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On Oscillatory Behavior of the Nonlinear Differential Equation with Deviating arguments

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Abstract: The purpose of the paper is to study nonlinear oscillation generated by general deviating arguments and to study on the asymptotic behavior of its solution the results obtained are generalization of some recent papers.

1. Introduction:

In the asymptotic theory on n^{th} order (n>1) nonlinear differential equation, an interesting problem is that the study of solutions with prescribed asymptotic behavior via solution of the equation, $x^{(x)}(t) = 0$. This problem has been extensively investigated during the last four decades for the case of second order nonlinear differential equations; see [2,4,5 ,6,and 8].

Recently, Agawal and Grace [1], studied the superliner differential equation of the special form

$$x^{(n)}(t) + q(t)|x(t)|^{\lambda}\, sign\, x(t) = e(t) \quad where\, q(t) < 0\, and\;\; \lambda > 1$$

Taksik and Hirotthi [9], studied the solutions of the equation

$$x^{(n)}(t) + q(t)|x(g(t))|^{\lambda}\, sign\, x\big(g(t)\big) = 0 \qquad 0 < \lambda < 1 \tag{*}$$

Which are oscillatory when n is even and are either oscillatory or strongly monotonic when n is odd, they investigated second order delay of equation (*), with n=2 and is the λ ratio odd positive number $\lambda < 1$.

In this paper we are concerned with the oscillatory behavior of the nonlinear differential equation with deviating arguments of the form

$$x^{n}(t) = f\left(t, x(g_1(t)), \ldots . x(g_m(t)), \ldots . x^{n-1}(g_1(t)), \ldots . x^{n-1}(g_m(t))\right) \tag{I}$$

We always assume that the functions $g_j : R_+ \to R,\ j = 1, \ldots . m;\ R_+ = [0, \infty)$ are continuous, $g_j(t) \le t\ for\ t \in R_+$ and $lim_{t\to\infty} g_j(t) = +\infty$, and the function

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$f: R_+ \times R^{mn} \to R$ Satisfies either the conditions

$$-f(t, u_{11}, \dots, u_{1m}, \dots, u_{n1}, \dots, u_{nm}) \, sign\, u_{11} \geq a(t) \prod_{i=1}^{n} |u_{1i}|^{\lambda_i} \quad (2)$$

Or $$f(t, u_{11}, \dots, u_{1m}, \dots, u_{n1}, \dots, u_{nm}) \, sign\, u_{11} \geq a(t) \prod_{i=1}^{m} |u_{1i}|^{\lambda_i} \quad (3)$$

For $t \in R$, $u_{ij} \in R$, $u_{1j}\, u_{11} > 0, i = 1, \dots, m;\ j = 1, \dots m$ where

$a: R_+ \to R_+$, $\lambda = \sum_{j=1}^{m} \lambda_j > 1$, $\lambda_j \geq 0$, $j = 1, 2, \dots, m$

We also assume that

$g_*(t) = inf\{s: g_i(\xi) \geq t \ for\ \xi \geq s,\ i = 1, \dots, m\}$

Let $t_0 \in R_+$. A function $x: [t_0, \infty) \to R$ well be called a regular solution of (1), if it is absolutely continuous together with the $x^{(i)}$, $i = 1, \dots, n-1$, on each finite segment of R_+ it satisfies (1) for almost all $t \in (g_*(t_0), \infty)$ and

$sup\{|x_i(s)|: s \in [t, \infty)\} > 0 \ for \ t \in [t_0, \infty)$

A regular solution of (1) will be said to be oscillating if it has a sequence converging to ∞; otherwise, it is said to be non-oscillating.

