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# Strongly $\theta$-continuous mappings between bitopological ordered spaces

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**ABSTRACT**: This paper is devoted to introduce and study the notion of increasing and decreasing pairwise strongly $\theta$-continuous mappings between bitopological ordered spaces. Examples and counter examples are given.Some properties and characterizations of the suggested are obtained. Relations between the new notions and corresponding concepts are proved.

# 1. Introduction

The study of partially order relation in topological ordered spaces was initiated by Nachbin in 1965 [8]. Nachbin's results were extended and generalized by several authors [3,4,11]. The notion of bitopological ordered space was introduced by M. K. Singal and A. R. Singal [12]. In [1,9,10], the work on bitopological ordered spaces was continued. A set equipped with two topologies and a partially order relation is said to be a bitopological ordered space.

## 1. Preliminaries

We would like to remark in the context of the present paper that $i, j \in \{1,2\}, i \neq j$. Also, we will use *P* to denote pairwise, *bto*-space to denote bitopological ordered space and *nbd* to denote neighborhood. Let $(X, \tau_1, \tau_2, \rho)$ be a bitopological ordered space. The closure and interior of a subset *A* of *X* with respect to (w. r. t., for short) $\tau_i$ are denoted by $\tau_i.cl(A)$ and $\tau_i.int(A)$ respectively.

Definition 2.1 [6]. A mapping $f:(X,\tau_1,\tau_2) \to (Y,\Delta_1,\Delta_2)$ is said to be

(a) P-continuous if $f:(X,\tau_i) \to (Y,\Delta_i)$ is continuous, $i=1,2$.

(b) P-open if $f:(X,\tau_i) \to (Y,\Delta_i)$ is open, $i=1,2$.

(c) P-closed if $f:(X,\tau_i) \to (Y,\Delta_i)$ is closed, i=1,2.

In the following, let $\rho_X$ denotes a partially order on a non-empty set X.

Definition 2.2 [8]. A subset A of X, where $(X,\rho)$ is a partially ordered set is said to be increasing (decreasing) if for all $a \in A$ and $x \in X$ such that $a\rho x (x \rho a)$ imply $x \in A$.

Definition 2.3 [2]. A mapping $f:(X,\tau,\rho_X)\rightarrow(Y,\Delta,\rho_Y)$ is said to be increasing (decreasing) mapping if $x_1\rho_X x_2$ leads to $f(x_1)\rho_Y f(x_2)(f(x_2)\rho_Y f(x_1))$.

Definition 2.4 [13]. A mapping $f:(X,\tau,\rho_X)\rightarrow(Y,\Delta,\rho_Y)$ is said to be order embedding if we have $x_1\rho_X x_2$ iff $f(x_1)\rho_Y f(x_2)$.

Definition 2.5 [12]. A bto-space $(X,\tau_1,\tau_2,\rho)$ is said to be pairwise $T_2(PT_2$, for short), if for all $a,b\in X$ such that $a\overline{\rho}b$, there exist an increasing $\tau_i-$ open nbd U of a and a decreasing $\tau_j-$ open nbd V of b such that $U\bigcap V=\phi$

Definition 2.6 [12]. A bitopological ordered space $(X,\tau_1,\tau_2,\rho_X)$ is said to be lower (upper) pairwise strongly regular ordered if for each decreasing (increasing) $\tau_i-$ closed subset F of X and for all $x\notin$ F there exists an increasing (decreasing) $\tau_i-$ open nbd U of x and a decreasing (increasing) $\tau_j-$ open nbd V of F such that $U\bigcap V=\phi$. $(X,\tau_1,\tau_2,\rho_X)$ is said to be pairwise strongly regular $(PSR_2$, for short) if it is both lower and upper pairwise strongly regular ordered.

Theorem 2.7 [12]. A bitopological ordered space $(X,\tau_1,\tau_2,\rho_X)$ is said to be pairwise strongly regular $(PSR_2$, for short) if for all $x\in X$ and for every increasing (decreasing) $\tau_i-$ open nbd U of x, there exists an increasing (decreasing) $\tau_i-$ open nbd U* of x such that $\tau_j.cl(U^*)\subseteq U$.

