right)^2 +$$

$$+ \sum_{t_i \in \Omega_2} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2} \times \left(\begin{array}{l} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)\left(\alpha \lambda t_i^{\alpha-1}\left(\log(t_i)\right)^2 + 2\lambda t_i^{\alpha-1}\log(t_i)\right) \\ -\left(\alpha \lambda t_i^{\alpha-1}\log(t_i) + \lambda t_i^{\alpha-1}\right)\left(\alpha \lambda t_i^{\alpha-1}\log(t_i) + \lambda t_i^{\alpha-1}\right) \end{array} \right) +$$

$$- \lambda \sum_{t_i \in \Omega_2} t_i^{\alpha} \left(\log(t_i)\right)^2 - r_3 \lambda \tau_m^{\alpha} \left(\log(\tau_m)\right)^2$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \frac{\partial^2 l}{\partial \lambda \partial \alpha} = -\sum_{t_i \in \Omega_{11}} t_i^{\alpha} \log(t_i) - \sum_{t_i \in \Omega_{12}} t_i^{\alpha} \log(t_i) - \sum_{t_i \in \Omega_2} t_i^{\alpha} \log(t_i) - r_3 \tau_m^{\alpha} \log(\tau_m) +$$

$$+ \sum_{t_i \in \Omega_2} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2} \left(\begin{array}{l} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)\left(\alpha t_i^{\alpha-1}\log(t_i) + t_i^{\alpha-1}\right) \\ -\left(\alpha \lambda t_i^{\alpha-1}\log(t_i) + \lambda t_i^{\alpha-1}\right)\left(\alpha t_i^{\alpha-1}\right) \end{array} \right)$$

$$\frac{\partial^2 l}{\partial \alpha \partial \sigma^2} = \frac{\partial^2 l}{\partial \sigma^2 \partial \alpha} = \sum_{t_i \in \Omega_2} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2} \left(\left(\alpha \lambda t_i^{\alpha-1}\log(t_i) + \lambda t_i^{\alpha-1}\right)\left(\frac{t_i}{\sigma^4}\right)\right)$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{r_{11}}{\lambda^2} - \sum_{t_i \in \Omega_2} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2} \left(\alpha^2 t_i^{2\alpha-2}\right)$$

$$\frac{\partial^2 l}{\partial \lambda \partial \sigma^2} = \frac{\partial^2 l}{\partial \sigma^2 \partial \lambda} = \sum_{t_i \in \Omega_2} \left(\alpha \lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2} \left(\left(\alpha t_i^{\alpha-1}\right)\frac{t_i}{\sigma^4}\right)$$

$$\frac{\partial^2 l}{\partial(\sigma^2)^2} = -\frac{2}{\sigma^6}\sum_{t_i \in \Omega_{11}} t_i^2 - \frac{r_{12}}{\sigma^4} - \frac{2}{\sigma^6}\sum_{t_i \in \Omega_{12}} t_i^2 +$$

$$-\sum_{t_i \in \Omega_2}\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)^{-2}\left(\left(\alpha\lambda t_i^{\alpha-1} + \frac{t_i}{\sigma^2}\right)\frac{2}{\sigma^6} t_i^2 + \frac{1}{\sigma^4} t_i^2\left(\frac{2}{\sigma^4} t_i\right)\right) +$$

$$-\frac{1}{\sigma^6}\sum_{t_i \in \Omega_2} t_i^2 + r_3\frac{\tau_m^2}{\sigma^6}$$

Since the values of $I_{iq}\left(\hat{\theta}\right)$ are not in closed form,their properties could only be obtained through simulation studies.

5- The relative risk rates:

In this section we present the relative risk (R.R) rates due to the two causes. Kundu and Basu (2000) considered the relative risk rate due to the first cause, π_1, as:

$$\pi_1 = P\left(T_{i1} < T_{i2}\right) = \int_0^\infty f_1(t_i)\, S_2(t_i) dt$$

$$= \int_0^\infty \alpha\lambda t_i^{\alpha-1} e^{-\lambda t_i^\alpha - \frac{t_i}{\sigma^2}} dt$$

and the relative risk rate due to the second cause, π_2, as:

$$\pi_2 = P\left(T_{i1} > T_{i2}\right) = 1 - \pi_1$$

This integral has no closed analytical solution. So, numerical integration is required to get $\pi_1$and π_2.We obtained the maximum likelihood estimates of the relative risks by replacing the unknown parameters in the above relations with their maximum likelihood estimates.Details of this will be given shortly from Table 1 through Table 6, in the Appendix .