Property I:

Equation (1) is said to be satisfy this property when n is even if each of its regular solutions is oscillating and when n is odd if each of its regular solution is either oscillating or solutions satisfies the condition

$$lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 0, \dots, n-1 \quad (4)$$

Property II:

Equation (1) is said to be satisfy this property when n is even when n is even and if each of its regular solutions is either oscillating or satisfies condition (4), or

$$lim_{t\to\infty} x^{(i)}(t) = \infty, \quad i = 0, \dots, n-1 \quad (5)$$

and which n is odd if each regular solution is either oscillating or satisfies (5). Consider the equation

$$x^{(n)}(t) \pm a(t)|x(g(t))|^{\lambda} sign\, x(g(t)) = 0 \quad (6^{\pm})$$

Where $\lambda > 1$ and $a: R_+ \to R_+$ is continuous functions for the case, $g(t) \leq t_1$, $g'(t) > 0$ and $lim_{t\to\infty} g(t) = \infty$. Bounded solutions of (6) are oscillatory if

$$lim_{t\to\infty} sup \int_{g(t)}^{t} (s - g(t))^{n-1} a(s) ds > 0 \quad (a)$$

Or $$lim_{t\to\infty} sup \int_{g(t)}^{t} (g(t) - g(s))^{n-1} a(s) ds > 0 \quad (b)$$

Or $$\int^{\infty} g'(t) \int_{g(t)}^{t} (s - g(t))^{n-2} a(s) ds\, dt = \infty \quad (c)$$

respectively.

Let in the equation (6) $a(t) = 1$ and $g(t) = t - t^{\frac{1}{(1-n)\lambda}}$. In this case the condition (a), (b) and (c) are not satisfied and on the other hand all bounded solutions of (6) are oscillatory since

$$\int^{\infty} a(t)[t - g(t)]^{\lambda(n-1)} dt = \int^{\infty} t^{-1} dt = \infty.$$

Equation (6) has property (I), (II) if $g: R_+ \to R$ is continuous functions,

$\lim_{t\to\infty} g(t) = \infty, \quad g(t) \le t \ for\ t \in R_+$

and $g'(t) \ge 0 \ for\ t \in R_+$ And $\int_0^\infty g^{n-1}(t)\, a(t)dt = \infty$ (7)

If $g: R_+ \to R$ is a continuous function satisfying condition (7) the following theorems hold:

Theorem 1:

If $n \ge 3$ and $\int_0^\infty g^{n-1}(t)a(t)dt = \infty$

Then equation (6)+ (equation (6)-) has property I (property II).

Theorem 2:

If $n \ge 3$

a) *If n is odd and for some $\epsilon > 0$, $\int_0^\infty g^{n-2+\lambda-\epsilon}(t)a(t)dt = \infty$, $\int_0^\infty t^{n-1}a(t)dt = \infty$*

Then equation (6)+ has property I.

b) *If n is even and $\int_0^\infty g^{n-2+\lambda}(t)a(t)dt = \infty$ and $\int_0^\infty t^{n-1}a(t)dt = \infty$*

Then equation (6)- has property II.

c) *If n is odd (even) and $\lim_{t\to\infty} \frac{g(t)}{t^\alpha} > 0$ where $\frac{(n-1)}{(n-2+\lambda)} < \alpha \le 1$, Then the condition $\int_0^\infty t^{n-1}\, a(t)dt = \infty$ is necessary and sufficient for equation (6)+ (equation (6)+) to have property I (property II).*

Lemma 1:

Let the function $x: [t_0, \infty) \to R$ and its derivatives up to order n-1 be absolutely continuous, and assume that

$x(t) \ne 0, \quad x^n(t) \le 0 \quad (\ge 0) \ for\ t_1[t_0, \infty).$

Then there are numbers $t_1 \in [t_0, \infty)$ and $L \in [0, \dots\dots, n]$ such that $L + n$ is odd (even) and

$$\left.\begin{array}{l} x^{(i)}(t)\, x(t) \ge 0 \ for\ (i = 0, \dots, L),\ t \in [t_0, \infty) \\ (-1)^{i+L} x^{(i)}(t)\, x(t) \ge 0 \ for\ (i = 0, \dots, L),\ t \in [t_0, \infty) \end{array}\right\} \quad (8)$$

Proof: See[3,7]

Theorem 3:

If x satisfies the condition of lemma 1 and $L \in \{1, \dots, n-1\}$,

$$\int_{t_1}^\infty t^{n-L-1} |x^n(t)| dt < \infty \quad (9)$$

$$|x(t)| \ge |x(t_1)| + \frac{1}{(n-1)!(n-L-1)!} \int_{t_1}^t (1-\xi)^{L-1} \int_\xi^\infty (s-\xi)^{n-L-1} \left|x^{(n)}(\xi)\right| ds d\xi \quad (10)$$

