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SPECIAL OPERATORS ONTO SOME HYPERPLANES OF THE BANACH SPACE l_n^1

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ABSTRACT

In this paper, we showed that the space l_n^1 contains hyperplanes Y with maximal but better relative projection constants than that given before. Use geometry to construct the exact minimal norm projection from l_n^1 onto Y, and give a positive answer to the question in a finite dimensional Banach space X with dimension n, is there hyperplane with the greatest exact relative projection constants $2-\frac{2}{n}$?

INTRODUCTION

Since the existence of a projection P from a Banach space X onto its closed subspace Y is equivalent to the existence of an extension $\bar{T}$ of any operator T from Y into W to an operator from X into W such that $\|\bar{T}\| \le \|P\| \, \|T\|$. The two equivalent problems [how small can the norm of the extended operator be made?] and [what is the projection of smallest norm?] are related to the study of the relative projection constant $\lambda(Y, X)$ of Y in X that is defined by

$$\lambda(Y, X) := \inf\{ \|P\| : P \text{ is a projection from } X \text{ onto } Y \}. \longrightarrow (1)$$

and the absolute projection constant of Y that is defined by

$$\lambda(Y) := \inf\{ \lambda(Y, X) : X \text{ contains } Y \text{ as a closed subspce} \}. \longrightarrow (2)$$

In a finite dimensional Banach space with dimension n, the question is there hyperplanes with the greatest exact relative projection constant $2-\frac{2}{n}$? has a positive answer. In this paper we gave examples of hyperplanes of exact relative projection constants equal $2-\frac{2}{n}$.

In (1938) Bohnenblust [1], it is shown that if X_n is a finite dimensional Banach space with $\dim(X_n)=n$ and Y_{n-1} is a subspace of dimension (n-1), then the relative projection constant of Y_{n-1} in the space X_n satisfies

$$\lambda(Y_{n-1}, X_n) \le 2-\frac{2}{n}. \longrightarrow (3)$$

In (1983) Konig, Lewis and Lin [8] gave the general estimation of the relative projection constants of a k-dimensional subspace Y_k of the $k \le n$-dimensional Banach space X_n, they showed that

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$$\lambda(Y_k,\ X_n) \le \sqrt{k}\left(\frac{\sqrt{k}}{n} + \frac{\sqrt{(n-1)(n-k)}}{n}\right). \longrightarrow (4)$$

In (1994) Konig and Tomczak-Jaegermann [9] gave the upper estimate for the absolute projection constant $\lambda(Y)$ of a finite dimensional space Y with dimY=n is found in the form

$$\lambda(Y) \le \begin{cases} \sqrt{n} - \frac{1}{\sqrt{n}} + O(n^{\frac{-3}{4}}),\ \text{in the real field}, \\ \sqrt{n} - \frac{1}{2\sqrt{n}} + O(n^{\frac{-3}{4}}),\ \text{in the complex field}. \end{cases} \longrightarrow (5)$$

Even the two results given in equations (4) and (5) are more recent than that given in equation (3), but for the hyperplanes the estimation given in (3) is more better than that in (4) and (5).

The precise values of $l_1^n,\ l_2^n$ and $l_p^n,\ p \ne 1,\ p \ne 2$ have been calculated by Grunbaum [7], Gordon [6], Garling and Gordon [5], Rutovitz [11] and Konig [10].

In [4] interesting results have been given for the injective and projective tensor products.

In [12] and [5] results for finite CO-dimensional subspaces of Banach spaces.

A relative projection constant of the closed subspace Y in the space X is said to be exact if and only if there is a projection P from X onto Y at which the infimum of equation (1) is attained.

A subspace Y of the space X is said to be a hyperplane (maximal proper subspace) of the space X if and only if X contains Y as a subspace of deficiency 1.

It is known that a subspace Y is a hyperplane of the space X if and only if there is a functional
$f \in X^*$ *such that* $Y = f^{-1}(0)$, see [2].

Let P be an operator on the space X. Then the point x in X is said to be maximal point of the operator P if and only if $\|P\| = \|P(x)\|$.

