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ON WEIERSTRASS-STONE THEOREM FOR ALGEBRAS AND MODULES OF CONTINUOUS FUNCTIONS

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ABSTRACT

This paper studies from the viewpoint of vector fibrations and vector spaces of cross-sections, endowed with appropriate topologies, the realm of ideas and results centered around the Weierstrass-Stone Theorem for algebras and modules of continuous functions in the real and self-adjoint complex cases. The proof of the equivalence between the separating and the general cases of such results by means of a quotient construction motivate the use of vector fibrations.

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§ 1. INTRODUCTION

This paper is partly expository. Our aim is to motivate the introduction of "vector fibrations" and "vector spaces of cross-sections" in a fairly elementary context. These two terms are defined in § 5 below, the definitions being exactly the same ones given in [12] and [13]. However, we do not presuppose familiarity with these papers. The present paper is quite self-contained, Theorem 1 being the only result accepted without any proof: it is a version of the classical Weierstrass-Stone Theorem.

We hope to have succeeded in showing how very naturally vector fibrations and vector spaces of cross-sections, endowed with appropriate topologies, can be made to appear in the realm of ideas and results centering around the Weierstrass-Stone Theorem. We do this in §'s 3 and 4, to which we refer the reader to explain the interesting technical question that guided us in the reasoning therein.

The Weierstrass-Stone Theorem and the results we have in mind fit into the following very general description. It is given a "large" vector

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space F over K, where k is either the real field **R** or the complex field C. A vector subspace L of F is selected and provided with a structure of normed or of a locally convex space over K. Then one proves results describing the closures in L of sufficiently restricted vector subspaces W of L; in particular, one proves results stating conditions under which W is dense in L.

The point to be brought out is that in a vector fibration, by the definition of it, there is a "large" vector space F, of cross-sections, from which many interesting vector subspaces L may be selected.

Let us fix our attention on a given vector fibration, a context of which the settings for the usual versions of the Weierstrass-Stone Theorem are examples. There is all along a "base" topological Hausdorff space E naturally provided with its algebra ℓ $(E;k)$ of K-valued continuous functions on E. In the further selection of the vector subspace $A \subset \ell$ $(E;K)$ plat a crucial role.

Some results may then be presented in pairs; a so called "general version" of a result; and a special "separating version". The "separating version" is usually a statement of density of W in L. It occurs when the "convenient sub-algebra" A separating over E in the sense that, for all pairs x_1 , $x_2 \in E$ with $x_1 \neq x_2$, there exists $a \in A$ such that $a(x_1) \neq a(x_2)$.

The question now arises of deciding whether, in each pair of results as above, the implication

"separating case" $\Rightarrow$ "general case"

is true.

This is the question we study in §'s 3 and 4 and which we offer as a motivation for the introduction of vector fibrations. But we study it with our minds firmly set on the idea of using a certain "quotient construction" (see § 2) in tackling it. It is in this way that we succeed in arriving at vector fibrations and in obtaining an insight into the sort of theorems one might try to prove in their contexts - and this is the real crux of the matter.

However, this paper is written more in the spirit of an introduction to [12] and [13] to which we refer to reader for further comments of a general nature and connections with the literature. We have not striven to obtain the strongest possible results here. But we observe that Theorem 6, stated and proved in § 6, is strong enough to imply a simple proof of the following generalization of the classical Weierstrass Theorem of polynomial approximation (see § 6 for the proof as well as the explanation of some terms and the notation used in its statement).

THEOREM 1. Let E and F two locally convex Hausdorff spaces, both real. The vector space $P(E;F)$, of all continuous polynomials on E with values in F, is dense in $\ell\ (E;F)$ in the compact-open topology.

The proof of Theorem 6 given here is essentially a repetition of the arguments Nachbin used in proving his Weierstrass -Stone Theorem for modules ([11], § 19, Theorem 1). But Nachbin's theorem itself is not used in that proof and follows therefore as a special case of Theorem 6.

Nachbin spaces of a very special type occur throughout the paper but are not referred to by that name until last remark in the paper. This terminology is proposed by the authors.

§ 2. PRELIMINARIES

The letter E stands for a topological Hausdorff space throughout the paper; further restrictions on E are explicitly stated when needed. The letter K denotes neutrally either the real or the complex field. The symbol $\ell\ (E;K)$ denotes the algebra of continuous K-valued functions on E under the usual pointwise operations. A non-empty subset A of $\ell\ (E;k)$ is said to be : (i) separating over E if, for any pair $x, y \in E$ with $x \neq y$, there is $a \in A$ such that $a(x) \neq a(y)$; (ii) self-adjoint in the complex case, i.e. , when K is complex field, if for any function in A, its complex conjugate function lies also in A. If A and W are subsets of $\ell\ (E;k)$, we put $AW = (aw;\ a \in A,\ w \in W)$. If A is a sub-algebra and W is a vector subspace, we say that W is a module over A, or an A-module, if $A\ W \subset W$.

For any non-empty subset A of ℓ $(E; k)$, we shall donate by E/A the equivalence relation defined on E as follows : if x , $y \in E$ then x is equivalent to y modulo **E/A**, if and only if $a(x) = a(y)$, for all $a \in A$. Let F denote the quotient topological space of E modulo E/A and π the quotient map of E onto F; π is continuous and for each $x \in \boldsymbol{E}$, $\pi(x)$ is the equivalence class of x modulo E/A. Each $\pi(x)$ is a closed subset of $\boldsymbol{E}$ and every $a \in A$ is constant on $\pi(x)$. Hence, for each $a \in A$ there is a unique function b : $F \to K$ such that $a = b.\pi$. It is easy to see that $b \in \ell$ $(F; k)$ we put $B = \{b;\ a = b.z,\ a \in A\}$. It follows that B is a sub-algebra of ℓ $(F; k)$, which contains the constants, if A does also ; moreover B is self-adjoint in the complex case if $\boldsymbol{A}$ also is. An all important fact is that B is separating over F, which implies that F is a Hausdorff space and therefore compact if $\boldsymbol{E}$ is compact.

If X is any subset of $\boldsymbol{E}$ and $f \in \ell$ $(E;k)$ we denote by $f \mid X$ the restriction of f to X , and if L is a subset of ℓ $(E;k)$ then $L \mid X = \{f \mid X; f \in L\}$. Also, if $x \in E$, then $L(x) = \{f(x)\ ;\ f \in L\} \subset K$.

If ℓ is a subset of a Cartesian product $\Pi\,(F_y\ ;\ y \in F)$, then $\ell\,(y) = \{f(y); f \in \ell\} \subset F_y$ is called the y-section of ℓ .

§ 3. WEIERSTRASS- STONE THEOREMS FOR ALGEBRAS AND MODULES OF CONTINUOUS FUNCTIONS

In this paragraph we state four results of the Weierstrass-Stone type and indicate how vector fibrations arise in a natural way in the study of the equivalence of the case in which the sub-algebra A separating over $\boldsymbol{E}$ and the general case. For the sake of simplicity we shall assume in most of what follows the following hypothesis:

Hypothesis H. Let $\boldsymbol{E}$ be a compact space and A a sub-algebra of ℓ $(E;k)$, containing the constants, which is assumed to be self-adjoint in the complex case. Assume that ℓ $(E;k)$ is provided with the topology of uniform convergence given by the norm f : sup $\{|f(x)|\ ;\ x \in E\}$ for all $f \in \ell$ $(E;k)$.

The first two theorems of this paragraph are a separating and a general version of the classical Weierstrass-Stone Theorem on sub-algebras of $\ell\ (E;k)$.