Definition 2.8 [5]. Let $(X,\tau_1,\tau_2,\rho_X)$ be a bitopological ordered space, $x\in X$ and $A\subseteq X$.

The $\theta_{ij}-$ closure of A, denoted by $\theta_{ij}.cl(A)$, is defined by :

$\theta_{ij}.cl(A)$ ={$x\in X$ : for every $\tau_i-$ open nbd U of $x,\tau_j.cl(U)\bigcap A\neq\phi\}$

The $\theta_{ij}-$ interior of A, denoted by $\theta_{ij}.int(A)$, is defined by :

$\theta_{ij}.int(A)=\{x\in X\exists\ \tau_i-$ open nbd U of x such that $\tau_j.cl(U)\subseteq A\}$.

3. Increasing and decreasing P.s. θ-continuous mappings

Definition 3.1. A mapping $f:(X,\tau_1,\tau_2,\rho_X)\rightarrow(Y,\Delta_1,\Delta_2,\rho_Y)$ is said to be:

Increasing P.s. $\theta$-continuous if for any increasing (decreasing) $\Delta_i -$ open nbd V of f(x) there exists an increasing (decreasing) $\tau_i -$ open nbd U of x such that $f(\tau_j.cl(U)) \subseteq V$ for all $x \in X$.

Decreasing P.s. $\theta$-continuous if for any increasing (decreasing) $\Delta_i -$ open nbd V of f(x) there exists a decreasing (increasing) $\tau_i -$ open nbd U of x such that $f(\tau_j.cl(U)) \subseteq V$ for all $x \in X$.

Definition 3.2. A mapping $f:(X,\tau_1,\tau_2,\rho_X) \to (Y,\Delta_1,\Delta_2,\rho_Y)$ is said to be:

Increasing P. $\theta$-continuous if for any increasing (decreasing) $\Delta_i -$ open nbd V of f(x) there exists an increasing (decreasing) $\tau_i -$ open nbd U of x such that $f(\tau_j.cl(U)) \subseteq \Delta_j.cl(V)$ for all $x \in X$.

Decreasing P. $\theta$-continuous if for any increasing (decreasing) $\Delta_i -$ open nbd V of f(x) there exists a decreasing (increasing) $\tau_i -$ open nbd U of x such that $f(\tau_j.cl(U)) \subseteq \Delta_j.cl(V)$ for all $x \in X$.

Definition 3.3. A mapping $f:(X,\tau_1,\tau_2,\rho_X) \to (Y,\Delta_1,\Delta_2,\rho_Y)$ is said to be:

(a) Increasing P-continuous if corresponding to every increasing (decreasing) $\Delta_i -$ open nbd V of f(x), there exists an increasing (decreasing) $\tau_i -$ open nbd U of x such that $f(U) \subseteq V$, for all $x \in X$.

(b) Decreasing P-continuous if corresponding to every increasing (decreasing) $\Delta_i -$ open nbd V of f(x), there exists a decreasing (increasing) $\tau_i -$ open nbd U of x such that $f(U) \subseteq V$, for all $x \in X$.

Definition 3.4. An increasing (decreasing) mapping $f:(X,\tau_1,\tau_2,\rho_X) \to (Y,\Delta_1,\Delta_2,\rho_Y)$ is said to be increasing (decreasing) P-continuous, P-open and P-closed if it is P-continuous, P-open and P-closed, respectively.

It is easy to prove the following theorems.

Theorem 3.5. A mapping $f:(X,\tau_1,\tau_2,\rho_X) \to (Y,\Delta_1,\Delta_2,\rho_Y)$ is

Increasing (decreasing) P-continuous iff $f^{-1}(V)$ is an increasing (decreasing) $\tau_i -$ open subset of X, for every increasing $\Delta_i -$ open subset V of Y.

Increasing (decreasing) P-continuous iff $f^{-1}(F)$ is an increasing (decreasing) $\tau_i -$ closed subset of X, for every increasing $\Delta_i -$ closed subset F of Y.

Theorem 3.6. Every increasing (decreasing) P.s.$\theta$-continuous mapping is increasing (decreasing) P-continuous.