If X is either of the Banach spaces l_∞ the Banach space of all bounded scalar valued functions $\{x_n\}_{n=1}^{\infty}$ on a countably infinite set N or l_p the Banach space of all scalar valued functions $\{x_n\}_{n=1}^{\infty}$ on a countably infinite set N such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ or c_0 the closed subspace of the Banach space l_∞ of all convergent to zero sequences, then the norms on X are defined as follows:

$$\|x\|_X := \begin{cases} \sup_{n=1}^{\infty} |x_n|,\ \text{if } X = l_\infty, \\ (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}},\ \text{if } X = l_p. \end{cases} \longrightarrow (6)$$

Main result is given in the following theorem:

<u>Theorem:(1)</u>

Let $f=\delta\{\varepsilon_k\}_{k=1}^{n}\in l_n^1,\ n>2$ be a sequence of the space l_n^1, where δ is a non zero scalar and $\varepsilon_k=\pm1$. Then the relative projection constant of the (n-1) dimensional subspace $Y=f^{-1}(\{0\})$ in the space l_n^∞ equals $2-\frac{2}{n}$. Moreover, the minimal norm projection is given by

$$P_0(x)=x-\frac{f(x)}{\|f\|_2^2}f.$$

<u>Proof:</u> The general formula of any projection from X onto Y is given by

$$P=I_{l_p}-f\otimes z\quad for\quad some\quad z=\{\alpha_k\}_{k=1}^{n}\in X\quad with\ f(z)=1.$$

The norm of the projection corresponding to the element $z=\{\alpha_k\}_{k=1}^{n}$ is given by

$$\|P\|=\sup_{k=1}^{n}\{\left|1-\alpha_k f_k\right|+\left|\alpha_k\right|\left[\|f\|_1-\left|f_k\right|\right]\}$$

$$=\sup_{k=1}^{n}\{\left|1-\delta\alpha_k\varepsilon_k\right|+\left|\alpha_k\right|[n-1]\delta\ \}\ \longrightarrow(7)$$

Assume that the minimal projection is a norm one projection. Then there is $z\in l_n^1$ and

$$\left|1-\delta\alpha_k\varepsilon_k\right|+\left|\alpha_k\right|[n-1]|\delta|\quad\le 1\quad for\quad every\quad k=1,\ 2,\ \ldots$$

In this case, we have

$$1-\left|\delta\alpha_k\right|+\left|\alpha_k\right|[n-1]|\delta|\quad\le 1\quad for\quad every\quad k=1,\ 2,\ \ldots\ n.$$

Therefore $\left|\alpha_k\right|[n-2]|\delta|\quad\le 0\quad for\quad every\quad k=1,\ 2,\ \ldots\ n$ for n>2. This is true only if

$\left|\alpha_k\right|\ \le 0\ for\ every\ k=1,2,\ldots n$ This is an obvious contradiction, thus there is no norm 1 projection from l_n^1 onto Y.

Now, let $x=\{x_k\}_{k=1}^{n}$ be an arbitrary point in the space l_n^∞. To project this point to the point $x^0=\{x_k^0\}_{k=1}^{n}$ in the space Y with a minimal available distant between the points $x=\{x_k\}_{k=1}^{n}$ and $x^0=\{x_k^0\}_{k=1}^{n}$, the sequence $x^0-x=\{x_k^0-x_k\}_{k=1}^{n}$ must be parallel to the line passing through $\{f_k\}_{k=1}^{n}$ and perpendicular to the plane Y. Thus there is a scalar λ such that $x_0-x=\lambda f$. On the other hand since $x_0\in Y,\ f(x_0)=0$, thus $0=f(x_0)=f(x)+\lambda\|f\|_2^2$ and so $\lambda=\frac{-f(x)}{\|f\|_2^2}$, it follows that $x_0=x-\frac{f(x)}{\|f\|_2^2}f$. The required projection P_0 from l_n^∞ onto Y is defined by the formula

$$P_0(x = \{x_k\}_{k=1}^{n}) = x_0 = x - \frac{f(x)}{\|f\|_2^2} f.$$

(Note that the element z_0 corresponding to P_0 is $z_0 = \frac{f}{\|f\|_2^2}$ and also $\|P_0\| = 2 - \frac{2}{n}$.