For $t \in [t_1, \infty)$

Moreover, if

$$\int_{t_1}^\infty t^{n-1} \left|x^{(n)}(t)\right| dt = \infty \quad (11)$$

Then there is a number $t_2 \in [t_1, \infty)$ such that

$$|x(s)| \ge \left(\frac{s}{t}\right)^L |x(t)| \quad for\ t \ge s \ge t_2 \quad (12)$$

Proof:

Relation(9) follows from the equality

$$\sum_{j=1}^{\gamma-1} \frac{(-1)^j t^{j-1}}{(j-i)!} x^{(j)}(t) = \sum_{j=i}^{n-1} \frac{(-1)^j s^{j-1}}{(j-i)!} x^{(j)}(s) + \frac{(-1)^{n-1}}{(n-i-1)} \int_s^t \xi^{n-i-1} x^{(n)}(\xi) d\xi \qquad (13)$$

With $i = L$ and (8) , using (9), we conclude from the relation

$$x^{(i)}(t) = \sum_{j=1}^{k-1} \frac{x^{(j)}(s)}{(j-i)!} (t-s)^{j-i} + \frac{1}{(k-i-1)!} \int_s^t (t-\xi)^{k-i-1} x^{(k)}(\xi) d\xi \qquad (14)$$

With $i = L$, $k = n$, and $s > t$ that

$$\left|x^{(L)}(t)\right| \geq \frac{1}{(n-L-1)!} \int_t^{\infty} (\xi - t)^{n-L-1} \left|x^{(n)}(\xi)\right| d\xi \quad for\ t \in [t,\infty) \qquad (15)$$

From (14) with $s = t$ that

$$|x(t)| \geq |x(t_1)| + \frac{1}{(L-1)!} \int_{t_1}^{t} (t-\xi)^{L-1} \left|x^{(l)}(\xi)\right| d\xi \quad for\ t \in [t_1,\infty) \qquad (16)$$

Thus (15) implies (10).

Now assume that (11); then by (8), it follows from (13) that

$$\lim_{t\to\infty} \left(\left|x^{(L-1)}(t)\right| - t\left|x^{(L)}(t)\right|\right) = \infty \qquad (17)$$

Let $$\rho_i(t) = i\left|x^{(L-1)}(t)\right| - t\left|x^{(L-i+1)}(t)\right| \qquad (18)$$

Since

$\dot{\rho}_{i+1}(t) = \rho_i(t) \quad for\ t \in [t_1,\infty),\ \ i = 1, \dots\dots, L-1$

We conclude from (17) that

$\lim_{t\to\infty} \rho_i(t) = \infty \qquad i = 1, \dots \quad L$

Hence t_2 can be assumed to be such that

$i\left|x^{(L-i)}(t)\right| \geq t\left|x^{(L-i+1)}(t)\right| \qquad for\ \ t \in [t_2,\infty) \qquad (i = 1, \dots\dots, L)$

Using this inequality for $i = l$, we find that $t^{-L}|x(t)|$ is non-increasing; thus, (12) holds and theorem 3 follows

Theorem 4:

Suppose that $\varphi : R \times R^m \to R$ satisfies conditions

$\varphi(t, u_1, \dots, u_n)\ sign\ u_1 \geq \varphi(t, v_1, \dots\dots, v_m)\ sign\ u_1 \geq 0$

$t \geq 0,\ \ u_i\, u_1 > 0,\ \ u_i\, v_i > 0 \quad and\ \ |u_i| \geq |v_i|,\ \ i = 1, \dots .m$

If the inequality

$x^{(n)}(t)\ sign\ x(t) \leq -\varphi\left(t, x(g_1(t)), \dots , x(g_n(t))\right) sign\ x(t)$

has the solution $x: [t_1,\infty) \to R \ \ (x(t) \neq 0\ \ for\ t \in [t_1,\infty))$

which satisfying condition (8) for some $L \in \{1, \dots, n-1\}$, then the equation

$$\gamma^{(n)}(t) = -\varphi\left(t, \left(\frac{g_1(t)}{t}\right)^L \gamma(t), \dots\dots, \left(\frac{g_m(t)}{t}\right)^L \gamma(t)\right) \qquad (19)$$

has the solution γ satisfying the following conditions

$$\left.\begin{array}{l} \gamma^{(i)}(t)\,\gamma(t) > 0 \quad for\ t \in [t_1,\infty),\ \ i = 0, \dots\dots, L \\ (-1)^{i+L} \gamma^{(i)}(t)\,\gamma(t) \geq 0 \quad for\ \ t \in [t_1,\infty),\ \ i = 0, \dots\dots, L \end{array}\right\} \qquad (20)$$

