A STUDY ON MEAN RESIDUAL LIFE AND FAILURE RATE FUNCTIONS FOR EXPONENTIATED WEIBULL DISTRIBUTION

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Abstract

The mean residual life (MRL) and failure rate (FR) functions are two important characteristics in reliability analysis. In this paper, a study of the MRL and FR of the exponentiated Weibull distribution (EW), with bathtub-shaped failure rate (BFR), is introduced. The comparison between MRL and FR functions is described and the changing points are obtained.

Key Words: *Exponentiated Weibull distribution, Failure Rate Function, Mean Residual Life, Bathtub Curve, Changing points.*

1 Introduction

The mean residual life (MRL) and failure rate (FR) functions have been widely used in fields of reliability, statistics, and insurance. In literature, many useful results have been derived in various aspects of the MRL and FR, such as the properties, the shape, the estimation, and the application etc.

For an item that has survived a period of time, its residual life is defined as a random variable conditioning on the time it has experienced. This measure contains two aspects of information, the lifetime of an item and the fact that this item has been working for some time period without failure. Because of its dual characters, residual life is widely applied in reliability engineering [see Salvia

(1996)]. A recent and detailed review of the MRL in reliability analysis was presented by Shen et al. (2009).

MRL function as well as FR function is very important since each of them can be used to determine a unique corresponding and descriptors for the life time distribution. This is because these two characteristics are closely related to each other and the comparison between them is helpful in decision-making and estimation.

The relationship between the behavior of the MRL and FR functions of a distribution was studied by many authors. Aarset (1987) discussed how to identify a bathtub failure rate. In (1994), Ruiz and Navarro discussed characterization of distributions by relationships between FR and MRL. Mi (1995) showed that the MRL of a component is an upside-down bathtub shape on condition that FR is bathtub-shaped with a unique change point. Besides, Salvia (1996) has discussed situations on discrete MRL. Ghai and Mi (1999) derived sufficient conditions for the unimodal MRL to imply the bathtub-shaped failure rate (BFR) function. Tang et al. (1999) summarized and discussed the general characteristics and results of the MRL and FR functions for both continuous and discrete lifetime distributions. Nassar and Eissa (2003) obtained the general formula for the MRL function and find the relationship between MRL and FR functions graphically. Xie et al. (2004) investigated the difference of the change points of MRL with bathtub shape and FR function with upside-down bathtub shape for some generalized Weibull distributions and found the relation to the flatness of the bathtub curve. Tang (2004) found the change points of MRL and FR functions for extended Weibull distributions with bathtub shaped failure rate. Bebbington et al. (2006) proposed using the curvature of FR function to evaluate the length of the useful period for a bathtub curve of the additive Weibull. Shen et al. (2009) discussed a model for upside-down bathtub-shaped MRL and its properties. Shen (2009) studied MRL and FR functions in reliability analysis and modeling, and focused on the relations between MRL and FR functions by studying the effect of the change of one characteristic on the other characteristics, and he obtained the relationship between change points for two functions for systems (parallel-series) and components.

This paper consists of six sections. Various classes of MRL and FR functions are defined in Section (2). In Section (3), the FR and MRL for exponentiated Weibull distribution (EW) with three parameters are discussed. The relationship between MRL and FR functions is described graphically in Section (٤). In Section (5), the change points of MRL and FR functions for EW distribution with BFR and upside-down bathtub-shaped MRL are investigated and the relationship between change points and parameters are found numerically. Finally, Section (6) is a conclusion.

2 Mean Residual Life and Failure Rate Function Classes

Assume that the random variable T follows an EW distribution. The cumulative distribution function (cdf), is given by

$$F(t;\theta,\alpha,\lambda) = [1 - e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta} \quad ; \quad t > 0, \theta, \alpha, \lambda > 0$$

(1)

where $(\theta, \alpha) > 0$ denote the shape parameters and $\lambda > 0$ is a scale parameter.

