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On A CERTAIN UNIFORM STRUCTURE

Siham J. Al. Sayyad
Department of mathematics, Faculty of sciences,King Abdel Aziz University, P.O. Box 30305
Jeddah 21477, Saudi Arabia
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ABSTRACT

A regular structure for a topological space is defined. Also, a relation between a regular structure and an Abian's structure is investigated.

INTRODUCTION

In [1], Abian defined a uniform structure for a topological space so that the theorem that a continuous function on a compact space is uniformly continuous holds. As uniform spaces are well- known [e.g. 1, 4], one would naturally like to compare a uniform structure with a uniformity when the former is defined. However, it does not seem easy to do this between a uniform structure defined by Abian (we shall use the name "Abian's structure" in the sequel) and a uniformity. In this paper, we define a regular structure (Definition 2.1 below) for a space so that the above mentioned theorem holds and it is easy to compare a regular structure with a uniformity. Moreover, a relation between a regular structure and an Abian's structure is found and it follows that a topological space which admits an Abian's structure must be regular.

1. PRELIMINARIES

We state here definitions occurred in [1] and a simple result for latter use. Let X be a topological space and I an index set. For each $i \in I$, let χ_i be a collection of open sets of X. Moreover, for each $i \in I$ and each $x \in X$, let $\chi_i(x) = \{X_i \in \chi_i : x \in X_i\}$ and $S_i(x) = \cup\{X_i : X_i \in \chi_i(x)\}$. An ordering $\leq$ for I is defined as follows. For i, j in I, $i \leq j$ if and only if every $X_j \in \chi_j$ is a subset of some $X_i \in \chi_i$. It should be noted that $\leq$ is reflexive and transitive. The family $\{\chi_i : i \in I\}$ is called an Abian's structure for the space X if it has the following properties (I), (II) and (III).

(I) If x is a point of an open set G in X, then there is an $i \in I$ such that $\cup\{S_i(y) : y \in S_i(x)\} \subset G$.

(II) For any i in I, we have either $i \leq j$ or $j \leq i$.

(III) $\cup\{X_i : X_i \in \chi_i\} = X$ for each $i \in I$.

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Next property (IV) is an easy consequence of (I) and the fact that each member in each χ_i is open in X.

(IV) A subset G of X is open if and only if for each $x \in G$ there is an $i \in I$ such that $S_i(x) \subset G$.

Now, let X and Y be two topological spaces with Abian's structures $\{\chi_i : i \in I\}$ and $\{\widetilde{Y}_j : j \in J\}$ respectively. A function f from X to Y is said to be uniformly continuous if for each $\chi_i \in \chi_i$, is mapped into some $Y_j \in \widetilde{Y}_j$.

2. DEFINITIONS

In this section we define regular structures for a non-empty set and introduce the concept of uniformly continuous functions.

***Definition* 2.1.** Let $X \neq \emptyset$ be given and Δ denote the diagonal of $X \times X$. A collection $\widetilde{U}$ of subsets of $X \times X$ is called a regular structure for X if the following conditions are fulfilled:

(1) Δ is a subset of each $U \in \widetilde{U}$.

(2) The transpose U^{-1} of each member $U \in \widetilde{U}$ contains some $V \in U$.

(3) The intersection $U \cap V$ of each two members in $\mathscr{U}$ contains a third one.

(4) For each $U \in \widetilde{U}$ and $x \in X$, there exists a $V \in \widetilde{U}$ such that $V \in \widetilde{U}$ such that $V \circ V[y] \subset \widetilde{U}$ $[y]$ for all $y \in V[x]$.

The above notations are as in [2, 3, 4].

It is clear that $\widetilde{U}$ would be a base for a uniformity for X if condition (4) was suitably strengthened. It is also clear that a topology for X can be induced by defining a set $G \subset X$ to be open if for each $x \in G$ there is a $U \in \widetilde{U}$ with $U[x] \subset G$.

***Defininition* 2.2.** Let $(x, \widetilde{U})$ and $(Y, \widetilde{V})$ be two spaces (with this notation, it is understood that $\widetilde{U}$ and $\widetilde{V}$ are regular structure for X and Y respectively) and let f be a function from X to Y. f is said to be uniformly continuous if for each $V \in \widetilde{V}$ there exists a $U \in \widetilde{V}$ such that $f(U[x]) \subset V[f(x)]$ for every $x \in X$.

3. RESULTS

***Theorem* 3.1** If a function f from $(X, \widetilde{U})$ to $(Y, \widetilde{V})$ is continuous and the space $(X, \widetilde{U})$ is compact (topological properties are relative to the topologies induced by $\widetilde{U}$ and $\widetilde{V}$), then f is uniformly continuous.