***Theorem* 1.** Assume hypothesis H. Then A is dense in $\ell\ (E;k)$ if and only if A is separating over E.

***Theorem* 2.** Assume hypothesis H and let $f \in \ell\ (E;k)$. Then f belongs to the closure of A in $\ell\ (E;k)$ if and only if f is constant on each equivalence class of E modulo E/A.

Before stating the last two theorems of this paragraph we make the following remark:

Remark 1. Let E be the closed unit interval [0,1]. The vector subspaces P ([0,1]; K) and A_0, of ℓ ([0 ,1]; K), generated by the functions $t \mapsto t^n$ $(n = 0, 1, 2, \ldots)$ and $t \mapsto t^{2n}$ $(n = 0,1,2,\ldots)$, respectively, are, in fact, sub-algebras of to which Theorem 1 applies : they are dense sub-algebras of ℓ ([0,1]; K).

However, the assumption that the algebras involved in Theorems 1 and 2 must contain the constants is an annoying restriction. For instance, it seems obvious that $L = \{f \in \ell\ ([0,1]: K); f(0)=0\}$ is the closure of the sub-algebra A_2 generated by the functions $t \mapsto t^{2n}$ $(n=1,2,\ldots)$ which is self-adjoint in the complex case but does not contain the constants. This fact does not follow directly from Theorems 1 and 2.

The above restriction can be dropped at the price of slightly complicating the necessary and sufficient conditions involved. But then one observes that it also seems obvious that L is the closure of the vector subspace W generated by the functions $t \mapsto t^{2n}$ $(n = 0,1,2\ldots)$ – and W is not a sub-algebra of ℓ ([0,1]; K). Notice the following facts, however : (i) A_2 and W are vector subspaces of L which is by itself a vector subspace of ℓ ([0,1]: K); (ii) A_2 and W are A_0-modules and A_0 is a sub-algebra containing the constants which is separating over [0,1]and is self-adjoint in the complex case; (iii) for each $x \in [0,1]$ the equalities $A_2(x) = L(x)$ and $W(x) = L(x)$ results. The fact that L is the closure of both A_2

and W is then made obvious by the following two versions of Nachbin's Weierstrass-Stone Theorem for modules)[11)], §19, Theorem 1). With an eye on the possible applications of these results to algebras, we still add the following comment:

The Relative standing of A_2 on A_0 above is typical of a more general situation. Let B be a sub-algebra, self-adjoint in the complex case, of ℓ $(E;K)$ where E is a Hausdorff space, say. If B does not contain the constants, then the sub-algebra A generated by B and the constant does, and A is also self adjoint in the complex case. Now B is always an A-module; and the E/A equivalence relation and the E/B equivalence relation, as defined in § 2, coincide. Observe also that a vector subspace $W \subset \ell\ (E; K)$ is a B-module if, and only if, it is an A-module.

Therorem 3. Assume hypothesis H and that A is separating over E. Let L and W be vector subspaces of $\ell\ (E;K)$ with $W \subset L$, and assume that W is an A-module. Then W is dense in L if and only if $W(x) = L(x)$, for all $x \in E$.

Theorem 4. Assume hypothesis H. Let W be a vector subspace of ℓ $(E;k)$, with $AW \subset W$. Then $f \in \ell\ (E;K)$ belongs to the closure of W in ℓ $(E;K)$ if and only if $f \mid X$ is in the closure of W/X in $\ell\ (X; K)$ for every equivalence class X of E modulo E/A.

Therorem 1 (resp. Theorem 3) is a corollary of Theorem 2 (resp. Theorem 4), and there arises the question of their equivalence. We treat first the algebra case.

Proposition 1. Theorem 1 implies Theorem 2.

Proof. The necessity part of Theorem 2 is evident. Let E, A and f be as in Theorem 2. To E and A there corresponds F, π and B by the quotient construction of §2. Since f is constant on each equivalence class of E modulo E/A, there is a unique $g \in \ell\ (F;K)$ such that $f = g.\pi$ By Theorem 1, B is dense in $\ell\ (F;K)$. Therefore g belongs to the closure of B in $\ell\ (F;K)$. Since the mapping $h \mapsto h.\pi$ is an isometry of $\ell\ (F;K)$ into $\ell\ (E;K)$, it follows that f belongs to the closure of A in $\ell\ (E;K)$.

Remark 2. Suppose that A is closed. By Theorem 2, A consists then of all $f \in \ell\ (E;\ k)$ which are constant on each equivalence class of E modulo E/A. The map $j(f) = g$, where g is the unique $g \in \ell\ (G;K)$ such that $f = g.\pi$, *is* then a linear isometry of A onto $j(A) = B$. Since B is dense, it follows that $B = \ell\ (F;k)$. Hence we have proved the following "representation result".

Proposition 2. Assume hypothesis H and that A is closed. Then there exists a compact space F jointly with a continuous map π of E onto F and a linear isometry j of A onto the algebra $\ell\ (F;k)$. The map j is also a * - algebra homomorphism.

We now turn our attention to the case of modulls. Let E, A, W and f be as in Theorem 4. We obtain F, π and B via the quotient construction. However, since f is notnecessarily constant on each equivalence classf E modulo E/A we cannot factor f through π in general as before. By insisting on the quest for natural "quotient objects" to attach to f and W one is led naturally to consider vector fibrations. Indeed, to each $f \in \ell\ (E;K)$ there corresponds a cross-section $(f \mid \pi^{-1}(y);\ y \in F)$, i.e. an element of the Cartesian product $P = \Pi\ (F_y;\ y \in F)$, where $F_y = \ell\ (\pi^{-1}(y);k)$ for each $y \in F$. Let j denote the map $f \mapsto (f\pi^{-1}(y);\ y \in F)$, *and let* $\ell = j\ (\ell\ (E;k))$ and $\mathscr{W} = j(W)$. ℓ and $\mathscr{W}$ are provided with the structure of vector subspaces of the product vector space $\Pi\ (F_y;\ y \in F)$, and $\mathscr{W}$ is then a vector subspace of ℓ. Notice that

$$F_y = \{f \mid \pi^{-1}(y);\ f \in \ell\ (E;K)\} = \ell\ (y)$$

by the Tietze- Yrhsohn Extension Theorem. The following additional results are true:

a. For each $y \in F$, let $v(y)$ be the sup-norm on $\ell\ (\pi^{-1}(y);k) = \ell\ (y)$. Since $\pi^{-1}\ (y)$ is closed in E, hence compact, $(\ell\ (y),\ v(y))$ is a Banach space.

b. Since $B \subset \ell\ (F;k)$, $P = \Pi\ (F_y;\ y \in F)$ has the natural structure of a B-module : if $b \in B$ and $(f_y\ ;\ y \in F)$ is an element of P, then $b\ (f_y\ ;$

$y \in F)$ is the element $(b(y)f_y, y \in F)$. It follows that $\mathcal{W}$ is a B-submodule, since W is and A-module.

c. Let $v = (v(y); y \in F)$ and for any $f \in \ell(E;k)$ put

$$\|j(f)\|_v = \sup\{v(y)[f|\pi^{-1}(y)]; y \in F\}.$$

Then $0 \leq \| j(f) \|_v < \infty$ and $j(f) \mapsto \| j(f) \|_v$ is a norm on ℓ. The map $j: \ell(E;k) \to \ell$ is a linear isometry of $\ell(E;k)$ onto $(\ell, \| \ \|_v)$ is a Banach space and the closure of $\mathcal{W}$ in $(\ell, \| \ \|_v)$ is $j(\overline{W})$, where $(\overline{W})$ denotes the closure of W in $\ell(E;K)$.

d. The condition on f of Theorem 4 is equivalent to the assertion that $j(f)(y)$ belongs to the closure of $\mathcal{W}(y)$, in $(\ell(y), v(y))$ for each $y \in F$. This leads to the introduction of the vector subspace $\mathcal{L} \subset \ell$ of all cross-sections $(f|\pi^{-1}(y); y \in F)$ in ℓ such that $f|\pi^{-1}(y)$ belongs to the closure of $\mathcal{W}(y)$ in $(\ell(y), v(y))$ for each $y \in F$. The vector subspace $\mathcal{L}$ is the image under j of the vector subspace $L = \{f \in \ell(E;K); {}^{f}/_{x}$ belongs to the closure of W/x in $\ell(X;K)$ for each equivalence class X of E modulo $E/A\}$. Hence the thesis of Theorem 4 is equivalent to the assertion that $\mathcal{L}$ is the closure of $\mathcal{W}$ in $(\ell, \| \ \|_v)$.

e. The condition in Theorem 4 is equivalent to the assertion that $\mathcal{W}(y)$ is dense in $\mathcal{L}(y)$ for each $y \in F$, as subset of $(\ell(y), v(y))$.