The following example shows that the converse of the previous theorem is not true in general.

Counterexample 3.7. Let $X=\{x_1,x_2\},\ \tau_1=\{X,\phi,\{x_1\},\{x_2\}\},\ \tau_2=\{X,\phi,\{x_2\}$ and $\rho=\{(x_1,x_1),(x_2,x_2),(x_1,x_2)\}$. Let $Y=\{y_1,y_2\},\ \Delta_1=\{Y,\phi,\{y_1\}\},\ \Delta_2=\{Y,\phi\}$ and $\rho^*=\{(y_1,y_1),(y_2,y_1),(y_2,y_2)\}$. Define the mapping f as $f(x_1)=y_2,\ \ f(x_2)=y_1$. We find that f is increasing P-continuous but not increasing P.s.$\theta$-continuous as f is not increasing P.s-continuous at $x_2$.

Theorem 3.8. Every increasing (decreasing) P-continuous mapping.is increasing (decreasing) P.$\theta$-continuous.

Proof . We are present the case of increasing and the other is similar. Let $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ be increasing P-continuous. Let $x\in X$ and V be an increasing (decreasing) $\Delta_i-$open nbd of f(x). Then by increasing P-continuity of f, there exists an increasing (decreasing) $\tau_i-$open nbd U of x such that $f(U)\subseteq V$. Now, we have $f(U)\subseteq V\subseteq \Delta_j.cl(V)$. Then $U\subseteq f^{-1}f(U)\subseteq f^{-1}(\Delta_j.cl(V))$. Therefore $\tau_j.cl(U)\subseteq f^{-1}(\Delta_j.cl(V))$. Thus $f(\tau_j.cl(U))\subseteq \Delta_j.cl(V)$. Hence f is increasing P.$\theta$-continuous.

The following counterexample shows that the converse of the previous theorem is not true in general.

Counterexample 3.9. Let $X=\{a,b\}=Y,\ \rho=(a,a),(a,b),(b,b)\}=\rho^*$, $\tau_1=\{X,\phi,\{a\}\},\ \tau_2=\{X,\phi,\{b\}\},\ \Delta_1=\{X,\phi,\{a\},\{b\}\}$ and $\Delta_2=\{X,\phi\}$. We find that $f:(X,\tau_1,\tau_2,\rho)\to(Y,\Delta_1,\Delta_2,\rho^*)$, which is defined as $f(x)=x$ for all $x\in X$ is an increasing P.$\theta$-continuous but not increasing P-continuous as not increasing P-continuous at b.

Now, we have the following diagram.

$$inc\ \ P.s.\theta-cont.\Rightarrow inc\ \ P-cont.\Rightarrow inc\ \ P.\theta-cont.$$

Dually,

$$dec\ P.s.\theta-cont.\Rightarrow dec\ \ P-cont.\Rightarrow dec\ \ P.\theta-cont.$$

where inc (dec) $\mathrm{P.s.}\theta-\mathrm{cont.}$ means an increasing (decreasing) pairwise $s.\theta-$ continuous and $\mathrm{A}\Rightarrow\mathrm{B}$ denotes A implies B but not conversely.

Although the converse of Theorems 3.6 and 3.8 is not true, we have the following results.

Theorem 3.10. Every increasing (decreasing) P-continuous mapping from a PSR2–ordered space into any bitopological ordered space is increasing (decreasing) P.s. $\theta$ -continuous.

Proof . Let $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ be an increasing P- continuous mapping and $(X,\tau_1,\tau_2,\rho_X)$ be PSR2–ordered space. Let $x\in X$ and V be an increasing (decreasing) $\Delta_i-$ open nbd of f(x). As f is increasing P-continuous, there exists an increasing (decreasing) $\tau_i-$ open nbd U of x such that $f(U)\subseteq V$. Also, as $(X,\tau_1,\tau_2,\rho_X)$ is PSR2 –ordered space, there exists an increasing (decreasing) $\tau_i-$ open nbd U* of x such that $\tau_j.cl(U^*)\subseteq U$. Therefore $f(\tau_j.cl(U^*))\subseteq f(U)\subseteq V$. Hence f is increasing P.s. $\theta$ - continuous.