Proof. Let $V \in \widetilde{V}$ be given. For each $x \in X$, since $f(x) \in Y$, by condition (4) in 2.1, there exists a $V_x \in V$ such that $V_x \circ V_x[y] \subset V[y]$ for all $y \in V_x[f(x)]$. Also, there exists a $W_x \in \mathscr{V}$ such that $W_x \subset V_x \cap V^{-1}{}_x$. Since f is continuous, there is for each $x \in X$ a $U_x \in \widetilde{U}$ such that $f(U_x[x]) \subset W_x[f(x)]$. By conditions (4) in 2.1

again, there is for each $x \in X$ a $\hat{U}_x \in \widetilde{U}$ such that $\hat{U}_x \circ \hat{U}_x[z] \subset U_x[z]$ for all $z \in \hat{U}_x[x]$. Clearly {int $(\hat{U}_x[x])$: $x \in X$} is an open covering for the compact space X. Thus $\cup[U_{x_p}\{x_p\} : p = 1, 2, ..., n\} = X$ for some finite set $\{x_1, x_2, ..., x_n\}$ in X. By condition (3) in 2.1, there exists a $U \in \mathscr{U}$ contained in $\cap\{U_{x_p} : p = 1, 2...n\}$. Assert $f(U[x]) \subset V[f(x)]$ for all $x \in X$. To see this, let $x \in X$ be given, $x \in \hat{U}_{xp}[x_p]$ for some $p \in \{1,2,......n\}$. Hence $U[x] \subset \hat{U}_{x_p}[x] \subset \hat{U}_{x_p} \circ \hat{U}_{x_p}[x_p] \subset U_{x_p}[x_p]$ and

$f(U[x]) \subset f(U_{x_p}[x_p]) \subset W_{x_p}[f(x_p)]$ (*)

In particular, we have $f(x) \in W_{x_p}[f(x_p)]$. Since $W_{x_p} \subset V_{x_p} \cap V_{x_p}^{-1}$, We get $f(x_p) \in V_{xp}[f(x)]$ and $f(x) \in V_{x_p}[f(x)]$. It follows that $W_{x_p}[f(x_p)] \subset V_{x_p} \circ V_{xp}[f(x)]$. This together with (*) gives $f(U[x]) \subset V[f(x)]$. This proof is complete.

To study the relation between an Abian's structure and a regular structure, let a topological X and an Abian's . structure $\{\chi_i : i \in I\}$ be given. For each $i \in I$, we define $U_i = \{(x,y): y \in S_i(x)\}$. With this notation , we have

Theorem 3.2. $\widetilde{U} = \{U_i : i \in I\}$ is regular structure for X (we shall call it the regular structure induced by the given Abian's structure) and the topology induce by $\widetilde{U}$ coincides with the given topology for X .

Proof. We have to check if $\widetilde{U}$ satisfies the four conditions in 2.1. condition (1) is clearly satisfied. We shall show that $U_i^{-1} \subset U_i$ for each $i \in I$ and condition (2) follows. It is easily seen that $U_i[x] = S_i(x)$ for all $i \in I$ and all $x \in X$. If $(x,y) \in U_i$, then $y \in S_i(x) = \cup\{X_i : X_i \in \chi_i(x)\}$, that is there is an $\hat{X}_i \in \chi_i(x)$ such that $y \in \hat{X}_i$. Thus $x \in \hat{X}_i \in \chi_i(y)$ and $x \in S_i(y)$ which implies $(y, x) \in U_i$ or $(x, y) \in U_i^{-1}$. To prove that condition (3) holds, let i and j in I be given. In view of property (II) in § 1, we may assume that $i \leq j$. Then it is clear that $U_i \cap U_j = U_i$. Finally, let $i \in I$ and $x \in X$ be given. Fix an $X_i \in \chi_i(x)$. Since X_i is open and $x \in X_i$, by (1), there is a $j \in I$ such that $\cup\{S_j(y): y \in S_j(x)\} \subset X_i \subset S_i(x)$. Similarly, there is a $k \leq I$ (we may assume $k \leq j$) such that $\sim\{S_k(y): y \in S_k(x)\} \subset S_j(x)$. It is routine to show that $U_k \circ U_k[y] \subset U_i[y]$ for all $y \in U_k[x]$.Thus condition (4) is also satisfied. It remains to show that the topology induced by $\widetilde{U}$ is the same as the given one for X. But this is obvious by (IV) and the fact that $S_i(x) = U_i(x)$ for all x $Y X$. The theorem is now proved.

Concerning the definition of uniformly continuous function, we have the following.

Theorem 3.3. Let X, Y be topological spaces with Abian's structures $\{\chi_i : i \in I\}$ and $\{\widetilde{Y}_j : j \in J\}$ respectively, and let f be a function X from to Y. We have :

(a) if f is uniformly continuous according to § 1, then f is uniformly continuous according to Definition 2.2 relative to the regular structures induced by the given Abian's structures ;

(b) if the space X is compact, then the converse of (a) is true.