Proof:

Theorem (3) implies that either

$$\int_{t_1}^{\infty} t^{n-1} \left|x^{(n)}(t)\right| < \infty \text{ and } |x(t)| \geq c\, t^{L-1} \quad for\ t \in [t_1, \infty) \tag{21}$$

Where c is a positive constant, or there is $t_2 \geq t_1$ such that (12) holds.

Let $t_3 \geq t_2$ be such that $g_i(t) \geq t_2$ for $t \geq t_3$, $i = 1, \dots, m$; then

$$|x(g_i(t)| \geq c\, g_i^{L-1}(t) \quad for\ t \in [t_3, \infty).$$

and

$$\varphi\left(t, x(g_1(t)), \dots, x(g_m(t))\right) sign x(t) \geq \varphi\left(t, c\, g_1^{L-1}(t), \dots, c\, g_m^{L-1}(t)\right), t \in [t_3, \infty)$$

It follows from the last inequality here and (21) that

$$\int_{t_3}^{\infty} t^{n-1}\, \varphi\left(t, c\, g_1^{L-1}(t), \dots, c\, g_m^{L-1}(t)\right) dt < \infty$$

Assume that t_3 is such that

$$\int_{t_3}^{\infty} t^{n-1}\, \varphi\left(t, c\, g_1^{L-1}(t), \dots, c\, g_m^{L-1}(t)\right) dt < \frac{c}{2}$$

Let $C([t_2, \infty); R)$ be the space of continuous functions $v: [t_2, \infty) \to R$ with the uniform convergence on finite intervals, let S be the set of function $v \in C([t_2, \infty) \to R)$ satisfying the conditions

$$\frac{c}{2} t^{l-1} \leq v(t) \leq c\, t^{l-1} \quad for\ t \in [t_2, \infty)$$

and let $\psi: C([t_2, \infty); R)$ be defined by the following

$$\psi(v)(t) = c t^{l-1} \frac{1}{(l-1)!\,(n-l)!} \int_{t_3}^{t} (s-t)^{n-1} \varphi\left(s, \left(\frac{g_1(s)}{s}\right)^l v(s), \dots, \left(\frac{g_m(s)}{s}\right)^l v(s)\right) ds$$

for $t \in [t_3, \infty)$ and $\psi(v)(t) = c t^{l-1}$ for $t \in [t_2, t_1)$

For $L > 1$ and by the relation

$$\psi(v)(t) = c - \frac{1}{(n-l)!} \int_{t_3}^{t} (s-t)^{n-1} \varphi\left(s, \left(\frac{g_1(s)}{s}\right)^l v(s), \dots \left(\frac{g_m(s)}{s}\right)^l v(s)\right) ds \quad for\ t \in [t_3, \infty),$$

$$\psi(v)(t) = \psi(v)(t_3) \qquad for\ t \in [t_2, t_3).$$

Let (12) holds , then

$$|x(g_i(t))| \geq \left(\frac{g_i(t)}{t}\right)^l |x(t)| \quad for\ t \in [t_3, \infty), \quad i = 1, \dots, m$$

and

$$x^n(t)\, sign\, x(t) \leq -\varphi\left(t, \left(\left(\frac{g_1(t)}{t}\right) x(t), \dots\dots, \quad \left(\frac{g_1(t)}{t}\right)^L x(t)\right)\right) sign\, x(t)$$

it follows by (10) that

$$|x(t)| \geq$$
$$sign\, |x(t_3)| +$$
$$\frac{1}{(l-1)!(n-l-1)!} \int_{t_3}^{t} ((t-\xi)^{l-1} \int_{\xi}^{\infty} (s-\xi)^{n-l-1} \varphi\left(s, \left(\frac{g_1(s)}{s}\right)^l x(s), \dots, \left(\frac{g_m(s)}{s}\right)^l x(s)\right) sign\, x(s) ds$$