Analogously, we can prove the case of decreasing mapping.

Theorem 3.11. Every increasing (decreasing) P. $\theta$ -continuous mapping from an arbitrary bitopological ordered space into PSR2 – ordered space is increasing (decreasing) P-continuous.

Proof . We are present the case of increasing and the other is similar. Let $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ be an increasing P. $\theta$ -continuous and Y be PSR2–ordered space. Let $x\in X$ and V be an increasing (decreasing) $\Delta_i-$ open nbd of f(x). As Y is PSR2–ordered space, then there exists an increasing (decreasing) $\Delta_i-$ open nbd V* of f(x) such that $V^*\subseteq\Delta_j.cl(V^*)\subseteq V$. By increasing P. $\theta$ -continuity of f, there exists an increasing (decreasing) $\tau_i-$ open nbd U of x such that $f(\tau_j.cl(U))\subseteq\Delta_j.cl(V^*)$. Now, we have $f(U)\subseteq f(\tau_j.cl(U))\subseteq\Delta_j.cl(V^*)\subseteq V$. Therefore $f(U)\subseteq V$. Hence f is increasing P-continuous.

Definition 3.12. Let $(X,\tau_1,\tau_2,\rho_X)$ be a bitopological ordered space, $x\in X$ and $A\subseteq X$. An increasing $\theta_{ij}-$ closure of A, denoted by $\theta_{ij}{}^1.cl(A)$, is defined by :

$\theta_{ij}{}^{1}.cl(A)=\{x\in X:$ for every increasing $\tau_i-$ open nbd U of x, $\tau_j.cl(U)\bigcap A\neq\phi\}.$

A decreasing $\theta_{ij}-$ closure of A, denoted by $\theta_{ij}{}^{D}.cl(A),$ is defined by :

$\theta_{ij}{}^{D}.cl(A)=\{x\in X:$ for every decreasing $\tau_i-$ open nbd U of x, $\tau_j.cl(U)\bigcap A\neq\phi\}.$

(c) A set A is said to be increasing (decreasing) $\theta_{ij}-$ closed if A= $\theta_{ij}{}^{1}.cl(A)(A=\theta_{ij}{}^{D}.cl(A)).$

Definition 3.13. Let $(X,\tau_1,\tau_2,\rho_X)$ be a bitopological ordered space, $x\in X$ and $A\subseteq X.$

(a) An increasing $\theta_{ij}-$ interior of A, denoted by $\theta_{ij}{}^{1}.int(A),$ is defined by :

$\theta_{ij}{}^{1}.int(A)=\{x\in X:$ there exists an increasing $\tau_i-$ open nbd U of x, $\tau_j.cl(U)\subseteq A.$

(b) A decreasing $\theta_{ij}-$ interior of A, denoted by $\theta_{ij}{}^{D}.int(A),$ is defined by :

$\theta_{ij}{}^{D}.int(A)=\{x\in X:$ there exists a decreasing $\tau_i-$ open nbd U of x, $\tau_j.cl(U)\subseteq A\}.$

(c) A set A is said to be increasing (decreasing) $\theta_{ij}-$ open if A= $\theta_{ij}{}^{1}.int(A)(A=\theta_{ij}{}^{D}.int(A)).$

Now, we introduce some characterizations of $P.s.\theta-$ continuity.

Definition 3.14. In a bitopological ordered space $(X,\tau_1,\tau_2,\rho_X)$, the point $x\in X$ is said to be an increasing (decreasing) $\tau_i-$ cluster point of A if each increasing (decreasing) $\tau_i-$ open nbd of x intersects A.

Theorem 3.15. For any mapping $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$, the following statements are equivalent :

(a) f is increasing $P.s.\theta-$ continuous.

(b) The inverse image of an increasing (decreasing) $\Delta_i-$ closed set is increasing (decreasing) $\theta_{ij}-$ closed.

(c) The inverse image of an increasing (decreasing) $\Delta_i-$ open set is increasing (decreasing) $\theta_{ij}-$ open.