Remark 3. It is now clear that we have all the ingredients for a version of Theorem 3 that would imply Theorem 4 by the sort of quotient argument we decided to use. Roughly speaking, It should state that the assertion "$\mathcal{L}$ is the closure of $\mathcal{W}$ in $(\ell, \| \ \|_v)$" follows because B is separating over F and $\mathcal{W}$ is a vector subspace of $\mathcal{L}$ which is a B-module and is such that $\mathcal{W}(y)$ is dense in $\mathcal{L}(y)$ for $y \in F$. The conjecture that such a result might be true is reinforced by the fact that we do know that $\mathcal{L} = \overline{\mathcal{W}}$ in $(\ell, \| \ \|_v)$: this is equivalent to $L = \overline{W}$ by item d, and $L = \overline{W}$ is true because Theorem 4 is true as a particular case of Theorem 1, § 19 [11], through we are not assuming it to have been proved.

§ 4. THE SEMICONTINUITY CONDITION

We consider the "module case" closed. There is however a somewhat subtle point still left untouched and which is most important for the selection of the proper hypothesis in Theorem 5 of the next paragraph.

We reconsider the situation in item c) of § 3 above. It is stated there that for each $f \in \ell\ (E;K)$ it is true that

$$0 \leq \sup \{v(y)[f \setminus \pi^{-1}(y);\ y \in F\} < \infty \qquad (*)$$

and, in particular, we may state that $y \mapsto \sup \{|f(x)|;\ x \in \pi^{-1}(y)$ is a finite non-negative function of F. The question is : is (*) valid because $y \mapsto \sup \{|f(x)|;\ x \in \pi^{-1}(y)\}$ is a continuous real-valued function of the compact space F? The answer is no. These functions are only upper semicontinuous in general, as the following example and lemma below show.

Example 1. Let E be the closed unit interval [0,1]. Let A be the set of all $f \in \ell\ (E;k)$ such that f is constant on [0,1/2]. A is clearly a sub-algebra of $\ell\ (E;k)$ which is self-adjoint in the complex case and contains the constants. Applying the quotient construction to A and with $E = [0,1]$ we obtain $F = \{[0,1/2]\} \cup \{\{x\};x \in (1/2,1]\}$ and the map $h : [1/2\ ,1] \rightarrow F$, such that $h\ (1/2) = [0,1/2]$ and $h(x) = \{x\}$ for all $x \in (1/2,1]$, is a homeomorphism of [1/2, 1] onto F. The function $x \mapsto f(x) = 1\text{-}x$ of $\ell\ (E;K)$ is such that $v(y)\ [f \setminus \pi^{-1}(y)] = 1$ at $y = [0,1/2]$ and $v(y)\ [f|\pi^{-1}(y)] = 1\text{-}x$ for all $y = \{x\}$ with $x \in (1/2,1]$. The function $y \mapsto \sup \{|\ f(x)|x \in \pi^{-1}(y)\}$ is clearly upper semicontinuous on F but it is not continuous at the point $y = [0,1/2]$ of F.

Of course, a non-negative finite upper semicontinuous function on a compact space is bounded and attains its maximum in at least one point of the space. The relevant aspects of item c are a special case of the situation covered by the following result :

Lemma **1.** Let E and F be (non-empty) compact Hausdorff spaces and π a continuous mapping from E onto F. For each upper semicontinuous function g : E$\mapsto$R define $h(y) = \sup \{g(x)\ ;\ x \in \pi^{-1}(y)\}$

for all $y \in F$. Then the correspondence $y \mapsto h(y)$ is a real-valued upper semicontinuous function on F.

Proof. For each $y \in F$, the set $\{y\}$ is closed in F, therefore $\pi^{-1}(y)$ is compact in E and non-empty because π is non-empty. Hence there is an a $\pi^{-1}(y)$ *such that* $h(y) = g'(a) = \sup \{|g(x)|; x \in \pi^{-1}(y)\}$ because g is upper semicontinous , so that $h(y)$ is finite.

Denote by h the real-valued function on F which has the value $h(y)$ in each $y \in F$. Let r be some real number. The set $\{x \in E; g(x) \geq r\} = Y$ is compact, because g is upper semicontinuous and E is compact, hence $\pi(Y)$ is compact in F. We claim that $\pi(Y) = \{y \in F; h(y) \geq r\}$, which proves that h is upper semicontinuous. Indeed, If $y \in \pi(Y)$, then, $y = \pi(x)$ for some $x \in Y$, and then $h(y) \geq g(x) \geq r$. Conversely, if $y \notin \pi(Y)$ and $t \in \pi^{-1}(y)$, then $g(t) < r$; it follows that $h(y) < r$ because there is $t_0 \in \pi^{-1}(y)$ such that $g(t_0) = h(y)$ and the proof is finished.

§ 5. BASIC DEFINTIONS AND RESULTS

The considerations in §'s 3 and 4 justify the introduction of the following notions.

***Definition* 1.** A vector fibration is a pair $(E; (F_x; x \in E))$ where E is a Hausdorff space and $(F_x; x \in E)$ is a family of vector spaces over the same scalar field K. Such a vector fibration is also said to be a vector fibration over E. The product set $\Pi (F_x; x \in E)$ is always provided with the structure of a product vector space; it is the vector space of , or attached to, the vector fibration. A cross-section is then any element $f = (f(x); x \in E)$ of the vector space $\Pi (F_x; x \in E)$, and a vector space of cross-sections is any one of its vector subspaces. For any cross-section $f = (f(x); x \in E)$ and any K-valued function a on E, we define the cross-section $a f$ by putting $(a f)(x) = a(x) f(x)$ for each $x \in E$. A vector space of cross-sections, W, is said to be a module over a sub-algebra $A \subset \ell(E; K)$, or an A-module, if $A W = \{aw; a \in A, w \in W\} \subset W$.

Any family $v = (v_x ; x \in E)$ such that v_x is a semi-norm on F_x for each $x \in E$ is called a weight of the vector fibration $(E (F_x; x \in E))$; then, if $y \in F_x$, $v_x[y]$ denotes the value of v_x at y. If $f = (f(x);\ x \in E)$ is a cross-section, $v\,[f]$ denotes the function on E which has the value $v_x\,[f(x)]$ at each $x \in E$.

