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Analysis of Chassis Flexibility for Half Car Model

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ABSTRACT

When the researchers before studied the behaviors of a vehicle chassis and the suspension due to the irregular road and the different driving conditions, they met some challenge like ensures the contact the wheels with the road all the time especially at cornering at all conditions of the road and the comfort of a ride. in most researches was assumed the chassis is rigid. but in the present paper, the flexible chassis included made a more realistic case. The Bernoulli beam is governed by a fourth differential equation and resolved by aid FEM which gives the time histories at every node of the beam including its flexibility. to make a more accurate image. Results are generated for Half Car elastic models at the ends and midpoints.

1. Introduction

The design of the vehicle systems from the standpoint of ride comfort and the degree of stability of the vehicle on the road must be in a safe way compatible with the stresses that may be occurred in the vehicle parts. The vehicle chassis and its parts are the most important parts exposed to dangerous stresses. Some researchers consider the vehicle chassis presented is a flexible model. This work giving consider the model beam as 2 Dof like the usual 2 degrees of freedom half-car model (bounce and pitches) [1]. The chassis (sprung mass) is a flexible beam 2 Dof half-car model which standing on a spring system at the front and rear. Three types of suspension systems categorize as passive, semi-active, and active vehicle suspension systems are modeled. The analyses of a half car model of a vehicle by a semi-active suspension system [2]. are performed considering the chassis of the half-car model is rigid under random road excitations [3]. Using a rigid half-car model, active vehicle suspension systems are modeled simulated using Matlab /Simulink [4]. The ride vibration of the truck that affected frame flexibility was studied using the finite element method, FEM, [5]. They found that the driver and ride comfort, affected by frame flexibility particularly in the important range of human sensitivity. Also, it has been found that flexibility affects the acceleration levels in the frame structure and vehicle safety. In the present work, the time a history of any point along the vehicle chassis with a passive suspension system has been theoretically investigated using a half car model include the chassis is flexible

2. System modeling

Figure (1) shows simply supported have two degrees of freedom (DOF) which acts as the sprung mass of a half-car model standing on its ends. to make the system more understanding in boundary condition application. Without specifying convenient support conditions, the system will be free to move as a rigid body.

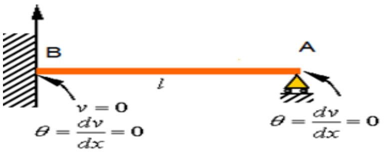

Fig: (1) propped cantilever beam

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According to Euler beam theory, Fig (2) [6] and Lagrangian Mechanics [7] which is used to model the beam with the concept of the frame [8] are used to derive the equations of motion. FEM of the flexible two Dof model is presented in the time domain [9]. To understand the underlying dynamics of the system [10-11]. The important step to this understanding numerical modeling has become established. The motivation of the study is to find out the behavior of cars for different types of road profiles [12]. The development of computers has provided the computational power to have software for mathematical modeling. Multibody dynamics, (MBS) [13], FEM [14], and Matlab software [15-16], etc., are widely used in the analysis of the mechanical design.

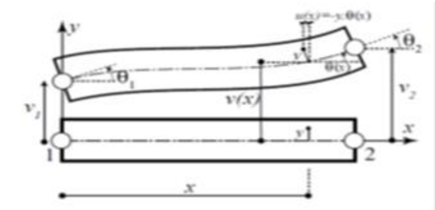

Figure (2) *Bernoulli-Euler Beam Element*

the beam equation solved with an assistant of Matlab. Must use Complex partial differential equations of Direct method Finite Element Method (*Appendix B*) which describe this system can be reduced to a collection of linear equations easily be using this method. In the Direct method finite element method exchanged into the governing equations and the unknown node, values are determined.

By using the concept of tracking frame [4] to derive the equations of motion. FEM of the flexible two Dof model is presented in the time domain Fig (2). To understand the underlying dynamics of the system. Numerical modeling has become an important step [8-9]. for different types of road profiles [16] is the motive to study to find out the behavior of the car. The mathematical results of Vibration analysis of a cantilever beam with load at the tip and simply supported beam with the center load**.** The development of computers has provided the computational power to have software for mathematical modeling. Multibody dynamics system (MBS) [5], Finite Element Method (FEM) [6] and Matlab software [14-15], Etc. are widely Euler beam theory [2] and Lagrangian Mechanics [3] are used to model the beam and along used in mechanical design and analysis. In this paper, we use the Finite Element Method to solve the beam equation with the helper of Matlab.