The reliability function is given by

$$R(t;\theta,\alpha,\lambda) = 1 - [1 - e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta} \quad ; \quad t,\theta,\alpha,\lambda > 0$$

(2)Different MRL classes describe different aging properties. In general, the MRL classes can be divided into two groups based on the behavior of the MRL function: monotonic and non-monotonic. The monotonic aging classes for the MRL function include distributions with decreasing mean residual life (DMRL) and with increasing mean residual life (IMRL). The non-monotonic MRL classes have much more types of distributions. Some known classes are upside-down bathtub-shaped MRL (UBMRL), bathtub-shaped MRL (BMRL). As the MRL function is closely related to the FR function, the MRL classes are also linked to the classes defined via the FR function, such as increasing failure rate (IFR), decreasing failure rate (DFR), bathtub-shaped failure rate (BFR) and upside-down bathtub-shaped failure rate (UBFR) [Nassar and Eissa (2003) , Shen (2009)].

3 Failure Rate and Mean Residual Life Functions

The FR function for EW distribution is given by

$$Z(t;\theta,\alpha,\lambda) = \frac{\theta\alpha\,[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta-1}\,e^{-(\frac{t}{\lambda})^{\alpha}}\,(\frac{t}{\lambda})^{\alpha-1}}{\lambda\,[1-[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta}]} \quad ; \quad t,\theta,\alpha,\lambda > 0$$

(3)

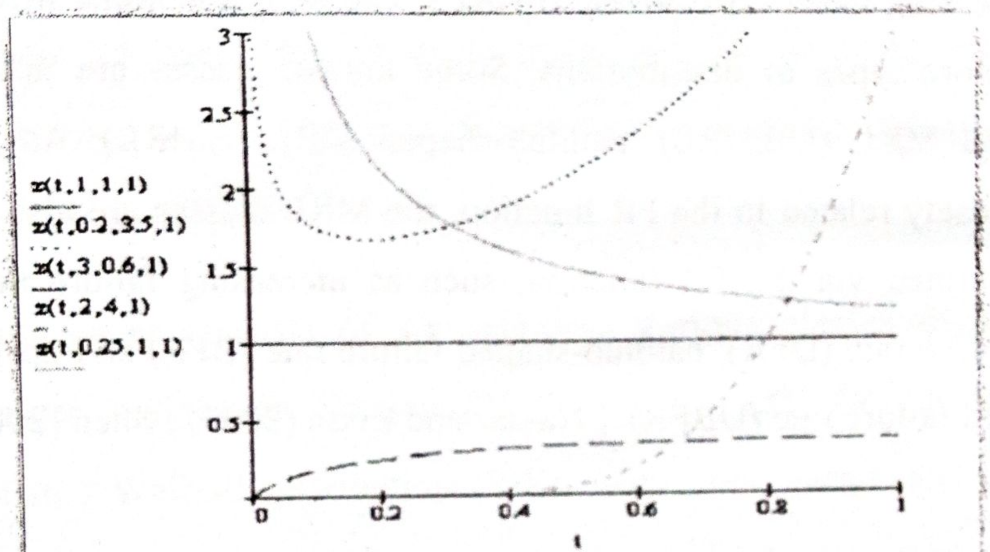

Figure (1): Failure rate function

One of the main properties of the EW distribution is that it can has different types of failure rate shapes. As it can be observed in Figure (1), when $\theta = \alpha = 1$ the FR function remains constant during the life time; if $\alpha \geq 1$ and $\alpha\theta \geq 1$ the FR function is monotone increasing; FR function is monotone decreasing for $\alpha \leq 1$ and $\alpha\theta \leq 1$; if $\alpha > 1$ and $\alpha\theta < 1$ we get the bathtub-shaped failure rate (BFR) function; upside-down bathtub-shaped failure rate (UBFRF) if $\alpha < 1$ and $\alpha\theta > 1$.