6- Simulation study:

In this section we present some simulations results to estimate the parameters included in independent competing risks in the presence of complete, incomplete and censored data according to *Type-II* censoring from some of Weibull models. These results show the behavior of different sample sizes and also different parametric values. We usedthe *MathCad-14* package for random number generations. We considered the case when the competing risks (two causes) have the two-parameter Weibull distribution and Rayleigh distribution respectively. We mainly observed the behavior of the *MLEs* in terms of their biases and in terms of their variances.

In this section the results when censored observation obtained according to *Type-II* censoring are obtained. The simulation experiments are studied according to the following steps:

1- We took sample sizes n= 25, 30, 40, 50 and 100.
2- The number of censored data is fixed, namely m, and thetime of failure forcensored data is random variable, namely τ_m.
3- The samples are drawn randomly for different values of n and different parameters of Weibull distribution and Rayleigh distribution, the same sample size.
4- The minimum lifetime value from the two distributions is selected to create a new sample.
5- The different values of the different parameters, for generated samples, were selected to allow for actual competition between the two causes. The curves of the p.d.f , c.d.fand h.fof the Weibull distributionand Rayleigh distribution for the competing area with different values of parameters are shownin Figure 1 through Figure 6.
6- The *MLEs* of α , λ and σ^2 for a new sample are computed.
7- The process are replicated ten thousand times, N=10000.

8- The average values of *MLEs*, their biases, their relative risks and the variance-covariance matrix are computed.

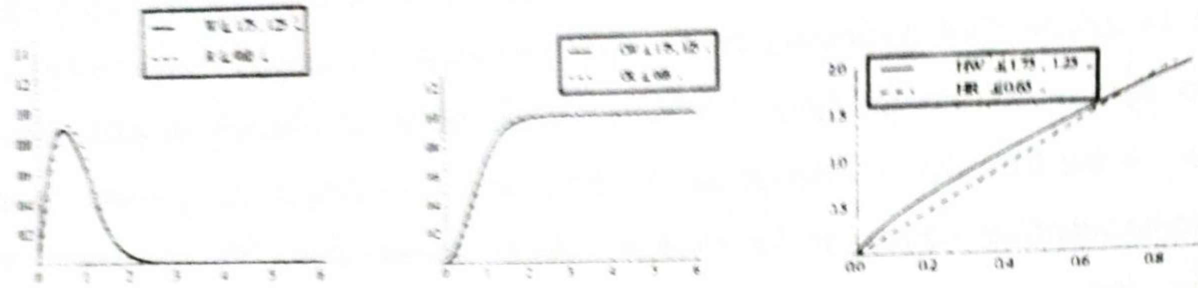

Figure (1): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 1.75$, $\lambda = 1.25$ and Rayleigh distribution at $\sigma = 0.65$

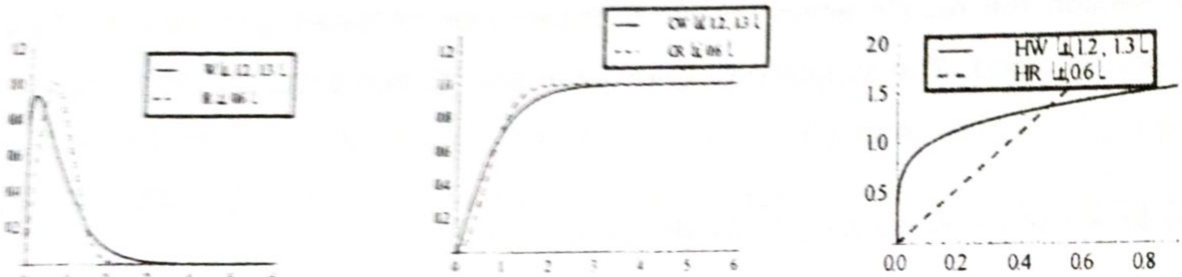

Figure (2): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 1.2$, $\lambda = 1.3$ and Rayleigh distribution at $\sigma = 0.6$