3. stiffness matrix Derivation of the spring element:

The local axis acts in the direction of the spring to be able to directly measure displacement and force along with the spring. the local nodal force at node one for the spring element associated with the local axis.

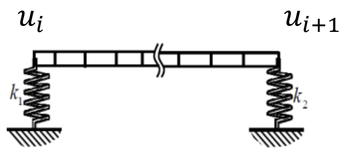

Fig:(3) Finite element model

The local nodal displacements are $\tilde{u}_1\tilde{u}_2, \tilde{u}_3, \tilde{u}_4$ for the nodal of the elements as shown in Fig. (3). the force at node one can be written as follow:

$$\tilde{f}_{y1} = (\tilde{u}_1 - \tilde{u}_2)k \qquad (1)$$

For node two as follow.

$$\tilde{f}_{y2} = (\tilde{u}_2 - \tilde{u}_1)k \qquad (2)$$

Or we can write the equation 13, 14 as follow.

$$\begin{Bmatrix} \tilde{f}_{y1} \\ \tilde{f}_{y2} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} \qquad (3)$$

So, we can write the stiffness matrix for element spring assuming K1 equal K2

$$k_1^e = k_2^e = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (4)$$

4. Finite Element modeling:

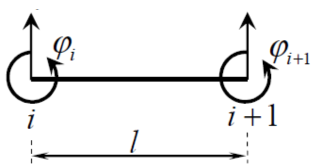

Fig:(4) one finite element

We can calculate the displacement of nodes of the beam and its associated modes of the flexible beam supported by two springs at both ends by the aid Finite element method. The damping is not considered for finite element analysis. the way to drive The elemental equations of stiffness and mass-spring at both ends. the beam is discretized (N-1) two noded elements generating N nodes. Fig. (4) shows matrices for the bending motion of the beam depend upon Euler Bernoulli bending theory. Fig. (2) shows a slender beam supported by the arbitrary element with the two degrees of freedom per node, u_i, φ_i represents displacement and deflection of $(i)^{th}$ node, u_{i+1}, and φ_{i+1} represents displacement and deflection of $(i+1)^{th}$ node respectively.

$$[k] = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & * 6L & 4L^2 \end{pmatrix} \qquad (5)$$

6. The global stiffness matrix for our case: -

From equation (1), (4) for the beam element and spring element respectively (her for the study assume the beam is one element) we can get the global stiffness matrix for all the system as follow:

Hence the stiffness matrix is (see appendix B)

$$K_G = \frac{EI}{L^3}\begin{bmatrix} \frac{kL^3}{EI} & -\frac{KL^3}{EI} & 0 & 0 & 0 & 0 \\ -\frac{KL^3}{EI} & 12+\frac{kL^3}{EI} & 6L & -12 & 6L & 0 \\ 0 & 6L & 4L^2 & -6L & 2L^2 & 0 \\ 0 & -12 & -6L & 12+\frac{kL^3}{EI} & -6L & -\frac{KL^3}{EI} \\ 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & -\frac{KL^3}{EI} & 0 & \frac{kL^3}{EI} \end{bmatrix} \qquad (6)$$

By

using the boundary condition for the system. Then the global stiffness matrix becomes as follow: -

$$K_G = \frac{EI}{L^3}\begin{bmatrix} 12+\frac{kL^3}{EI} & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12+\frac{kL^3}{EI} & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} (7)$$

We must know the global stiffness for our modeling will take another form because the beam divides into 100 elements and assuming the boundary condition for the simply supported beam.