The mean residual life function is

$$MRL(t;\theta,\alpha,\lambda) = \frac{1}{1-[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta}} \int_t^{\infty} 1-[1-e^{-(\frac{x}{\lambda})^{\alpha}}]^{\theta}\,dx$$

(4)

For positive values of θ, we can write (4) in the form

$$MRL(t;\theta,\alpha,\lambda)$$

$$= \theta \sum_{i=0}^{\theta-1} \frac{(-1)^i \binom{\theta-1}{i}}{(i+1)R(t)} \left\{ \lambda \Gamma\left(\frac{1}{\alpha}+1\right)\left[(i+1)^{-1/\alpha} - (i+1)^{-1}\Gamma(i+1)\left(\frac{t}{\lambda}\right)^{\alpha}\left(\frac{1}{\alpha}\right)\right] + te^{-(i+1)\left(\frac{t}{\lambda}\right)^{\alpha}} \right\} - t \tag{5}$$

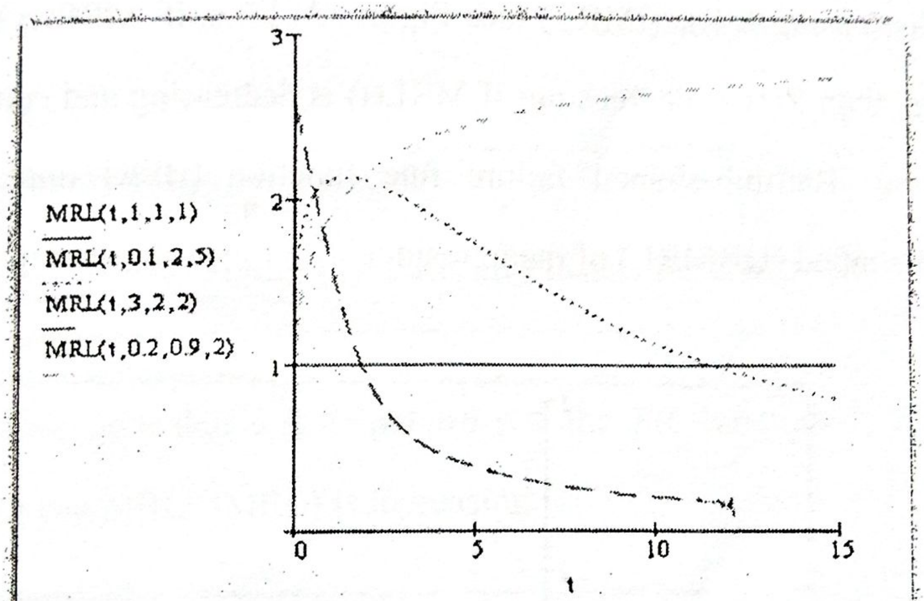

Figure (2): Mean residual life function

Figure (2) illustrates to shape of some mean residual life for EW where (θ, α) are the shape parameters and (λ) is a scale parameter. when $\theta = \alpha = 1$ the mean residual remains constant during the life time; if $\alpha \geq 1$ and $\alpha\theta \geq 1$ the mean residual function is monotone decreasing; mean residual function is monotone increasing for $\alpha \leq 1$ and $\alpha\theta \leq 1$; if $\alpha > 1$ and $\alpha\theta < 1$ we get the BFR function; UBMRL if $\alpha > 1$ and $\alpha\theta < 1$.

4 Mean Residual Life Relation with Failure Rate Function

The MRL function and the FR function are two important measures used to describe failure times. Although these two functions depict aging behaviors in different ways, both of them are in fact equivalent to the reliability function in the sense of probability; moreover, the characteristic of one function is always related to that of the other [for more details see Tang (2004)].