for $t \in [t_3, \infty)$

If S is the set of function $v \in ([t_2, \infty); R)$ such that

$$v(t)x(t) \geq 0, \ |v(t)| \leq |x(t)| \ for\ t \in [t_2, \infty).$$

and $\psi: s \to c_1([t_2, \infty); R)$ is defined by

$$\psi(y)(t) = x(t_3) + \frac{1}{(L-1)!\,(n-L-1)!} \int_{t_2} (t-\xi)^{n-1-1}\, \varphi\left(s, \left(\frac{g_1(s)}{s}\right) v(s), \dots \left(\frac{g_m(s)}{s}\right)^L v(s)\right) ds$$

for $t \in [t_3, \infty)$ and

$$\psi(y)(t) = x(t) \ for\ t \in [t_2, t_3)$$

Then it can be proved eq^{n} *(19) has a regular solution satisfying conditions (20). This completes the proof of theorem.*

We shall prove the following more general theorem

Theorem 5:

If $n \geq 3$*, condition (2) is satisfied, and*

$$\int_0^\infty a(t) \prod_{i=1}^m g_i^{\mu_i(n-L)} dt = \infty \qquad (22)$$

for $\mu_i \in [0, \lambda_i)$ *such that* $\sum_{i=1}^m \mu_i = 1$*, then equation (1) satisfy property I.*

Proof:

Let $x : [t_0, \infty) \to R$ *be regular non-oscillating solution of (1), and assumed that* $a(t) \neq 0$ *for* $t \in [t_0, \infty)$*, then by equation (2) and lemma 1, there are numbers* $t_1 \in (g_2(t_0), \infty)$ *and* $L \in [0, \ldots\ldots, n-1]$ *such that* $L + n$ *is odd and (9) holds.*

First assume that $L \in \{1, \ldots\ldots, n-2\}$ *and that the numbers*

$\epsilon_i \in [0, \mu_i] \cap [0, \lambda_i - \mu_i]$ *are such that* $\epsilon = \sum_{i=1}^m \epsilon_i > 0$*, for some* $c_1 > 0$*,*

$$|x(g_i(t))| \geq c_i\, g_i^{L-1}(t) \quad for\ t \in [t_1, \infty), \qquad i = 1, \ldots, m \qquad (23)$$

Hence, in (2) and (1) it follows.

$$x^{(n)}(t) sign\, x(t) \leq -c_2\, a(t) \prod_{i=1}^m (g_i(t) |x(g_i(t))|^{\mu_i + \epsilon} \quad for\ \epsilon \in [t_1, \infty)$$

where $c_2 = c_1^{L-1-\epsilon} > 0$*. by the theorem 3, follows*

$$y^{(n)}(t) = -c_2\, a(t) \prod_{i=1}^m (g_i(t))^{\lambda_i(L-1)+\mu_i+\epsilon_i}\ t^{-L(1+\epsilon)} |y(t)|^{1+\epsilon} sign\, y(t)$$

has a solution satisfying (20) hence

$$\int_0^\infty a(t) \prod_{i=1}^m (g_1(t))^{\mu_i(n-1)+(L-1)(\lambda_i-\mu_i-\epsilon_i)} dt \leq \int_0 t^{n-1} a(t) \prod_{i=1}^m (g_1(t))^{\lambda_i(L-1)+\mu_i+\epsilon_i\, t^{-L(1+\epsilon)}} dt < \infty$$

Which contradicts (22), putting $L = n - 1$ *and applying (2) and (22), we find that*

$$x^{(n)} sign\, x(t) \leq -c_2 a(t) (g_i(t))^{(\lambda_i - \mu_i + \epsilon_i)(n-2)} |x(g_i(t))|^{\mu_i - \epsilon_i}$$

Hence, theorem 3 implies

$$y^{(n)}(t) = -c_2\, a(t) t^{-(n-1)(1-\epsilon)} \prod_{i=1}^m (g_i(t))^{\lambda_i(n-2)+\mu_i-\epsilon_i} |y(t)|^{1-\epsilon} sign\, y(t) \qquad (24)$$

Has a solution satisfying (20) with $L = n - 1$.