Example 2. $\ell\ (E;K)$ is a subset of $\Pi\ (F_x;\ x \in E)$ where $F_x = K$ for each $x \in E$. If $F_x = K$ for each $x \in E$, $(E;\ (F_x\ ;\ x \in E))$ is, by definition, the scalar fibration over E - also said to be the real fibration over E if $K = \mathbb{R}$, or the complex fibration over E if $K = \mathbb{C}$. Its vector spaces of cross-sections are simply vector spaces of K-valued functions on E under the usual pointwise operations, and $\ell\ (E;K)$ is perhaps the most important among them.

We now state and prove result conjectured in Remark 3.

***Theorem* 5.** Assume hypothesis H with A separating over E. Let $(E;\ (C_x;\ x \in E))$ be a vector fibration over E, and $v = (v_x;\ x \in E)$ and ℓ respectively a weight and a vector space of cross-sections pertaining to it such that $v\,[f]$ is upper semicontinuous on E for each $f \in \ell$.

a. Then $f \in \ell \mapsto \|f\|_v = \sup\ \{v_x[f(x)];\ x \in E\}$ is a semi-norm on ℓ. It is a norm if v_x is also a norm for all $x \in E$.

b. Let $\mathscr{L}$ and $\mathcal{W}$ be vector subspaces of ℓ such that $\mathscr{L} \supset \mathcal{W}$ and $\mathcal{W}$ is an A-module. Then $\mathcal{W}$ is dense in $\mathscr{L}$ in the semi-normed space $(\ell\ ,\ \|\ \|_v)$, if and only if, for each $x \in E$, $\mathcal{W}(x)$ is dense in $\mathscr{L}(x)$ in the semi-normed space $(\ell\ (x),\ v_x)$.

Proof. The inequalities $0 \le \|f\|_v < \infty$ are true for each $f \in \ell$ because $x \mapsto v_x\ \ [f\,(x)]$ is a finite non-negative upper semicontinuous function on E for each $f \in \ell$ the remaining assertions in item (a) are also easily proved.

The necessity part of the assertion in item (b) is evident. We prove its sufficiency below.

Let $f = (f\,(x);\ x \in E)$ be an arbitrary but fixed element of $\mathscr{L}$. Let any $\epsilon > 0$ be given. For each $t \in E$, the assumed density of $\mathcal{W}(t)$ in $\mathscr{L}(t)$ as subsets of $(\ell\ \ (t),\ v_t)$ implies the existence of $w_t = (w_t(x);\ x \in E)$ in $\mathscr{W}$ such that

$$V_t [f(t) - w_t (t)] < \in.$$

Since $\mathcal{W} \subset \mathcal{L}$ and $\mathcal{L}$ is a vector space of cross-sections, we have $f - w_t \in \mathcal{L}$ for each $t \in E$, and the function $x \mapsto v_x[f(x) - w_t(x)]$ is upper semicontinuous on E because $\mathcal{L} \subset \ell$. This implies that $K_t = \{x \in E;\ v_x [f(x) - w_t(x)] \geq \in\}$ is compact for each $t \in E$; $U_t = E \setminus K_t$ is open and $(U_t : t \in E)$ is an open cover of the compact space E and there are finitely many points $t_1, \ldots, t_n$ in E such that $E \subset \bigcup_{i=1}^{n} U_{t_i}$.

Let $\{\varphi_i;\ i=1,\ldots,n\}$ be a continuous partition of the unity subordinated to the covering $\{U t_i ;\ i = 1,\ldots,n\}$ of E (see [10], Chapter I, § 12, Theorem 4). Then $\varphi \in \ell \ (E; \mathbf{R})$, $0 \leq \varphi \leq 1$, and $\varphi_i (K_{ti}) = \{0\}$ for each $i \in \{1,\ldots,n\}$; and for each $x \in E$ the equality $\Sigma_{i=1}^{n} \varphi_i(x) = 1$ results. Consider the cross-section $g = (g(x) ; x \in E) = \Sigma_{i=1}^{n} \varphi_i W_{t_i}$. We claim that

$$v_x [f(x) - g(x)] \leq \in \text{ each } x \in E. \tag{1}$$

To prove this we notice that

$$v_x[f(x) - g(x)] = v[\Sigma_{i=1}^{n} \varphi_i(x)(f(x) - w_{ti}(x))] \leq \Sigma_{i=1}^{n} \varphi_i(x) v_x[f(x) - w_{ti}(x)].$$

This shows that (1) follows from $\varphi_i (x)\, v_x [f(x) - w_{ti} (x))] \leq \in \varphi (x)$, $x \in E$; this estimate is proved by observing that $\varphi_i (x) = 0$ if $x \in K_{ti}$ and that $v_x [f(x) - w_{ti} (x)] < \in$ if $x \in U_{ti}$. This proves (1). Now, A is dense in $\ell \ (E; K)$ by the Weirstrass-Stone Theorem (§ 3, Theorem 1). Hence given any $\delta > 0$, there are $h_i \in A \ (i = 1,\ldots,n)$ such that for each $x \in E$ and $i \in \{1, \ldots, n\}$, $|\varphi_i (x) - h_i (x)| \leq \delta$. Since $\mathcal{W}$ is an A-module, the cross-section $w = \Sigma_{i=1}^{n} h_i w_{ti}$ belongs to $\mathcal{W}$. Letting $M = \max \{\|w_{ti}\|_v ;\ i = 1,\ldots,n\}$, choose $\delta \leq \in / Mn$, and use (1); it then follows that

$$\| f - w \|_v = \sup\{ v_x[f(x) - w (x)];\ x \in E\} \leq$$

$$\in + \Sigma_{i=1}^{n} \sup\{ |\varphi_i(x) - h_i(x)| v_x[w_{t_i}(x)]; x \in e\} \leq \in + \delta \Sigma_{i=1}^{n} \|W_{t_i}\|_v \leq 2 \in.$$

This completes the proof.

***Remark* 4.** In the remaining part of this paragraph we prove two propositions which together add up to an interesting representation result.

The importance of this result in the present context lies in the illuminating role it plays as regards semicontinuity assumptions as such as the one in Theorem 5. In Theorem 5, it is clear that, for each $x \in E$, the part of C_x outside of its vector subspace $\ell\ (x)$, indeed out of $\mathscr{L}\ (x)$, plays no essential role; that is, we might replace the original vector fibration by $(E; (\ell\ (x); x \in E))$, or even by $(E; \mathscr{L}(x); x \in E))$.

***Definition* 2.** Let $(E;(C_x;\ x \in E))$ be a vector fibration in which is given a fixed weight v such that v_x is a norm on C_x for each $x \in E$. The space E is assumed to be compact (and non-empty). Let $\mathscr{L} \subset \Pi\ (F_x; x \in E)$ be a vector space of cross-sections. A representation of $\mathscr{L}$ is a linear map r of $\mathscr{L}$ into $\ell\ (F;K)$ where F is a compact Hausdroff space provided with a continuous onto map $\pi : F \to E$ such that for all $f \in \mathscr{L}$ and all $x \in E$ the equality

$$v_x\ [f(x)] = \sup \{|(r(f))\ (y)|\ ; y \in \pi^{-1}(\{x\})\}$$

results. And $\mathscr{L}$ is said to be essential if. $\mathscr{L}\ (x) = C_x$ for all $x \in E$.

***Proposition* 3.** Under the conditions of Definition 2, if $\mathscr{L}$ admits a representation, then $f \in \mathscr{L} \mapsto \|f\|_v = \sup \{v_x\ [f(x)]\ x \in E\}$ is a norm on $\mathscr{L}$ and r is a linear isometry of $(\mathscr{L}\ \|\ \|_v)$ into $(\ell\ (F,K), \|\ \|_F)$ where $\|\ \|_F$ is the sup norm. Moreover, for each $f \in \mathscr{L}$, the function $x \mapsto v_x\ [f(x)]$ is upper semicontinuous on E.