Relationship between the FR and MRL functions has been investigated by several authors. It was shown that the FR function is increasing (IFR) imply decreasing MRL (DMRL) (see Figure (3)) and decreasing FR function (DFR) imply increasing MRL (IMRL) (see Figure (4)), i.e. if MRL(t) is increasing and concave, then Z(t) is decreasing; if MRL(t) is decreasing and convex, then Z(t) is increasing. Bathtub-shaped failure rate function (BFR) implies upside-down bathtub-shaped (UBMRL) of mean residual life function (see Figure (5)).

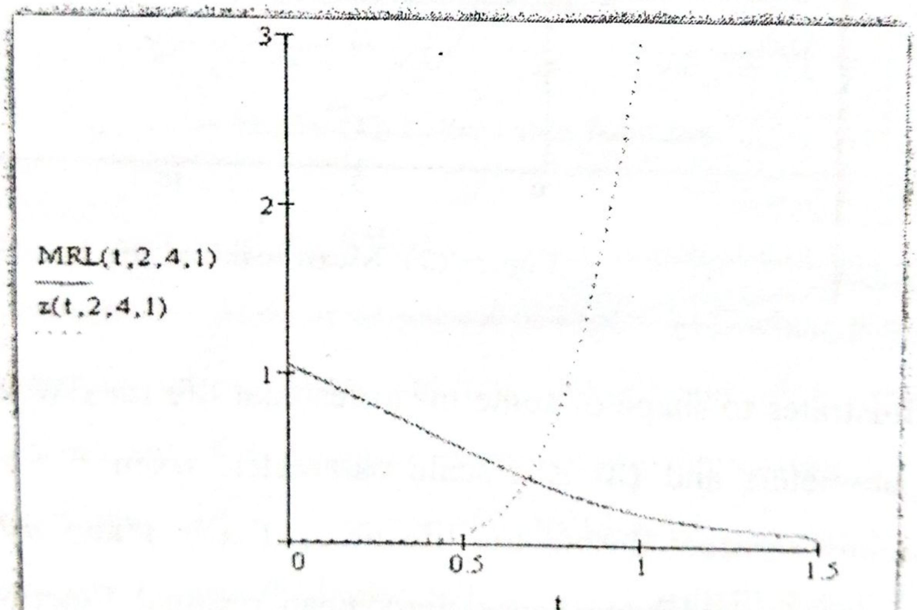

Figure (3): IFR and DMRL for EWD

Figure (3) illustrates FR and MRL for $\alpha \geq 1$ and $\alpha\theta \geq 1$ the FR is monotone increasing (IFR) and MRL is decreasing (DMRL).