Thus $L \in \{1, \ldots, n-1\}$ *and theorem follows for even n. Now let n be odd; then* $L = 0$ *and (8) implies that*

$$\lim_{t\to\infty} |x^{(i)}(t)|\ c_i \geq 0 \quad i = 0, \ldots\ldots, n-1$$

It is clear that $c_i = 0$ $i = 1, \ldots\ldots n-1$*, and, if we assume that*

$c_0 > 0$ *and* $|x(t)^{(n)}| \geq c_0^{\lambda} a(t)$ *for* $t \in [t_1, \infty)$ *that* $\int_t^\infty s^{n-1} a(s) ds < \infty$

This is a contradiction with (22), hence $c_0 = 0$*, x satisfies condition (4), and the theorem follows.*

Remark 1:

It follows from the proof of this theorem that, if n is odd, (22) can be replaced by the condition that

$$\int_0^\infty a(t)\prod_{i=1}^{m}(g_i(t))^{\mu_i(n-2)+\lambda_i-\epsilon_i}\,dt=\infty, \quad \text{and} \quad \int_0^\infty t^{n-1}\,a(t)\,dt=\infty,$$

for some $\epsilon_i\in[0,\mu_i]\cap[0,\lambda_i-\mu_i)$ *such that* $\sum_{i=1}^{m}\epsilon_i>0$.

Remark 2:

In $n=2$, *it can be proved that (1) has property (I) if*

$$\int_0^\infty a(t)\left(\prod_{i=1}^{m}(g_i(t))^{\mu_i-\epsilon_i}+\tau^{-\epsilon}(t)\prod_{i=1}^{m}\beta_i^{\mu_i+\epsilon_i}(t)\right)dt=\infty$$

Where the function $\tau: R_+\to(0,\infty)$ *is continuous non-decreasing, and unbounded,* $\epsilon_i\in[0,\mu_i]$, $\epsilon=\sum_{i=1}^{m}\epsilon_i>0$ *and* $\beta_i(t)=min\{g_i(t),\tau(t)\}$

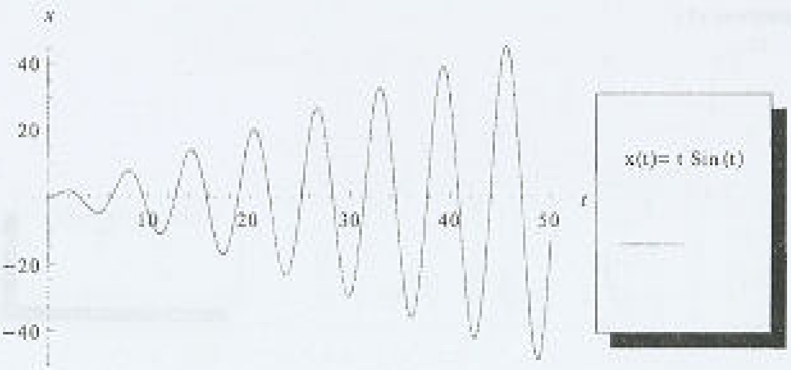
Example 1:

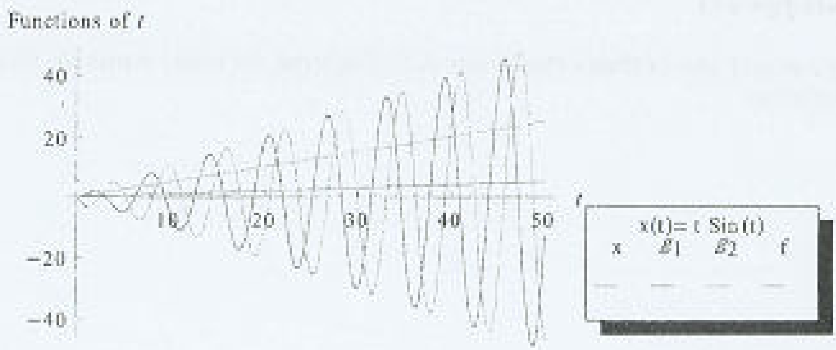
$$\begin{aligned} x^{(3)}(t) = {} & -t\,\mathrm{Cos}(t)+6(0.01t^2\mathrm{Cos}(0.1t)+0.25t^2\mathrm{Cos}(0.5t)+x(g_1(t)) \\ & +x(g_2(t)))(0.02\mathrm{Cos}(0.1t)+0.5\mathrm{Cos}(0.5t)-0.01x(g_1(t)) \\ & -0.25x(g_2(t)))+-\frac{3x(t)}{t}-6(0.1x'(0.1t)+0.5x'(0.5t))(0.01x''(0.1t) \\ & +0.25x''(0.5t) \end{aligned}$$