Proof. Let the induced regular structures be denoted by $\widetilde{U} = \{U_i : i \in I\}$ and $\widetilde{V} = \{V_j : j \in J\}$ respectively, where :

$$U_i = \{(x, x') : x' \in S_i(x)\} \text{ and } \qquad V_i = \{(y, \hat{y}) : \hat{y} \in S_i(y)\}.$$

To prove (a), let $j \in J$ be given. By assumption there is an $i \in I$ such that, for each $X_i \in \chi_i$, $f(X_i) \subset Y_j$ for some $Y_j \in \widetilde{Y}_j$. Obviously, for any fixed $x \in X$, $f(X_i) \subset S_j(f(x))$ for every $X_i \in \chi_i(x)$. Thus, $f(U_i[x]) = f(S_i(x)) = \cup \{f(X_i) : X_i \in \chi_i(x)\} \subset S_j(f(x)) = V_j[f(x)]$.

We proceed to prove (b).For given $j \in J$ and $x_0 \in X$, fix a $\hat{Y}_j \in \widetilde{Y}_j(f(x_0))$. Since $f(x_0) \in \hat{Y}_j$ and $\hat{Y}_j$ is open in Y, there is an $l_0 \in J$ such that $V_{l_0}[f(x_0)] \subset \hat{Y}_j$. Uniform continuity of f according to Definition 2.2 implies the existence of an $m \in I$ such that $f(U_m[x]) \subset V_{l_0}[f(x)]$ for all $x \in X$. Now $x_0 \in X$ and $U_m \in \widetilde{U}$, imply that there is a $k_0 \in I$ with $U_{k_0} \circ U_{k_0}[x_0] \subset U_m[x_0]$, we see that for every $x \in U_{k_0}[x_0]$, $f(U_{k_0}[x]) \subset f(U_m[x_0]) \subset V_{l_0}[f(x_0)] \subset \hat{Y}_j$. Noting that x_0 is an arbitrary point in X, we have for each $x \in X$ a $k(x) \in I$ such that $f(U_{k(x)}[z] \subset Y_j$ for some $Y_j \in \widetilde{Y}_j$ and all $z \in U_{k(x)}[x]$. By compactness of X, there is a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that $\cup \{U_{k(x_p)}[x_p] : p = 1, 2, \ldots, n\} = X$. By (II) in § 1, there is a $k \in \{k(x_1), k(x_2), \ldots, k(x_n)\}$ such that $k \preceq k(x_p)$ for all $p = 1, 2, \ldots, n$. Assert that each $X_k \in \chi_k$ is mapped into some $Y_j \in \widetilde{Y}_j$. Since $X_k \in \chi_k$ must be a member of $\chi_k(x)$ for some $x \in X$ (the case $X_k = \emptyset$ is trivial and is not considered), $X_k \subset S_k(x) \subset S_{k(x_p)}(x) = U_{k(x_p)}[x]$, where p is in $\{1, 2, \ldots, n\}$ with $x \in U_{k(x_p)}[x_p]$. It follows that $f(X_k) \subset f(U_{k(x_p)}[x]) \subset Y_j$ for some $Y_j \in \widetilde{Y}_j$. the theorem is proved.

4. REMARKS

***Remark* 4.1.** Owing to Theorem 3.3, Abian's result follows from our Theorem 3.1. If a regular structure $\widetilde{U}$ for a topological space X (that is, $\widetilde{U}$ induces the topology for X) is induced by an Abian's structure, then every two members of $\widetilde{U}$ are comparable with respect to $\subset$. However, in general a regular structure need not have this property and hence is not necessarily induce by an Abian's structure. Thus we can hardly say that Abian's result implies ours.

***Remark* 4.2.** A regular structure for a space X is similar to a " symmetric indexed neighborhood system with ' local triangle inequality ' for X " defined by Davis [2]. Similar to his proof [2, Theorem4], we can show that a topological space admits a regular structure if and only if it is regular. Consequently, a topological space that has an Abian's structure must be regular. It is unknown to the author whether a regular topological space admits an Abian's structure or not.

REFERENCES

[1] Abian, On uniform structures, Rend. Circ. Mat. Palermo, Ser. II, 14(1965), 202-206

[2] A. S. Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly, 68(1961), 886-893.

[3] J. Davis and P. Kirk : Lecture Notes in Topology, Graduate studies in Math. Vol. 31, 2001

[4] J. L.Kelly, General Topology, Van Nostrand, Princeton, 1955.

عن بنية منتظمة مؤكدة

سهام جلال السيد

قسم الرياضيات – كلية العلوم – جامعة الملك عبد العزيز- جدة

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