Proof. $(a)\Rightarrow(b)$ Let F be an increasing $\Delta_i-$ closed subset of Y and $x\notin f^{-1}(F)$. Then $f(x)\notin F$, and so there exists an increasing $\Delta_i-$ open nbd V of f(x) such that $V\bigcap F=\phi$. By (a), there exists an increasing $\tau_i-$ open nbd U of x such that $f(\tau_j.cl(U))\subseteq V$. Hence $f(\tau_j.cl(U))\bigcap F=\phi$, and then $\tau_j.cl(U)\bigcap f^{-1}(F)=\phi$ This implies that $x\notin\theta^1{}_{ij}.cl(f^{-1}(F))$. Thus $\theta^1{}_{ij}.cl(f^{-1}(F))\subseteq f^{-1}(F)$. As $f^{-1}(F)\subseteq\theta^1_{ij}.cl(f^{-1}(F))$, then $f^{-1}(F)$ is increasing $\theta_{ij}-$ closed.

$(b)\Rightarrow(c)$ Let U be an increasing $\Delta_i-$ open subset of Y. Then Y\U is a decreasing $\Delta_i-$ closed subset of Y. By (b), $f^{-1}(Y\setminus U)$ is decreasing $\theta_{ij}-$ closed. Now, $X\setminus f^{-1}(U)=$ $f^{-1}(Y\setminus U)$ is decreasing $\theta_{ij}-$ closed. Hence $f^{-1}(U)$ is increasing $\theta_{ij}-$ open

$(c)\Rightarrow(a)$ Let $x\in X$ and V be an increasing $\Delta_i-$ open subset of Y such that $f(x)\in V$. By (c), $f^{-1}(V)$ is an increasing $\theta_{ij}-$ open and $x\in f^{-1}(V)$. Then there exists an increasing $\tau_i-$ open nbd U of x such that $\tau_j.cl(U)\subseteq f^{-1}(V)$. Hence $f(\tau_j.cl(U))\subseteq ff^{-1}(V)\subseteq V$. Therefore f is increasing P.s.$\theta$-continuous.

Theorem 3.16. For an onto increasing P.$\theta$-continuous mapping $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$, $f(\theta^D_{ij}.cl(A))\subseteq\theta^D_{ij}.cl(f(A))$ ($f(\theta^1_{ij}.cl(A))\subseteq\theta^1_{ij}.cl(f(A))$,) for every subset A of X.

**Theorem 4.1.** *Let $X \in$ | L-TOP | be an extremally predisconnected . Then $\lambda \in L^X$ is preclopen if and only if $\lambda$ is a regular preopen L-set in X .*

**Proof. Necessity .** It follows from Thoerem 3.3.

**Sufficiency .** Since $X$ is an extremally predisconnected and $\lambda$ is a regular preopen L-set in $X$ , then $\lambda$ is preopen and so $pCl\ (\lambda)$ is a preopen L-set in $X$ . Hence $\lambda = pInt\ (pCl\ \lambda) = pCl\ \lambda$ which implies that $\lambda$ is a preclosed L-set in $X$ .

**Theorem 4.2.** *Let $X \in$ | L-TOP | be an extremally predisconnected . For $\lambda \in L^X$ , the following statements are equivalent :*

(i) $\lambda$ *is a preclopen L-set in X ;*

(ii) $\lambda = pCl\ (pInt\ \lambda)$ ;

*(iii)* $\lambda'$ *is a regular preopen L-set in X ;*

*(iv)* $\lambda$ *is a regular preopen L-set in X .*

**Proof.** (i)$\Rightarrow$ (ii) is obvious .

(ii) $\Rightarrow$(iii) . Let $\lambda = pCl\ (pInt\ \lambda)$ . Then $\lambda' = (pCl\ pInt\ \lambda)' = pInt\ (\ pInt\ \lambda\ )' = pInt\ pCl\ \lambda'$ . Hence $\lambda'$ is a regular preopen L-set in $X$ .