$$x^{(3)}(t)=f\left(t,x(t),x(g_1(t)),x(g_2(t)),x'(g_1(t)),x'(g_2(t)),x''(g_1(t))x''(g_2(t))\right)$$

Where $g_1(t)=0.1\,t,\quad g_2(t)=0.5\,t\qquad 0\le t\le 50$

The solutiom which satisfies the above example is $x(t)=\ t\,sin(10\,t)$



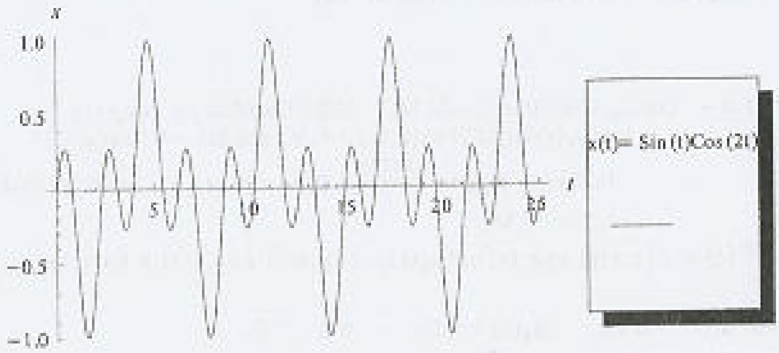


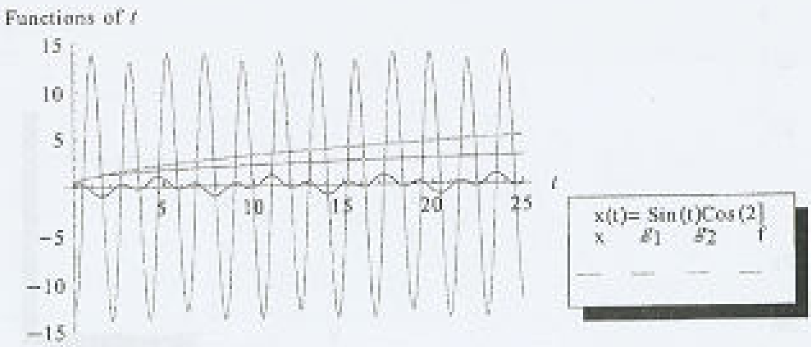
Example 2:

$$x^{(3)}(t)=\frac{1}{648t^{8/3}x(t)^2}\Big(-40\mathrm{Cos}(t^{1/3})+120\mathrm{Cos}(3t^{1/3})-\frac{81}{2}t^{1/6}(\mathrm{Cos}(\sqrt{t})-3\mathrm{Cos}(3\sqrt{t}))$$
$$+96t^{1/3}(\mathrm{Sin}(t^{1/3})+\mathrm{Sin}(3t^{1/3}))+162t^{2/3}(\mathrm{Sin}(\sqrt{t})+\mathrm{Sin}(3\sqrt{t}))$$
$$+240t^{1/3}x(g_1(t))+405t^{2/3}x(g_2(t))+324t^{8/3}\mathrm{Cos}(t)x(t)^2$$
$$-8748t^{8/3}\mathrm{Cos}(3t)x(t)^2-80x'(g_1(t))-81t^{1/6}x'(g_2(t))$$
$$+48t^{1/3}x''(g_1(t))+81t^{2/3}x''(g_2(t))\Big)$$

$$x^{(3)}(t)=f\left(t,x(t),x(g_1(t)),x(g_2(t)),x'(g_1(t)),x'(g_2(t)),x''(g_1(t))x''(g_2(t))\right)$$

$$g_1(t)=t^{1/3},\qquad g_2(t)=t^{1/2}\qquad 0\le t\le 25$$





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