Proof. For each $f \in \mathscr{L}$, write $\tilde{f} = r(f)$. By the definition of representation, the equality $v_x\ [f(x)] = \sup \{|\tilde{f}\ (y)|;\ y \in \pi^{-1}(\{x\})\}$ is true for each $x \in E$. Since π is continuous and onto, $(\pi^{-1}\ (\{x\});\ x \in E)$ is a partition of F into non-empty, compact subsets; and since $\tilde{f} \in \ell\ (F;K)$ for each $f \in \mathscr{L}$ it follows that

$$\sup \{v_x\ [f(x)];\ x \in E\} = \sup \{\sup\{|\tilde{f}\ (y)|; y \in \pi^{-1}(\{x\})\}; x \in E\} =$$

$$= \sup \{|\tilde{f}\ (y)|; y \in F\} = \|\ \tilde{f}\ \|_F < \infty.$$

It is now clear that r is a linear isometry of $(\mathscr{L}, \|\ \|_v)$ into $(\ell\ (F;\ k, \|\ \|_F)$. Let f be a fixed element of $\mathscr{L}$. Since $\tilde{f} = r\ (f) \in \ell\ (F;K)$ is related to $v\ [f] : E = \pi\ (F) \to \mathbf{R}$ by the set of equations

$$V_x[f(x)] = \sup\{|\tilde{f}(y)|;\, y \in \pi^{-1}(\{x\})\}, \qquad x \in E,$$

the upper semicontinuity of $v\,[f]$ follows from Lemma 1. The proof is now complete.

The following converse of Proposition 3 is true.

Proposition 4. Let $(E;\ (C_x;\ x \in E))$, v, and $\mathscr{L}$ be as above. Assume that $\mathscr{L}$ is essential and that $v\,[f]$ is upper semicontinuous for each $f \in \mathscr{L}$. Then there exists a representation of $\mathscr{L}$.

Proof. The proof consists of the construction of a compact Hausdorff space F jointly with a continuous onto map $\pi : F \to E$ and a linear map $r : \mathscr{L} \to \ell\ (F;K)$ such that

$$V_x[f(x)] = \sup\{|r(f)\,(y)|\ ;\ y \in \pi^{-1}(\{x\})\}$$

for all $f \in \mathscr{L}$ and all $x \in E$.

First, observe that the function $f \in \mathscr{L} \to \|f\|_v := \sup\{v_x[f(x)];\ x \in E\}$ is a norm on

$\mathscr{L}$. Let $\overline{B}(C_x^{'})$, $x \in E$, and $\overline{B}\,(\mathscr{L}')$ denote the unit balls of the continuous duals $C_x^{'}$, $x \in E$, and $\mathscr{L}'$, respectively, of the normed spaces $(C_x,\ v_x)$, $x \in E$, and $(\mathscr{L}',\ \|\ \|_v)$, as topological subspaces of these continuous duals provided with the respective weak-star topologies $\sigma\,(C_x^{'},\ C_x)$, $x \in E$, and $\sigma\,(\mathscr{L}',\mathscr{L})$. $\overline{B}\,(C_x^{'})$, $x \in E$, and $\overline{B}\ (\mathscr{L}')$ are compact Hausdorff spaces by the Alaoglu-Bourbaki Theorem. F, purely as a set, is defined through the equation

$$F = \cup\{\{x\} \times \overline{B}(C_x^{'});\, x \in E\},$$

F is a disjoint union, or sum, of the sets $\overline{B}(C_x^{'})$, $x \in E$. This set is provided with natural onto map $\pi : F \to E$ such that for each $(x,\ \varphi) \in F$, $\pi\,(x, \varphi) = x$.

The bulk of the proof consists of providing F with a topology. Define $\varpi : F \to \overline{B}\,(\mathscr{L}')$ as follows. For each $(x\ ,\ \varphi) \in F$, that is, for each $x \in E$ and $\varphi \in \overline{B}\ (C_x^{'})$, the value of ϖ at $(x\ ,\ \varphi)$ is the function $\varpi_{x,\varphi} : \mathscr{L} \to K$

such that $\omega_{x,\varphi}(f) = \varphi(f(x))$ for all $f \in \mathcal{L}$. This definition of ω is justified by the following assertion:

(1) $\omega_{x,\varphi} \in \overline{B}(\mathcal{L}')$ for each $x \in E$ and each $\varphi \in E\,\overline{B}(C_x')$.

Indeed, $\omega_{x,\varphi}$ is obviously linear and,

(2) for each $f \in \mathcal{L}$, $|\omega_{x,\varphi}(f)| = |\varphi(f(x))| \leq v_x[f(x)] \leq \|f\|_\infty$

This computation also proves the necessity part of the next assertion:

(3) A linear functional Φ on $\mathcal{L}$ belongs to the image set $\omega(F)$ if, and only if, there exists $x \in E$ such that $|\Phi(f)| \leq v_x[f(x)]$ for all $f \in \mathcal{L}$. If this condition is satisfied, then $\Phi = \omega_{x,\varphi}$ where $\varphi \in \overline{B}(C_x')$ is such that for any given $z \in C_x'$, $\varphi(z) = \Phi(f)$ for any $f \in \mathcal{L}$ for which $f(x) = z$.

Let x and φ be as in (2) for a given Φ. The function φ is well-defined: indeed if $f_1, f_2 \in \mathcal{L}$ and $f_1(x) = f_2(x)$, then $|\Phi(f_1) - \Phi(f_2)| = |\Phi(f_1 - f_2)|\, v_x\,[f_1(x) - f_2(x)] = 0$; on the other hand, given any $z \in C_x$ there is a $f \in \mathcal{L}$ such that $f(x) = z$, because $\mathcal{L}$ is essential. φ is obviously a linear functional on C_x. Actually, $\varphi \in B(C_x')$; for, if $z \in C_x$ and $f \in \mathcal{L}$ are such that $f(x) = z$, then $\varphi(z) = \Phi(f(x))$, and therefore $|\varphi(z)| = \Phi(f(x))\,| \leq v_x[f(x)]\, v_x[z]$ which proves that $\varphi \in \overline{B}(C_x')$ because v_x is the norm of C_x. For this φ, what is $\omega_{x,\varphi}$? It is such that for all $z \in C_x$ or, since $\mathcal{L}$ is essential, for all $f(x)$, $f \in \mathcal{L}$, it is true that $\omega_{x,\varphi}(f) = \varphi[f(x)] = \Phi(f)$. Hence $\Phi = \omega_{x,\varphi} \in \omega(F)$, and the proof of (3) is complete.

To the maps $\pi : F \to E$ and ω $F \to \overline{B}(\mathcal{L}')$ corresponds the map

$$\pi \times \omega : F \to E \times \overline{B}(\mathcal{L}').$$

(4) $\pi \times \omega$ is an injective map.

To see this let (x, φ) and (y, ψ) be elements of F such that $(x, \omega_{x,\varphi}) = (y, \omega_{y,\psi})$. Then $x = y$; hence $\omega_{x,\varphi} = \omega_{x,\psi}$, i.e. for all $f \in \mathcal{L}$, $\omega_{x,\varphi}(f) = \varphi(f)(x)) = \omega_{y,\psi}(f) = \Phi(f(x))$ implying that $\varphi = \Psi$ (because $L(x) = C_x$).

Providing $E \times \overline{B}(\mathcal{L}')$ with the product topology, it is then a compact Hausdorff space. The following assertion is true as a consequence (in particular) of the semicontinuity assumption.

(5) The image $(\pi\times\omega)(F)$ is a compact subset of the compact Hausdorff space $E\times\overline{B}(\mathscr{L}')$.