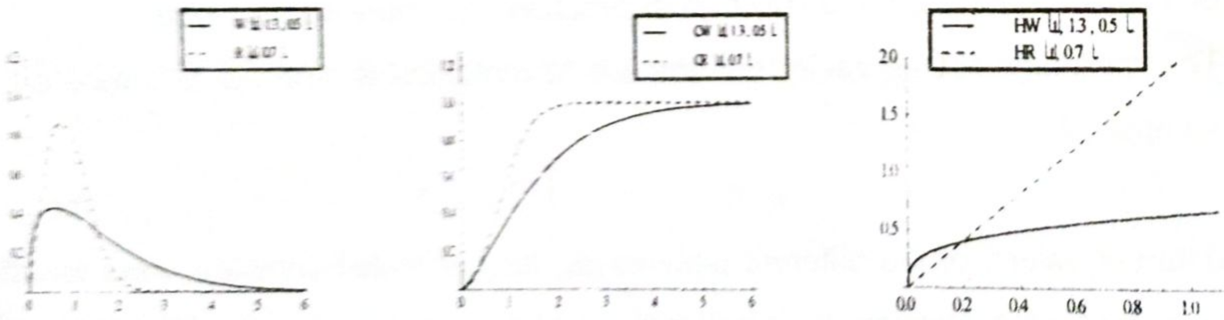

Figure (3): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 1.3$, $\lambda = 0.5$ and Rayleigh distribution at $\sigma = 0.7$

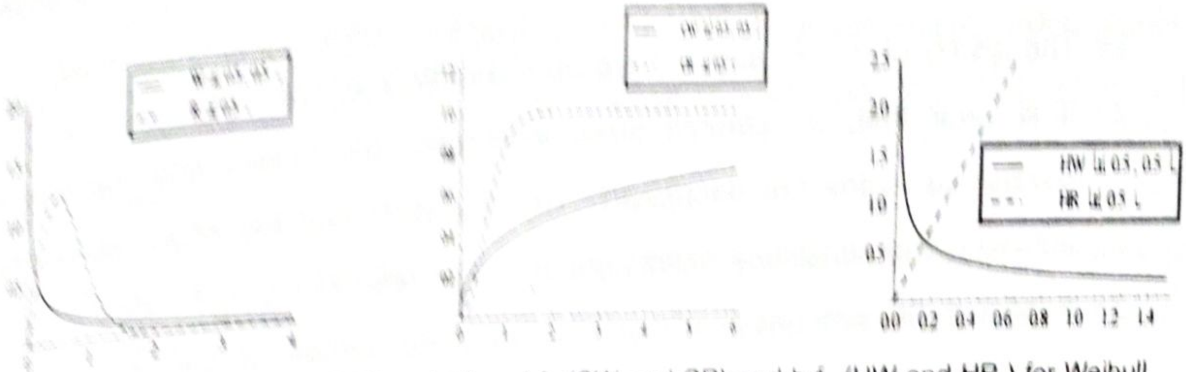

Figure (4): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 0.5$, $\lambda = 0.5$ and Rayleigh distribution at $\sigma = 0.5$

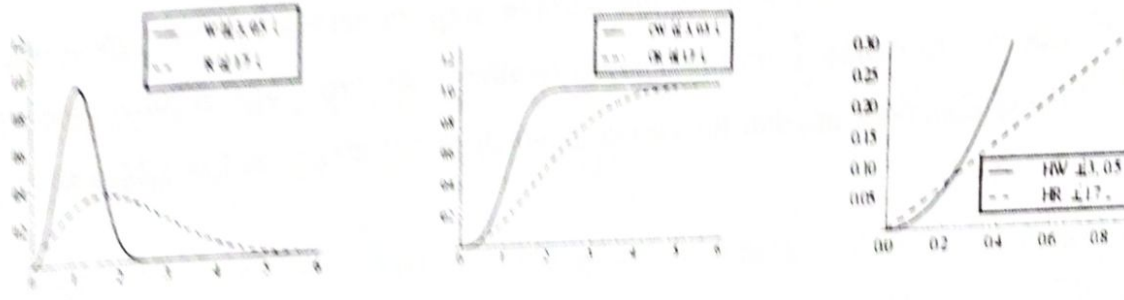

Figure (5): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 3$, $\lambda = 0.5$ and Rayleigh distribution at $\sigma = 1.7$