(iii) $\Rightarrow$(iv) . From Theorem 4.1, $\lambda'$ is a preopen and preclosed L-set in X . Hence $\lambda$ is a preopen and preclosed L-set in $X$ . Therefore $\lambda = pInt\ pCl\ \lambda$ and hence $\lambda$ is a regular preopen L-set in $X$ .

(iv) $\Rightarrow$(i) . It follows from Theorem 4.1.

**Theorem 4.3.** *Let $X \in$ | L-TOP | be an extremally predisconnected . Then for $\lambda \in L^X$ ; $Cl\ \lambda$ (resp. $PCl\ \lambda$ ) is a regular preopen L-set in X if and only if $\lambda$ is a preopen L-set in X .*

**Proof. Necessity.** It follows from Theorem 3.4.

**Sufficiency.** Let $\lambda$ be a preopen L-set in $X$ . Then $Cl\ \lambda$ is a preopen L-set in $X$ and hence preclopen . Thus $Cl\ \lambda$ is a regular preopen L-set in $X$ .

**Corollary 4.1.** *Let $X \in$ | L-TOP | be an extremally predisconnected . Then for $\lambda \in L^X$ , the L-sets $Cl\ (Int\ \lambda)$ , $Cl\ (pInt\ \lambda)$ , $pCl\ (Int\ \lambda)$ and $pCl\ (pInt\ \lambda)$ are regular preopen L-sets in X .*

**Theorem 4.4.** *Let $X \in$ | L-TOP | be an extremally predisconnected . If $\lambda \in L^X$ is either a regular open or a regular closed L-set , then $\lambda$ is a regular preopen L-set in X .*

**Proof.** If $\lambda$ is regular open , then $\lambda = Int\ Cl\ \lambda$ . Now

$$\lambda = Int\ Cl\ \lambda = Int\ Cl\ (pCl\ \lambda\ ) \geq pInt\ pCl\ \lambda\ .$$

Also

$$pInt\ pCl\ \lambda \geq pInt\ (Cl\ Int\ \lambda) \geq Int\ Cl\ (\ Int\ \lambda)$$
$$= Int\ Cl\ \lambda = \lambda\ .$$

Thus $\lambda = pInt\ pCl\ \lambda$ implies $\lambda$ is a regular preopen L-set .

Similarly if $\lambda$ is regular closed , then $\lambda = Cl\ Int\ \lambda$ , and we have

$$\lambda = Cl\ Int\ \lambda = pCl\ (\ Cl\ Int\ \lambda) \geq pInt\ pCl\ (Cl\ Int\ \lambda)$$
$$= pInt\ pCl\ \lambda\ .$$

Also

$$pInt\ pCl\ \lambda = pInt\ pCl\ (\ Cl\ Int\ \lambda\ ) \geq pInt\ pCl\ (\ Int\ \lambda)$$
$$= pCl\ (Int\ \lambda) \geq Cl\ Int\ (\ Int\ \lambda)$$

**Theorem 4.2.** If $(Y,\Delta_1,\Delta_2,\rho_Y)$ is $PT_2$-ordered space and $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ is P.s.$\theta$-continuous and order embedding mapping, then $(X,\tau_1,\tau_2,\rho_X)$ is $PT_{2\frac{1}{2}}$-ordered space.

**Proof.** Let $x_1,x_2\in X$ such that $x_1\overline{\rho}_X x_2$. Since *f* is order embedding mapping, then $f(x_1)\overline{\rho}_Y f(x_2)$. Since $(Y,\Delta_1,\Delta_2,\rho_Y)$ is $PT_2$-ordered space, then there exist an increasing $\Delta_i-$open nbd $U^*$ of $f(x_1)$ and a decreasing $\Delta_j-$open nbd $V^*$ of $f(x_2)$ such that $U^*\cap V^*=\phi$. Now, by P.s.$\theta$-continuity of *f*, there exist an increasing $\tau_i-$open nbd *U* of $x_1$ and a decreasing $\tau_j-$open nbd *V* of $x_2$ such that $f(\tau_j.cl(U))\subseteq U^*$ and $f(\tau_i.cl(V))\subseteq V^*$. Therefore $f(\tau_j.cl(U))\cap f(\tau_i.cl(V))=\phi$ and so $\tau_j.cl(U)\cap\tau_i.cl(V)=\phi$. Hence $(X,\tau_1,\tau_2,\rho_X)$ is $PT_{2\frac{1}{2}}$-ordered space.