It is enough to prove that $(\pi\times\omega)(F)$ is closed in $E\times\overline{B}(\mathscr{L}')$. To this end let $(x, \omega_{xj,\varphi})$ be a net in $(\pi\times\omega)(F)$ convergent to $(y, \Phi)\in\times\overline{B}(\mathscr{L}')$. Since $E\times\overline{B}(\mathscr{L}')$has the product topology, it follows that $x_j\to y$ in E and that $\omega_{xj,\varphi j}\to\Phi$ in $\overline{B}(\mathscr{L}')$. However, convergence in $\overline{B}(\mathscr{L})$ means weak convergence, therefore, for each $f\in\mathscr{L}$,

$$\omega_{xj,\varphi j}(f)=\varphi_j(f(x_j))\to\Phi(f)$$

results. Using (2), the inequality $|\omega_{xj,\varphi j}(f)|\le[f(x_j)]$ follows since $x_j\to y$ and $v[f]$ is upper semicontinuous for each $f\in\mathscr{L}$ it follows that

$$\lim_j|\omega_{xj,\varphi}(f)|=|\Phi(f)|\le\lim_j\sup v_{xj}[f(x_j)]=v_y[f(y)]$$

for all $f\in\mathscr{L}$. (3) now implies that $\Phi\in\omega(F)$; actually, $\Phi=\omega_{y,\varphi}$ where $\varphi\in\overline{B}(C_y')$ with $\varphi(z)=\Phi(f)$ for $z\in C_y$, and $f\in\mathscr{L}$ with $f(y)=z$. Hence $(y,\Phi)=(y, \omega_{y,\varphi})\in\omega(F)$ proving that $\omega(F)$ is closed in $E\times\overline{B}(\mathscr{L}')$.

By (4), $\pi\times\omega$ is an injective map so that the correspondence $(x,\varphi)\in F\to(x, \omega_{x,\varphi})\in(\pi\times\omega)(F)$ is bijective. F is topologized by transporting to it the structure of a compact Hausdorff space of $(\pi\times\omega)(F)$ via this bijection. With each $f\in\mathscr{L}$, associate $\tilde{f}: F\to K$ defined by $\tilde{f}(x,\varphi)=\omega_{x,\varphi}(f)=\varphi(f)=\varphi(f(x))$ for all $(x,\varphi)\in F$. Assume that it has been proved that $\tilde{f}\in\ell(f;K)$for all $f\in\mathscr{L}$ and consider $r:\mathscr{L}\to\ell(F;K)$ such that $r(f)=\tilde{f}$ for all $f\in\mathscr{L}$. The map is obviously linear. As a consequence of the Hahn-Banach Theorem, the equality $v_x[f(x)]=\sup\{|\varphi(f(x))|;\varphi\in\overline{B}(C_x')\}$ is obtained for any $x\in E$ and any $f\in\mathscr{L}$; but, clearly, $\sup\{|\varphi(f(x))|;\varphi\in\overline{B}(C_x')=\sup\{|\tilde{f}(x,\varphi)|;\varphi\in\overline{B}(C_x')=\sup\{|r(f))(y)|; y\pi^{-1}(\{x\})\}$, since $\pi^{-1}(\{x\})=\{x\}\times\overline{B}((C_x')$. Therefore r is a representation of $\mathscr{L}$ except that the assertion "$r(f)=\tilde{f}\in\ell(f;K)$ for all $f\in\mathscr{L}'$ still remains to be proved. The truth of the assertion follows from the following remarks.

By the definition of the topology of F, the map $\pi\times\omega$ is continuous. Fix $f\in\mathscr{L}$. The evaluation map$\in\overline{B}(\mathscr{L}'):\mapsto\Phi\{f\}'$ is continuous in the given topology of $\overline{B}\{\mathscr{L}'\}$ hence, $s_f: E\times\overline{B}(\mathscr{L}')\mapsto K$ such that s_f

$(x,\Phi)=\Phi(f)$ for all $(x,\Phi)\in E\times B(\mathscr{Z}')$ is continuous. Finally, $\tilde{f}=\tau(f)=s_f\cdot(\pi\times\omega)$ and the proof is complete.

§ 6. A MORE GENERAL RESULT - APPLICATION - COMMENTS

There corresponds to Theorem 5 a "general version" in which A is not assumed separating. Theorem 6 is such a version except that the conditions are relaxed and E is allowed to be a completely regular Hausdorff space.

Theorem 6 is a generalization of Nachbin's Weierstrass-Stone Theorem for modules of scalar-valued functions as stated in [11], § 19, Theorem 1, and of which Theorems 3 and 4 in our § 3 are special cases. The reader will notice that Nachbin's proof of this theorem in [11] was repeated almost verbatim depends only on Theorem 1 plus some algebraic-topological arguments, in particular a quotient-type argument. Nachbin's theorem, referred to above, may then be considered as a first application, in fact, an obvious specialization of Theorem 6; in this sense, Theorem 3 is an obvious special case of Theorem 5.

Theorem 6. Let C and v be, respectively, a vector space of cross-sections and a weight of a vector fibration $(E;(F_x;x\in E)$, such that $v[f]$ is upper semicontinuous for each $f\in C$. Assume that E is completely regular and that $A\subset \ell(E;K)$ is a sub-algebra containing the constants and self-adjoint in the complex case, and denote the set of the compact subsets of E by $\mathscr{K}_E$. The following results are true:

1) For each $K\in\mathscr{K}_E$ and $f\in C$, put $p_{v,k}(f)=\sup\{v_x[f(x)];x\in K\}$. Then $(p_{v,k};K\in\mathscr{K}_E)$ is a directed family of seminorms on C. Let $\omega_{v,c}$ be the topology on C defined by this family of seminorms. Then $(C,\omega_{v,c})$ is a locally convex space which is separated if v_x is a norm on F_x for each $x\in E$.

2) Let $W\subset C$ be a vector subspace which is an A-module and $f_0\in C$. Then f_0 belongs to the closure of W in $(C,\omega_{v,c})$ if, and only if, for any equivalence class $X\subset E$ modulo E/A, any compact subset $K\subset X$ and any $\in>0$, there is $w\in W$ such that $p_{v,K}(\omega-f_0)<\in$.

3) Under the conditions of item 2, suppose further that A is separating over E. Then f_0 belongs to the closure of W in $(C, \omega_{v,c})$ if, and only if, for any $\epsilon>0$ and $x \in E$ given, there is $w \in W$ such that $v_x [w(x)-f_0(x)] <\epsilon$. Also, W is dense in $(C, \omega_{v,c})$ if, and only if, $W(x)$ is dense in $(C(x), v_x)$ for each $x \in E$.

Proof. All assertions in (1) and the necessity part of (2) are easily proved. We treat the sufficiency part of (2) only. We may then assume that E is compact (compare [11], § 14, Remark 2). Now hypothesis H applies to E and A, and we obtain, F,B and π by the quotient construction; and $(C, \omega_{v,c})$ is the semi-normed space $(C, \| \|_v)$ where $\|f\|_v$=sup$\{v_x [f(x)]; x \in E\}$ for all $f \in C$. For each $y \in F$, $C_y = \{f | \pi^{-1}(y); f \in C\}$ is a vector space over K in a natural way and $(F; C_y; y \in F))$ is a vector fibration. Since $\pi^{-1}(y)$ is a compact subset of E, the function $f | \pi^{-1}(y) \mapsto \tilde{v}_y[f | \pi^{-1}(y)] = \sup \{v_x [f(x)]; x \in \pi^{-1}(y)\}$ is a sminorm on C_y. Then $\ell = \{ \tilde{f} = (f | \pi^{-1}(y); y \in F); f \in C\}$ and $\mathscr{W} = \{ \tilde{\omega}; w \in W\}$ are vector spaces of cross-sections of $(F; (C_y; y \in F)$ and $\tilde{\ell}(y) = C_y$ for each $y \in F$, and $\tilde{v} = \{\tilde{v}_y; y \in F)$ is a corresponding weight. It follows that $\tilde{v}[f]$ is uppersemicontinuous on F for each $f \in C$, and that $\| \|_{\tilde{v}} : \ell \mapsto \mathbf{R}$ such that $\|\tilde{f}\|_{\tilde{v}} = \sup\{ \tilde{v}_y[f | \pi^{-1}(y)]; y \in F\}, \tilde{f} \in \ell$ is a semi-norm on ℓ.