5. Drive the element mass matrix for the beam

Where: M$= \rho AL$,

N=$[N_1 \quad N_2 \quad N_3 \quad N_4]$, we can get the value of (N) from equation (8). Now, we can get the value of $(N^T N)$ as follow

$$N^T N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}[N_1 \quad N_2 \quad N_3 \quad N_4] \qquad \text{Or}$$

$$N^T N = \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_2N_1 & N_2N_2 & N_2N_3 & N_2N_4 \\ N_3N_1 & N_3N_2 & N_3N_3 & N_3N_4 \\ N_4N_1 & N_4N_2 & N_4N_3 & N_4N_4 \end{bmatrix} \qquad (8)$$

We can get the value of N from Appendix B as follow:

$$M = \int_0^L N^T m N \, dx$$

$$N_1N_1 = \left\{\frac{1}{L^3}(2x^3 - 3x^2L + L^3)\right\}^2 = \frac{1}{L^6}(2x^3 - 3x^2L + L^3)^2 \dots\dots\dots \qquad (9)$$

$$= 4x^6 - 12x^5L + 9x^4L^2 + 4x^3L^3 - 6x^2L^4 + L^6$$

Now,

$$\frac{1}{L^6}\int_0^L N_1N_1 dx = \frac{4}{7}L^7 - 2L^7 + \frac{9}{5}L^7 + L^7 - 2L^7 + L^7 = \frac{L^7}{L^6}\left(\frac{13}{35}\right) = \frac{L * 156}{420} \qquad (10)$$

For the element N_1N_1, and we can do all elements of equation (8) in the same way to get the global mass matrix (8) for the beam in our system as follow:

$$M = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad (11)$$

Equation (5) and (9) acts as the stiffness and mass matrix for elements.

7. System input

Fig. 5 shows the sketch half-car model system. The suspension model consists of a chassis as a beam [13] and two axles. Linear spring supports the chassis without includes the tires. The input displacement from the road equals sinusoidal wave at the front tire (x= X sin wt.) and the same value at the rear tire additional time delay (x=X sin wt. +L/v) as shown in figure (10) ***(****Appendix A****)***. Where X is the amplitude of displacement and equal 0.1, L the length of the chassis, and v is the velocity of the vehicle [1]. The system constraints are mainly due to the tire contacts. Thus, if the tires are assumed to be static and replaced by two pins. The model is excited through the tires by a base motion, resulting in a force through springs and dampers, on nodes 3 for the front and GDof-3 for the rear [14 to 17].

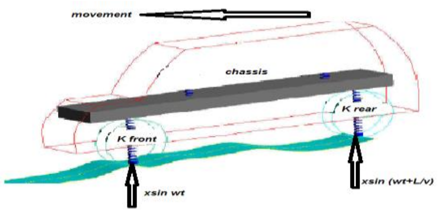

Fig: (5) System model

8. Results

The results were obtained for a system as a simply supported beam with springs at both ends. The Figures (From 6 to 12) represent behavior amplitude for a beam (chassis) under the load acted which was born from the road for 100 beam elements with two spring elements at both ends. The Matlab program used to get these results is given. The boundary conditions were obtained results for a claimed beam for deflection degree of freedom at its ends and Assume fixed point at the ends of spring. Fig (6) Shows the total elastic amplitude of the system at the total node of the beam. The fluctuating area showed the system behavior of the model. Figure (7) Show the only vertical elastic displacement at every node. Figure (8) Show the elastic deflection shape for the Chassis at all nodes. Figure (9) shows the elastic vertical displacement shape at the left tip of the Chassis. Figure (10) Show elastic vertical displacements respectively at the right tip of the Chassis. Figure (11) shows the elastic vertical displacement at the middle span of the chassis. Figure (12) shows the elastic deflection of the mid-span of the Chassis.