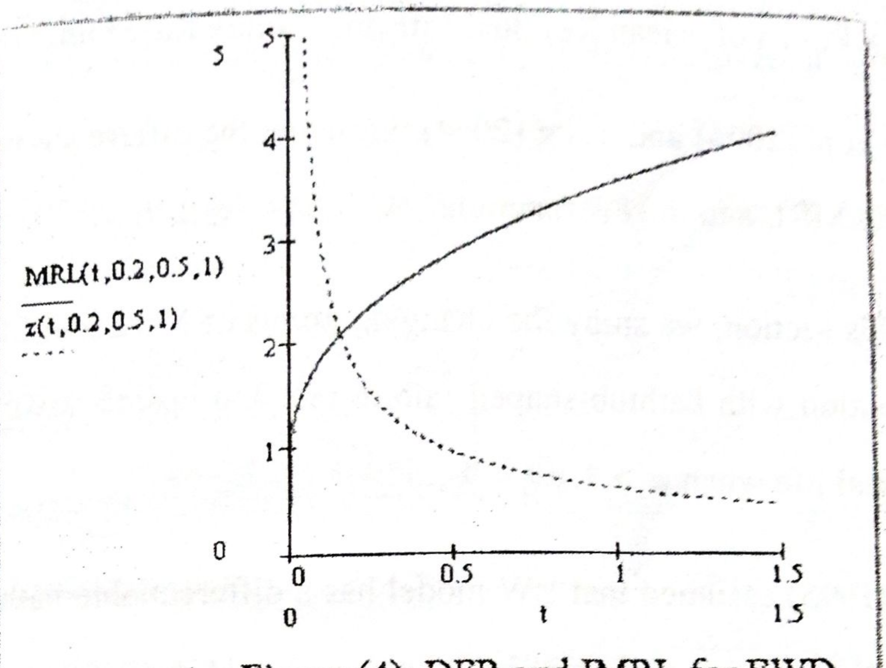

Figure (4): DFR and IMRL for EWD

From figure (4) we note that $\alpha \leq 1$ and $\alpha\theta \leq 1$ the FR function is monotone decreasing (DFR) and MRL (IMRL) is increasing.

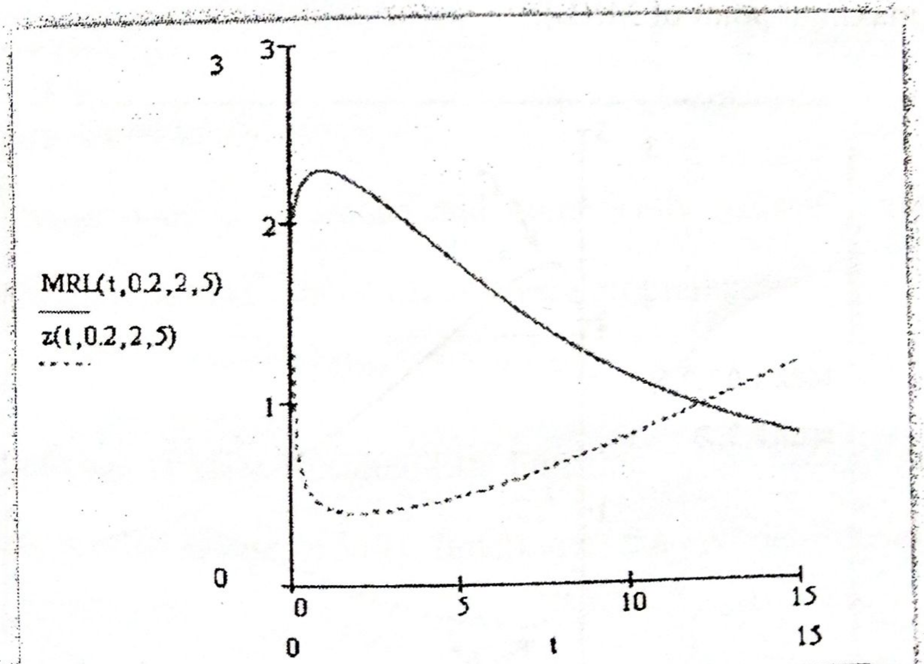

Figure (5): BFR and UBMRL for EWD

Figure (5) illustrates Bathtub-shaped failure rate function and upside-down mean residual life function if $\alpha > 1$ and $\alpha\theta < 1$.

5 Change Points of Mean Residual Life and Failure Rate Functions

Xie et al.(2004) and Tang (2004) discussed the difference between change points of the MRL and the FR functions.

In this section, we study the changing points of MRL and FR functions for EW distribution with bathtub-shaped failure rate and upside-down bathtub shape mean residual life when $\alpha > 1, \alpha\theta < 1$.

Mi (1995) assumed that EW model has a differentiable bathtub shaped FR function with a unique change point b^* where $0 < b^* < \infty$, the mean residual life has an upside-down bathtub shape with a unique change point t^* where $0 < t^* < b^*$. (see Figure (6)), b^* denotes the change point of the FR curve (minimal FR of the bathtub curve); t^* denotes the change point of the MRL function (maximal point of MRL).