Let $L \subset C$ be the set of all $f \in C$ such that $f | \pi^{-1}(y)$ belongs to the closure of $\mathscr{W}(y)$ in $\ell(y) = C_y$ provided with the semi-norm $\tilde{v}_y$ for each $y \in F$. Let $\mathscr{L} = \{ \tilde{f}; f \in \mathscr{L}\}$. Define $i : C \to \ell$ By putting $i(f) = \tilde{f}$ for each $f \in C$. Theorem 5 applies in an obvious way and the proof of (2) ends. Results in item (3) follow trivially from item 2 because the E/A – equivalence classes are the one point subsets of E. This completes the proof.

Example 3. Let F be a Banach space with norm $\| \|$ and for each point $x = (x_1, \ldots x_n)$ in real Euclidean n-space, $\mathbf{R}^n$, put $F_x = F$. Consider the vector fibration $(\mathbf{R}^n; (F_x; x \in \mathbf{R}^n)$; from its vector space of cross-sections, distinguish $\ell(\mathbf{R}^n; F)$, the vector space of all continuous F-valued functions on $\mathbf{R}^n$ under pointwise operations. For each $x \in \mathbf{R}^n$ let $v_x = \| \|$; then $v = (v_x; x \in \mathbf{R}^n)$ is a weight in the above vector fibration. For each $f \in \ell(\mathbf{R}^n ; F)$, the function $v[f]$ is upper-semicontinuous. Theorem

6 applies with $E = \mathbf{R}^n$ $C = \ell$ $(\mathbf{R}^n ; F)$ and v as above. Consider ℓ $(\mathbf{R}^n ; F)$ provided with the corresponding topology ω_{vc} :it is the compact-open topology on ℓ $(\mathbf{R}^n ; F)$ and Δ $(\mathbf{R}^n ; K)$ and Δ $(\mathbf{R}^n ; F)$ be the vector spaces of infinitely differentiable functions on $\mathbf{R}^n$ with values in K and in F, respectively. Δ $(\mathbf{R}^n ; K)$ is a separating sub-algebra containing the constants and self-adjoint in the complex case; the same assertions are true with respect to the sub-algebra Δ_c $(\mathbf{R}^n ; K)$ of the $f \in \Delta$ $(\mathbf{R}^n ; K)$ with compact support. Δ $(\mathbf{R}^n ; F)$ and Δ_c $(\mathbf{R}^n ; F) = \{f \in \Delta$ $(\mathbf{R}^n ; F)$; f has compact support$\}$ are vector subspaces of ℓ $(\mathbf{R}^n ; F)$ which are Δ $(\mathbf{R}^n ; K)$ – modules and also Δ_c $(\mathbf{R}^n ; K)$- modules. For each $x \in \mathbf{R}^n$ the x-section of Δ_c $(\mathbf{R}^n ; F)$ is equal to F : it is enough to consider $f_x \otimes y$ $\in \Delta_c(\mathbf{R}^n ; F)$ where y is an arbitrary element of F and $f_x \in \Delta_c(\mathbf{R}^n ; K)$ is such that $f_x(x) = 1$. Theorem 6 then implies that $\Delta_c(\mathbf{R}^n ; F)$ and Δ $(\mathbf{R}^n ; F)$ are dense in $(\ell$ $(\mathbf{R}^n ; F)$, $\phi_v c)$.

***Remark* 5.** Let L be a vector space over K and $(q_j, j \in J)$ be a filtrant family of semi-norms on it. Then the set of all semi-balls $\{f \in L \mid q_j(f) < \Delta\}$, $j \in J$, $\in > 0$, is a fundamental system of neighborhoods of the zero of L in the locally convex topology on L defined by the family $(q_j; \in J)$. Under such a topology on L, a subset $W \subset L$ is dense if, and only if, it is dense in the semi-normed space (L, q_j) for each $j \in J$.

***Theorem* 7.** Let E and F be two locally convex Hausdorff spaces, both real. Provide the vector space ℓ $(E;F)$ of all continuous maps from E into F with the compact open topology. Then the vector subspace of ℓ $(E;F)$ of all continuous polynomials defined on E with values in F and of finite type is dense in ℓ $(E;F)$.

Proof. First we explain what is the compact-open topology in ℓ $(E;F)$. To do this we place ourselves in the context of the vector fibration $(E; (F_x; x \in E))$ where $F_x = F$ for all $x \in E$ and, at first, think of ℓ $(E;F)$ as having only the structure of a vector space of cross-sections of this fibration. As in Theorem 6, we let $\mathscr{K}_E$ denote the set of compact subsets of E. Let $s(F)$ be the set of all continuous semi-norms on F. For each q $\in s(F)$, the constant function $x \in E \mapsto v_q(x) = q$ is a weight in the present vector fibration; for each q $s(F)$and each $K \in \mathscr{K}_E$, introduce $f \in \ell$ $(E;F)$

$\mapsto p_{v_q,k}(f) = \sup \{v_q(x)[f(x)];\ x \in K\} = \sup \{q(f(x));\ x \in K\}$. It is clear that $(p_{vq,k};\ q \in s(F),\ k \in \mathscr{K}_E)$ is a directed family of simi-norms on $\ell\ (E;F)$ $\ell\ (E;F)$ is provided with the Hausdorff locally convex topology defined by this family, and this topology is precisely the compact-open topology on $\ell\ (E;F)$ we have been referring to.

Next we introduce a set $a \subset \ell\ (E;\ \mathbf{R})$ of real continuous polynomials on E. By definition, A is the algebra generated by the real constant functions on E together with the elements of the continuous dual $E' \subset \ell\ (E;\ \mathbf{R})$ of E. The algebra A contains the constants and is separating over E because E is a real locally convex Hausdorff space and $A \supset E'$. This assertion is a corollary of the Hahn-Banach Theorem.

The vector subspace $W \subset \ell\ (E;F)$ of the continuous polynomials on E, values in F and of finite type is to be understood as being the set of all functions $w \in \ell\ (E;F)$ of the form : for some integer $n \geq 1$, there are $a_1, ..., a_n \in A$ and $v_1, ..., v_n \in F$ such that $w = a_1 v_1 + ... + a_n v_n$. To simplify the notation, write now $\ell\ (E:F) = C$ purely as a vector space of cross-sections. It is clear that for each $x \in E$ the equality $W(x) = C(x) = F$ results.

The proof of Theorem 7 is now reduced to a obvious application of Remark 5, after applying Theorem 6 for $(E;\ (F_x;\ x \in E)),\ C,\ W$ as above for each choice of $v = v_q,\ q \in s(F)$. The proof of Theorem 7 is now complete.

***Remark* 6.** The above arguments may be repeated to give a proof of a version of Theorem 7 for complex spaces E and F. The notion of "polynomial" must be interpreted as indicated in [12], § 7, Remark 8.