Now we are going to show that this projection is a minimal norm projection. Let us assume the contrary, i.e., there exists an element $z \in l_1^n$ such that f(z)=1 and the corresponding projection P satisfies $\|P\| < 2 - \frac{2}{n}$, according to equation (7), we have

$$|1 - \delta\alpha_k \varepsilon_k| + |\alpha_k|[n-1]\delta < 2 - \frac{2}{n}, \longrightarrow (8)$$

for every $k \in \{1, 2, ..., n\}$, from which we get $|\alpha_k|[n-2]\delta < 1 - \frac{2}{n}$,

then for such a z, we have

$$|\alpha_k| < \frac{1}{n|\delta|} \quad for \ \ all \ \ k \in \{1, 2, ..., n\}, \longrightarrow (9)$$

multiplying the inequality (9) by $|f_k||\delta|$ and summing with respect to k, we get $\sum_{k=1}^{n} |f_k||\alpha_k| < 1$. On the other hand the inequality $1 = \sum_{k=1}^{n} f_k \alpha_k \le \sum_{k=1}^{n} |f_k||\alpha_k| < 1$, gives a contradiction, hence no such z exists, from which we concluded the proof.

Example: The minimal norm projection of the subspace $Y := \left\{ y; y = \{y_i\}_{i=1}^{3}, \sum_{i=1}^{3} y_i = 0 \right\}$ of the space l_3^∞ is the projection given by $P_0(\{y_i\}_{i=1}^{3}) := \frac{1}{3}\{2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_2 - x_1\}$, with norm $\|P_0\| = \frac{3}{4}$.

Remark:

1- If there is a sequence $f = \{f_n\}_{n=1}^{\infty} \in l_1$ such that

$$\alpha := \frac{1}{\|f\|_2} \sup_{n\ge1} |f_n| [\|f\|_1 - 2|f_n|] \times \sum_{n=1}^{\infty} \frac{f_n}{[\|f\|_1 - 2|f_n|]} = 1, \qquad \longrightarrow (10)$$

then the relative projection constant $\lambda(Y, c_0)$ of the subspace $Y = f^{-1}(\{0\})$ in the space c_0 is exact. Moreover, the minimal norm projection is given by $P_0(x) = x - \frac{f(x)}{\|f\|_2^2} f$ and its norm equals $\lambda(Y, c_0) = 1 + \frac{1}{\|f\|_2} \sup_{n\ge1} |f_n| [\|f\|_1 - 2|f_n|]$.

2- The number α that given in equation (10) is independent on the choice of the given f for which $Y = f^{-1}(\{0\})$, in fact., this is a consequence of the fact that for the hyperplane if $Y = f^{-1}(\{0\}) = g^{-1}(\{0\})$, we have $f = \lambda g$ for some scalar λ.

3- For any bounded sequence $\{f_n\}_{n=1}^{\infty}$ it is true that $\frac{\inf_{n\ge1}|f_n|}{\sup_{n\ge1}|f_n|} \le 1 \le \alpha$. In fact.,

$$\frac{\inf_{n\ge1}|f_n|}{\sup_{n\ge1}|f_n|} \le \frac{\inf_{n\ge1}|f_n|}{\sup_{n\ge1}|f_n|\sum_{n=1}^{\infty}|f_n|}\left[\sum_{n=1}^{\infty}|f_n|\right]$$

$$\le \frac{\inf_{n\ge1}|f_n|}{\sum_{n=1}^{\infty}|f_n|^2}\left[\sum_{n=1}^{\infty}|f_n|\right]$$

$$\le \frac{1}{\|f\|_2^2}\inf_{n\ge1}|f_n|\sup_{n\ge1}\left[\|f\|_1 - 2|f_n|\right]\times\left\{\sum_{n=1}^{\infty}\frac{|f_n|}{\|f\|_1 - 2|f_n|}\right\}$$

$$\le \frac{1}{\|f\|_2^2}\sup_{n\ge1}|f_n|\left[\|f\|_1 - 2|f_n|\right]\times\left\{\sum_{n=1}^{\infty}\frac{|f_n|}{\|f\|_1 - 2|f_n|}\right\}$$

$$= \alpha.$$

And

$$\alpha \ge \frac{1}{\|f\|_2^2}\left\{\sum_{n=1}^{\infty}|f_n|\left[\|f\|_1 - 2|f_n|\right]\frac{|f_n|}{\|f\|_1 - 2|f_n|}\right\} = \frac{1}{\|f\|_2^2}\|f\|_2^2 = 1.$$

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