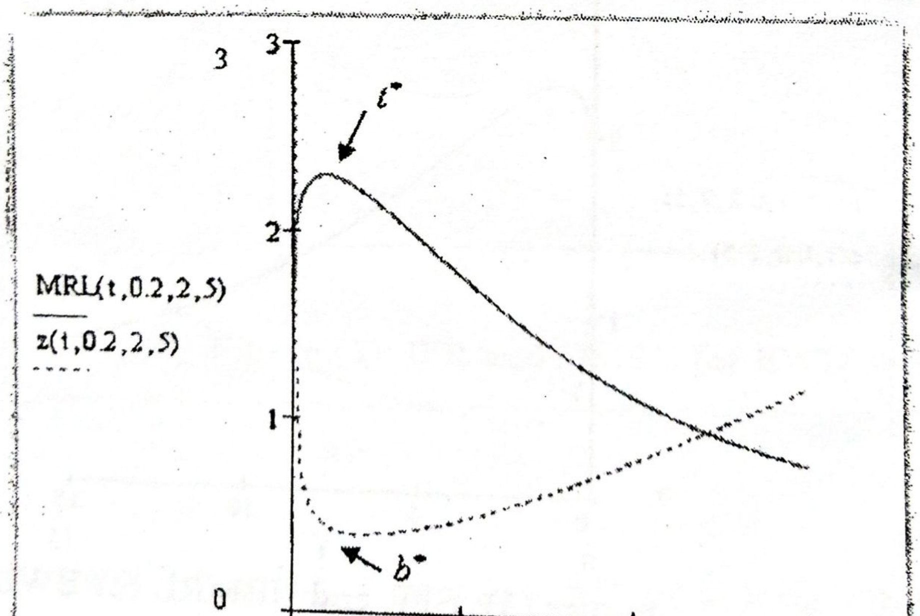

Figure (6): Maximum and minimum point

5.1 Change Points of Failure Rate Function

The EW distribution has cdf given in (1), FR function given in (3) and MRL function given in (4).

In order to study the change point, take the derivative of the FR function of the EW distribution.

$$\frac{dZ(t)}{dt} = \frac{d}{dt}\left[\frac{\theta\alpha\,[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta-1}\,e^{-(\frac{t}{\lambda})^{\alpha}}\,(\frac{t}{\lambda})^{\alpha-1}}{\lambda\,[1-[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta}]}\right]$$

Let $y = \left[1 - e^{-(\frac{t}{\lambda})^{\alpha}}\right]$.

The above derivative implies that solving the equation of $\frac{dZ(t)}{dt} = 0$ as

$$\alpha(\theta-1)t^{\alpha}e^{-(\frac{t}{\lambda})^{\alpha}} + (\alpha-1)y + \alpha y^{\theta}t^{\alpha}e^{-(\frac{t}{\lambda})^{\alpha}} + \alpha y^{\theta+1}t^{\alpha} - (\alpha-1)y^{\theta+1} - \alpha y t^{\alpha} = 0 \qquad (6)$$

The change point of FR function is given by

$$b^{*} = \lambda[-\ln(1-(y_1)^{\theta})]^{\frac{1}{\alpha}}$$

where y_1 is the solution to (6).

The change point b^{*} is unique and numerically traceable, and it can be obtained numerically with the statistical software programming.

5.2 Change Points of Mean Residual Life Function

Similarly, take the derivatives of MRL function of the EW distribution to find the change point t^{*}.

$$\frac{d\,MRL(t)}{dt} = \frac{d}{dt}\left[\frac{1}{1-[1-e^{-(\frac{t}{\lambda})^{\alpha}}]^{\theta}}\int_{t}^{\infty} 1-[1-e^{-(\frac{x}{\lambda})^{\alpha}}]^{\theta}\,dx\right]$$

Let $y = \left[1 - e^{-(\frac{t}{\lambda})^{\alpha}}\right]$.

The change point can be obtained by the equation of $\frac{dMRL(t)}{dt} = 0$ as

$$\frac{d\,MRL(t)}{dt} = \frac{d}{dt}\left[\frac{1}{1-[1-e^{-\left(\frac{t}{\lambda}\right)^{\alpha}}]^{\theta}}\int_{t}^{\infty} 1-[1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}]^{\theta}\,dx\right] = 0$$

(7)

Then the change point of MRL function is given by

$$t^{*} = \lambda[-\ln(1-(y_2)^{\theta})]^{\frac{1}{\alpha}}$$

where y_2 is the solution to (7).