These results complete and generalize a result of P.M. Prenter, Bull. Amer. Math. Soc. 75 (1969), 860-2.

As an application of Theorem 5, we shall prove the following result:

***Theorem* 8.** Suppose that E and F are real Banach spaces and that E is reflexive. Let B_E be the closed unit ball of E with center at the origin and $u : E \to F$ a compact linear map. Then, for any $\in > 0$ given, there is a continuous polynomial $w: E \to F$, of finite type such that

$$\|u(x) - w\|_F \leq \in \qquad \text{for all} \qquad x \in B_E$$

This statement remains true for complex Banach spaces E and F if w is taken to be an element of the vector space $W \subset \ell\ (E; F)$ defined below (see Remark 6 above).

We shall prove the version of the result which is valid for real or complex spaces.

We say that $a : E \to F$ is anti-linear if for all x , $y \in E$ and $c \in K$, $a(x+y) = a(x) + a(y)$ and $a(cx) = \bar{c}\, a(x)$. Let $\overline{E}'$ be the set of all continuous anti-linear maps $a: E \to F$. Let A be the sub-algebra generated by $E' \cup \overline{E}'$ be the set of all continuous anti-linear constant functions on E. It is clear that A coincides with the algebra introduced in Theorem 7 when E and F are real Banach spaces. It is also easily seen that A is a separating sub-algebra of $\ell\ (E;K)$, containing the constants, which is self adjoint in the complex case. Let W be the set of all continuous functions $w: E \to F$ of the form : for some integer $n \geq 1$, there are $a_1, \ldots, a_n \in A$ and $v_1, \ldots v_n \in F$ such that $w = a_1 v_1 + \ldots + a_n v_n$. The span of the range of each $w \in W$ is a finite dimensional vector subspace of F. Moreover, W is a vector subspace of $\ell\ (E;F)$ which is an A-module and $W(x) = F$ for all $x \in F$.

Let E_σ denote E provided with the weak (or weakened) topology $\sigma(E,E')$, $(B_E)_\sigma$ denoting B_E as a topological subspace of $E\sigma$. Consider the vector fibration $((B_E)_\sigma, (F_x;\ x \in (B_E)_\sigma))$ where $F_x = F$ for all $x \in (B_E)_\sigma$; distinguish its weight v such that $v(x) = \|\ \|_F$ for all $x \in (B_E)_\sigma$ and its vector space of cross-sections $C = \ell\,((B_E)_\sigma; F)$. $W \backslash B_E = \{w \backslash B_E;\ w \in W\}$ is a vector subspace of C such that $(W \backslash B_E)(x) = C(x) = F$ for all $x \in (B_E)_\sigma$ and which is an $A \backslash B_E$ - module, where A/B_E is the separating sub-algebra of $\ell\ (B_E)_\sigma; k)$. $W \backslash B_E = \{w \backslash B_E;\ w \in W\}$ containing the constants and self-adjoint in the complex case defined by putting $A \backslash B_E = \{a \backslash B_E - a \in A\}$.

Let C be normed by $\| \|_v$, where

$$\|f\|_v = \sup \{v_x[f(x)];\ x \in (B_E)_\sigma\} = \sup \{\|f(x)\|_F;\ x \in B_E\} \text{ for all } f \in C.$$

Theorem 5 now applies in an obvious way (with L $= C$ and W = $W \backslash B_E$) implying that $W \backslash B_E$ is dense in C. The proof of our result is finished when we observe that $u \backslash B_E$ belongs to C, that is , $u \backslash B_E : (B_E)_\sigma \to F$ is continuous.

Remark 7. In the statement of Theorem 6 only one weight, v, is mentioned as such. But it is clear that all weights of the form $X_B v = (X_B(x)\ v_E;\ x \in E)$, where B is an arbitrary compact subset of E and X_B is its characteristic function, enter into the definition of the topology ω_{v_E} on C.

In fact, the reader will easily convince himself that every time we topologize a vector space of cross-sections, even when they were only vector spaces of scalar continuous functions, the topology used could be obtained from a convenient set of weights through the following procedure.

In a given vector fibration $(E;(F_x, x \in E))$ there are given a vector space of cross-sections L and a set V of weights such that for each $f \in L$ and each $v \in V$ the function $v[f]$ is upper-semicontinuous and bounded on E. Then, for each $v \in V$, the function $\| \|_v : L \to \mathbf{R}$ defined by $\|f\|_v = \sup \{v_x[f(x)];\ x \in E\}$ for all $x \in E$, is a semi-norm on L : only the boundedness assumption plays a role here. V is assumed filtrant in the sense that for each $v_1, v_2 \in V$ there are $v \in V$ and $t > 0$ such that $v_i(x) \leq tv(x)$ for all $x \in E\}$, $I = 1,2$. Then $(\| \|_v;\ v \in V)$ is a filtrant family of semi-norms on L is provided with the topology ω_v defined by this family.

Under the above conditions, the space $(L,\ \omega_v)$ is denoted by LV_b in [12] and [13]. If it is further assumed that $v\,[f]$ is null at infinity for each $f \in L$ and $v \in V$, then $L \in V_b$ is denoted by LV_∞ in those papers. The considerations in this paper show that the upper semicontinuity assumption is a most reasonable hypothesis. Nullity at infinity was always satisfied : the functions $v\,[f]$ we considered, implicitly or explicitly, were always, indeed, null outside of a compact set or the whole domain space was itself compact. We again refer to [12] and [13] for further details.

Remark 8. The spaces LV_b and LV_∞ described in Remark 7 are called weighted locally convex spaces of cross-sections in [12] and [13]. We propose that they be called Nachbin spaces. Outstanding examples of such structures are the spaces $CV_b(E)$ and $CV_\infty(E)$ (see [11]) introduced by Nachbin in the course of his important work on weighted approximation.

Some recent contributions to the study of the space $CV_\infty(E)$ are found in Summers [22], [23], [24]. The vector-valued case has been studied recently by Bierstdt ω_v can be studied from the viewpoint of operator-induced topologies (see Prolla [18]). It should also be mentioned that the vector space C_b $(E;F)$ of all bounded continuous functions from the locally compact space E into the locally convex space F under the strict topology β is another important early example of a Nachbin space. This structure was defined by R. C. Buck (see Buck [2]) and has been the object of many interesting studies (see for example Collins and Dorroh [4], Dorroh [7]) and generalizations (see Sentilles and Taylor [19] and Sentilles [20]).

A final word about the notion of vector fibrations. This notion goes back to the work of von Neumann [14] (written in 1938) and Silov (see [21]). The general theory of such vector fibrations in the case where each fiber is a Banach space F_x with norm $\| \|_x$ and the functions $x \to \|f(x)\|_x$ are continuous was studied by Godement (see [8] and chapter III of [9]). The case in which F_x is a C^*-algebra was treated by Dixmier and Douady [6], and the case of Frechet spaces by Cathelineau [3]. For the connections with integration theory see Godement [9] and the references cited on the notes and remarks for § 8 of Dinculeanu's book (see [5], page 414).

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حول نظرية فيرشتراس لجبريات وموديولات الرواسم المتصلة

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هذا البحث يتعلق بدراسة نطاق الأفكار والنتائج المتصلة بنظرية فيرستراس – ستون لجبريات وموديولات الدوال المتصلة في الحالة الحقيقة وكذلك الحالة المركبة ذاتية الترافق وذلك من وجهة نظر التليف الأتجاهي وفراغات متجهات القطوع . لقد كان برهان التكافؤ بين الحالات المنفصلة والعامة لمثل هذه النتائج باستخدام تكوينات القسمة هو الدافع وراء استخدام التليف الأتجاهي.