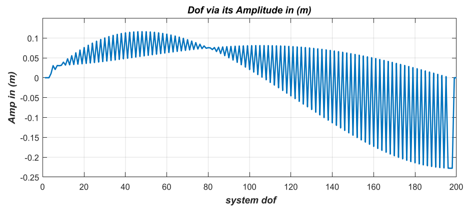

Fig: (6) System Elastic Amp. With element Dof for all model

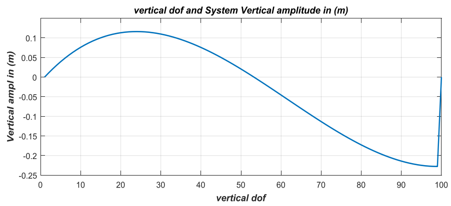

Fig: (7) Elastic System vertical Amplitude for all model

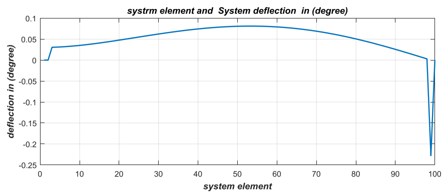

Fig. (8)elastic system deflection for all model

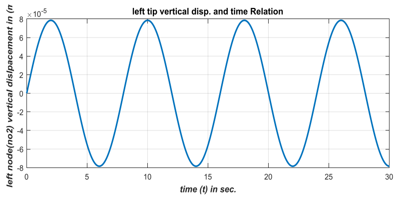

Fig: (9) elastic Vertical left tip node amplitude

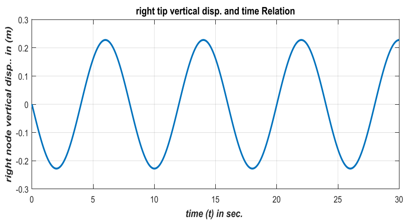

Fig: (10) right tip elastic Vertical amp for all model

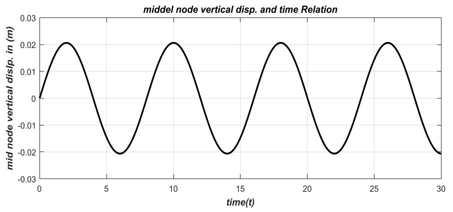

Fig: (11) Middle node vertical elastic amp

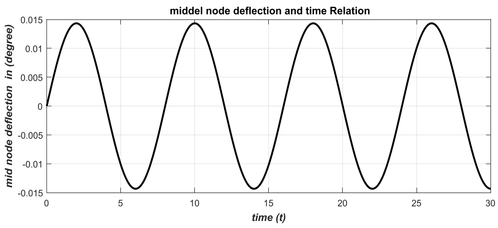

Fig: (12) Middle node elastic deflection amp.

9. Discussion

By finite element method theory, we can say the error reduces, as increasing the number of elements, that improves the accuracy of the solution. Also, the graphs display the results obtained for homogenous boundary conditions for every node of the Chassis. The fluctuating area (figure (6)) due to road irregular can be decreased by changing the properties of the chassis material. We notice the vertical amplitude at the right tip is greater than the left tip. This is seeming a realistic result because the deflection of the beam (chassis) makes as the spring which presses at the right end makes the right displacement increase as shown.

10. conclusion

The present paper study the vehicle behavior which directly influenced by road irregular and suspension system. The displacement of the half vehicle without tire via Matlab as simply supported beam at the right end and fixed at left considered under load acting from road hump and tire stiffness at the contact point between the tire and the road. the Bernoulli beam which is a fourth-order differential equation. Cubic elements are used as required for continuity by governing differential equation is that pre-described. Graphs are presented and discussed the elastic displacement at every node of the chassis.

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Appendices

- Appendix A

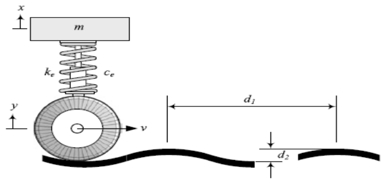

Fig: (13) Car model moving with speed v on a wavy road

$x = A1 \cos(\omega 1t) + A2 \sin(\omega 2t + \alpha)$. The required time to pass one wavelength $d1$ is the period of the excitation (fig. (13)). $T = d1/v$. Therefore, the frequency of excitation is $\omega = 2\pi/T = 2\pi v/d1$ and the excitation $y = Y \sin \omega t$ is $y = d2/2 \sin 2\, t\, \pi v/d1$

- Appendix B

Direct method

Starting with only one element beam which is subject to bending and shear forces. There are 4 nodal degrees of freedom. Rotation at the left and right nodes of the beam and transverse displacements at the left and right nodes. The following diagram shows the sign convention used for external forces. Moments are always positive when the anti-clockwise direction and vertical forces are positive when in the positive y direction. The two nodes are numbered 1 and 2 from left to right. $M1$ is the moment at the left node (node 1), $M2$ is the moment at the right node (node 2). $V1$ is the vertical force at the left node and $V2$ is the vertical force at the right node.