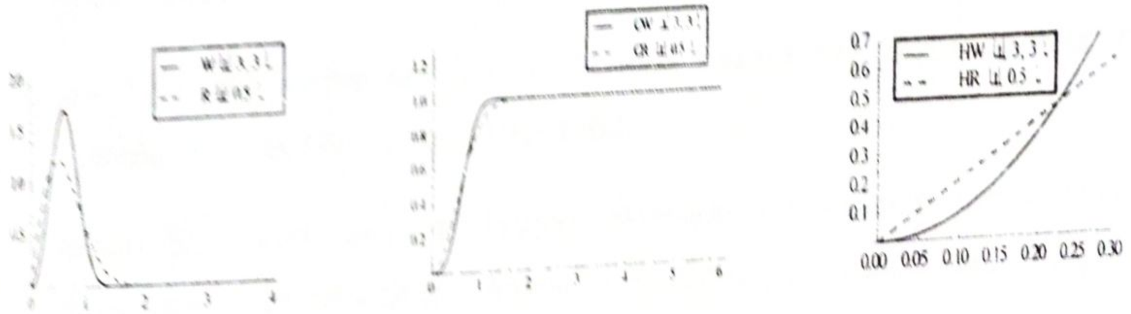

Figure (6): Curves of the p.d.f (W and R), c.d.f (CW and CR) and h.f. (HW and HR) for Weibull distribution at $\alpha = 3$, $\lambda = 3$ and Rayleigh distribution at $\sigma = 0.5$

Conclusions:

In this paper, we have introduced estimators of the parameters included in independent competing risks in the presence of complete, incomplete and *Type-II* censored data.Based on the results reported in Table 1 through Table 6, it is observed that:

1- The results of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}^2$ are approximately similar in nature.

2- It is clear that, as sample sizes increases, the biases and the variances covariance matrix are decreases. This suggests that the *MLEs* asymptotically unbiased and consistent estimators of the corresponding parameters.

3- It should be emphasized that, the selected values of the parameters for generated samples which were selected according to the competing area are very important to obtain.

4- Also, it is observed that as the sample size increases the theoretical relative risk due to cause 1 and simulated relative risks are close to each other. The same comment applies for cause 2 which is not shown in the tables since (π_1 + π_2)= 1

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APPENDIX

Table (1)

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 1.75$, $\lambda = 1.25$ and $\sigma = 0.65$, where $\pi_1 = 0.529$.

n *m*	*Estimated value*			*R.R*	*Bias*			*Variance-Covariance*
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 5	1.858	1.366	0.682	0.568	0.108	0.116	0.032	0.313 0.236 −0.9E − 3 0.609 8.527E − 4 0.016
30 6	1.845	1.351	0.676	0.562	0.095	0.101	0.026	0.25 0.174 −1.54E − 3 0.346 9.242E − 4 0.013
40 8	1.82	1.315	0.669	0.551	0.07	0.065	0.019	0.171 0.108 −1.25E − 3 0.207 4.425E − 4 9.965E − 3
50 10	1.808	1.301	0.665	0.547	0.058	0.051	0.015	0.132 0.084 −2.6E − 4 0.152 9.211E − 4 6.806E − 3
100 20	1.776	1.237	0.658	0.538	0.026	0.023	0.007	0.06 0.036 −2.4E − 4 0.063 4.835E − 4 3.34E − 3

Table (2)

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 1.2$, $\lambda = 1.3$ and $\sigma = 0.6$, where $\pi_1 = 0.554$.

n m	Estimated value			R.R	Bias			Variance-Covariance
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 5	1.273	1.413	0.637	0.594	0.073	0.113	0.037	0.135 0.117 -1.855E − 3 0.428 1.187E − 3 0.021
30 6	1.257	1.382	0.631	0.585	0.057	0.082	0.031	0.101 0.097 -1.59E − 4 0.286 3.447E − 3 0.013
40 8	1.241	1.357	0.621	0.575	0.041	0.057	0.021	0.07 0.063 1.759E − 5 0.153 1.421E − 3 8.948E − 3
50 10	1.234	1.352	0.617	0.573	0.034	0.052	0.017	0.085 0.031 1.705E − 4 0.12 1.755E − 3 6.734E − 3
100 20	1.216	1.322	0.608	0.562	0.016	0.022	0.008	0.026 0.022 2.011E − 4 0.061 7.427E − 4 2.225E − 3

Table (3)