The change point t^{*} is unique and numerically traceable, and it can be obtained numerically with the aid of statistical software programming.

5.3 Numerical Studies on Change Points

Define the absolute deviation between the change points of FR and MRL functions as $D = |b^{*} - t^{*}|$, relative deviation in the change points of FR function as $d\_FR = \frac{D}{b^{*}}$ and relative deviation in the change points of MRL function is $d\_MRL = \frac{D}{t^{*}}$.

The parameters are restricted with the conditional region that $\alpha > 1, \alpha\theta < 1$ which gave the BFR and MRL functions.

The regions of parameters are as follows:

- Fixing α ,λ and changing : $0 < \theta < 1$.
- Fixing θ ,α and changing λ: $\lambda > 0$.

Table (1) summarizes the numerical results of change points when we fix two parameters and vary the remaining parameter.

Table (1): Change points of FRF and MRL

θ	α	λ	b^*	t^*	D	d-FRF	d-MRL
0.1	0.5	100	1080	253.785	826.215	0.765	3.256
0.2	0.5	100	668.95	100.155	568.795	0.850	5.679
0.3	0.5	100	472.670	47.377	425.293	0.899	8.977
0.4	0.5	100	353.053	23.717	329.336	0.933	13.886
0.5	0.5	100	271.482	11.877	259.605	0.956	21.858
0.6	0.5	100	212.094	5.680	206.414	0.973	36.340
0.7	0.5	100	166.966	2.440	164.526	0.985	67.429
0.1	0.5	10	237.780	17.112	220.668	0.928	12.896
0.1	0.5	20	347.482	41.588	305.894	0.880	7.355
0.1	0.5	100	1080	253.785	826.215	0.765	3.256
0.1	0.5	150	1509	389.580	1119.420	0.742	2.873
0.1	0.5	200	1931	526.349	1404.651	0.727	2.669
0.1	0.5	250	2348	663.778	1684.222	0.717	2.537

From Table (1), we have the following conclusions:

The change points of FR and MRL are highly related to the selection of parameters. The minimal FR point is higher than the maximal MRL point i.e. $(0 < b^* < \alpha\ ,\ 0 < t^* < b^*)$. Change points of FR and MRL are decreasing as shape parameter θ increases and increasing as scale parameter λ increases.

The absolute difference between changing points is decreasing when increasing shape parameter θ. The relative deviation in changing points of FR and MRL functions are increasing when increasing shape parameter θ.

The absolute difference is greater when increasing the scale parameter λ. The relative deviation in changing points of FR and MRL functions are decreasing when increasing the scale parameter λ.

The change points are believed to be closely related to the bathtub curve. From the investigation, if the difference between the change points of failure rate and mean residual life functions is large, the corresponding bathtub curve tends to be flatter.

6 Conclusion

In this paper, we studied EW distribution with BFR and UBMRL functions, dealt with an interesting issue by studying the relationship between FR and MRL functions. The FR and MRL functions take different types of shape, and they are closely related to each other; if the FR function is increasing then the MRL function through the time is decreasing when $\alpha \geq 1$ and $\alpha\theta \geq 1$, both functions are constant for $\theta = \alpha = 1$, if the FR function is decreasing then the MRL function is increasing when $\alpha \leq 1$ and $\alpha\theta \leq 1$, if the FR function has bathtub shape then the MRL function has upside-down bathtub shaped when $\alpha > 1$ and $\alpha\theta < 1$.

Furthermore, we discussed the change points of two functions. . In fact, the difference of change points of FR and MRL functions is shown to be indicative of the flatness of bathtub curve. The difference is important from a number of points of view. We note that the change points are smaller when the shape parameter increases or the scale parameter decreases.

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