**Theorem 4.3.** The composition of an increasing (decreasing) P.s.$\theta$-continuous mapping and an increasing (decreasing) P- continuous mapping is increasing (decreasing) P.s-continuous .

**Proof.** Let $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ be increasing (decreasing) P.s.$\theta$-continuous and $g:(Y,\Delta_1,\Delta_2,\rho_Y)\to(Z,\gamma_1,\gamma_2,\rho_z)$ be increasing (decreasing) P-continuous. Let *V* be an increasing (decreasing) $\gamma_i-$open subset of Z. As *g* is increasing P-continuous then $g^{-1}(V)$ is an increasing (decreasing) $\Delta_i-$open subset of *Y*. Now, as *f* is an increasing P.s.$\theta$-continuous mapping, then by Theorem 3.15, we have $f^{-1}(g^{-1}(V))=(gof)^{-1}(V)$ is an increasing (decreasing) $\theta_{ij}-$open subset of *X*. Hence $g\,o\,f:(X,\tau_1,\tau_2,\rho_Y)\to(Z,\gamma_1,\gamma_2,\rho_x)$ is an increasing P.s.$\theta$-continuous.

By duality, one can prove the theorem in case of decreasing mapping.

**Theorem 4.4.** The $PT_{2\frac{1}{2}}$-ordered axiom is inverse invariant under P.$\theta$-continuous and order embedding mapping.

**Proof.** Let $f:(X,\tau_1,\tau_2,\rho_X)\to(Y,\Delta_1,\Delta_2,\rho_Y)$ be P.$\theta$-continuous and order embedding mapping from a bitopological ordered space $(X,\tau_1,\tau_2,\rho_X)$ into a $PT_{2\frac{1}{2}}$-ordered space $(Y,\Delta_1,\Delta_2,\rho_Y)$. Let $x_1,x_2\in X$ such that $x_1\overline{\rho}_X x_2$. Then $f(x_1)\overline{\rho}_Y f(x_2)$. Since $(Y,\Delta_1,\Delta_2,\rho_Y)$ is $PT_{2\frac{1}{2}}$, then there exist an increasing

$\Delta_i$ – open nbd $V_1$ of $f(x_1)$ and a decreasing $\Delta_j$ – open nbd $V_2$ of $f(x_2)$ such that $\Delta_j.cl(V_1)\cap\Delta_i.cl(V_2)=\phi$. By $P.\theta$ – continuity of *f*, there exist an increasing $\tau_i$ – open nbd $U_1$ of $x_1$ and a decreasing $\tau_j$ – open nbd $U_2$ of $x_2$ such that $f(\tau_j.cl(U_1))\subseteq\Delta_j.cl(V_1)$ and $f(\tau_i.cl(U_2))\subseteq\Delta_i.cl(V_2)$. Therefore $f(\tau_j.cl(U_1))\cap f(\tau_i.cl(U_2))=\phi$ and so $\tau_j.cl(U_1)\cap\tau_i.cl(U_2)=\phi$. Hence $(X,\tau_1,\tau_2,\rho_X)$ is $PT_{2\frac{1}{2}}$-ordered space.

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## الدوال ثنائية الإتصال من النوع θ القوى بين الفراغات ثنائية التوبولوجى المرتبة

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هذا البحث إختص بتقديم مفهومى الرواسم الثنائية المتصلة من النوع θ والرواسم قوية الإتصال من النوع θ بين الفراغات ثنائية التوبولوجى المرتبة. وأعطى بعض خصائص لهذين المفهومين بإستخدام المجموعات المغلقة من النوع $\theta_{ij}$. كذلك درس تأثير هذه الرواسم على بعض فرضيات الإنفصال . وقد تم بحث العلاقة بين هذين المفهومين والرواسم الثنائية المتصلة. كما تم إثبات تكافؤ هذه الأنواع من الاتصال تحت شروط معينة.