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 1.3$, $\lambda = 0.5$ and $\sigma = 0.7$, where $\pi_1 = 0.333$.

n m	Estimated value			R.R	Bias			Variance-Covariance
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 5	1.416	0.512	0.723	0.348	0.116	0.012	0.023	0.271 0.008 -1.94E − 3 0.041 9.25E − 4 9.52E − 3
30 6	1.431	0.522	0.72	0.352	0.131	0.022	0.02	0.264 0.025 -9.68E − 4 0.042 7.99E − 4 9.09E − 3
40 8	1.386	0.517	0.715	0.347	0.086	0.017	0.015	0.177 0.018 -6.55E − 5 0.031 6.014E − 4 6.517E − 3
50 10	1.369	0.511	0.712	0.343	0.069	0.011	0.012	0.124 0.014 -2.71E − 4 0.022 4.036E − 4 5.239E − 3
100 20	1.339	0.505	0.706	0.338	0.039	0.005	0.0059	0.054 0.006 6.707E − 7 0.01 1.683E − 4 2.306E − 3

Table (4)

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 0.5$, $\lambda = 0.5$ and $\sigma = 0.5$, where $\pi_1 = 0.313$.

n m	*Estimated value*			*R.R*	*Bias*			*Variance-Covariance*
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 5	0.567	0.532	0.513	0.324	0.067	0.032	0.013	0.079 0.013 −6.9E−5 0.045 6.056E−4 4.409E−3
30 6	0.556	0.525	0.512	0.322	0.056	0.025	0.012	0.04 0.01 2.48E−5 0.036 4.807E−4 3.782E−3
40 8	0.544	0.524	0.51	0.322	0.044	0.024	0.0031	0.028 0.009 3.619E−5 0.029 3.647E−4 3.35E−3
50 10	0.531	0.517	0.509	0.32	0.031	0.017	0.009	0.02 0.007 1.141E−4 0.022 3.81E−4 2.674E−3
100 20	0.516	0.509	0.505	0.317	0.016	0.009	0.005	0.008 0.003 9.78E−5 0.011 1.39E−4 1.34E−3

Table (5)

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 3$, $\lambda = 0.5$ and $\sigma = 1.7$, where $\pi_1 = 0.791$.

n m	*Estimated value*			*R.R*	*Bias*			*Variance-Covariance*
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 5	3.029	0.497	1.893	0.826	0.029	-0.003	0.193	0.368 −0.04 −8.35E−3 0.02 0.011 0.63
30 6	3.051	0.496	1.878	0.824	0.051	-0.004	0.178	0.414 −0.033 −6.73E−3 0.018 8.265E−3 0.55

40 8	3.042	0.497	1.816	0.813	0.042	-0.003	0.116	0.304 -0.024 3.053E − 3 0.013 3.894E − 3 0.238
50 10	3.036	0.496	1.79	0.809	0.036	-0.004	0.09	0.245 -0.019 2.66E − 3 0.01 3.211E − 3 0.168
100 20	3.007	0.498	1.74	0.799	0.007	-0.002	0.04	0.124 -0.009 3.74E − 3 0.004 1.44E − 3 0.037

Table 6

The average values of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\sigma}$, their average bias, covariances and their relative risks under *Type-II* censoring at $\alpha = 3, \lambda = 3$ and $\sigma = 0.5$, where $\pi_1 = 0.484$.

n *m*	*Estimated value*			*R.R*	*Bias*			*Variance-Covariance*
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\pi}_1$				
25 2	3.125	3.544	0.515	0.522	0.125	0.544	0.015	0.748 1.77 -1.47E − 3 8.675 -0.01 8.404E − 3
30 3	3.124	3.535	0.513	0.52	0.124	0.535	0.013	0.65 1.515 -2.63E − 3 7.112 -8.13E − 3 8.283E − 3
40 4	3.088	3.337	0.51	0.509	0.088	0.337	0.01	0.451 0.903 -2.22E − 3 3.409 -6.39E − 3 3.84E − 3
50 5	3.085	3.281	0.508	0.503	0.085	0.281	0.008	0.356 0.835 -1.39E − 3 2.207 -2.16E − 3 3.039E − 3
100 10	3.032	3.104	0.504	0.493	0.032	0.104	0.004	0.16 0.282 -1.5E − 5 0.712 8.722E − 4 1.403E − 3