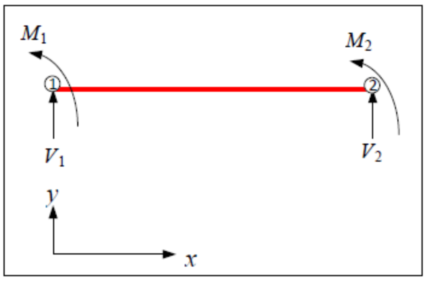

Fig. (14) applied forces direction

Figure (14) shows the signs used for the direction of the applied force when acting in the positive sense. Since this is a one-dimensional problem, the displacement field (the unknown being solved for) will be a function of one independent variable which is the x coordinate. The displacement field in the vertical direction is called $v\ (x)$. This is the vertical displacement of point x on the beam from the original $x - axis$. Figure (15) shows the notation used for the coordinates

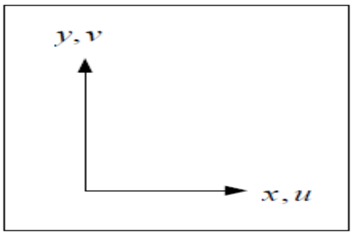

Fig. (15) notation used for the coordinates

Angular displacement at distance x on the beam is found using $\theta\ (x) = d(x)\ /dx$. At the left node, the degrees of freedom or the displacements, are called $v1$, $\theta1$ and at the right node, they are called $v2$, $\theta2$. At an arbitrary location x in the beam, the vertical displacement is $v\ (x)$ and the rotation at that location is $\theta\ (x)$. Figure (16) shows the displacement field $v\ (x)$

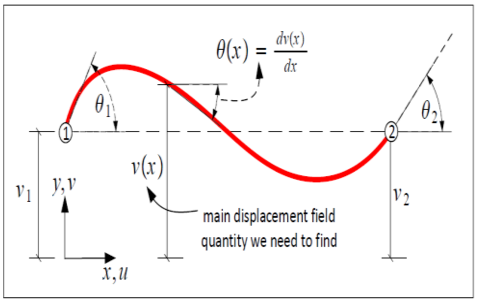

Fig. (16) displacement field

In the direct method of finding the stiffness matrix, the forces at the ends of the beam are found directly by the use of beam theory. In beam theory, the signs are different from what is given in the first diagram above. Therefore, the moment and shear forces obtained using beam theory (MB and VB in the diagram below) will have different signs when compared to the external forces. The signs are then adjusted to reflect the convention as shown in the diagram above using M and V.

For example, the external moment $M1$ is opposite in sign to $MB1$ and the reaction $V2$ is opposite to $VB2$. To illustrate this more, a diagram with both sign conventions is given below fig. (17)

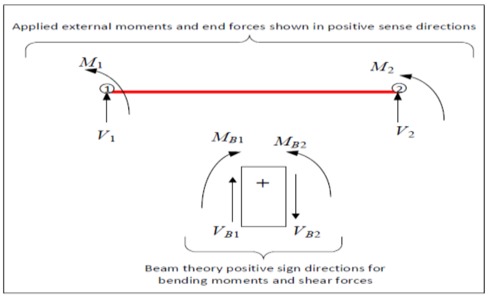

Fig. (17) sign conventions

The goal now is to obtain expressions for external loads Mi and Ri in the above diagram as a function of the displacements at the nodes.

$[d] = \{v_1,\ \theta_1, v_2,\ \theta_2\}^T$

In other words, the goal is to obtain an expression of the form $\{p\} = [K]\ \{d\}$ where $[K]$ is the stiffness matrix where

$[P] = \{v_1,\ \theta_1, R_2,\ M_2\}^T$

is the nodal forces or load vector, and $\{d\}$ is the nodal displacement vector.

In this case $[K]$ will be a 4×4 matrix and $\{p\}$ is a 4×1 vector and $\{d\}$ is a 4×1 vector.

Starting with V_1. It is in the same direction as the shear force V_{B1}.

Since $V_{B1} = \frac{dM_{B1}}{dx}$ then

$V_1 = \frac{dM_{B1}}{dx}$

Since from beam theory

$M_{B1} = -\sigma(x)\frac{1}{y}$, the above becomes

$$V_1 = -\frac{I}{y}\frac{d\sigma(x)}{dx}$$

But $\sigma\ (x) = E\varepsilon\ (x)$ and $\varepsilon(x)\frac{1}{y}\ \varepsilon\ (x)$ where ρ is the radius of curvature, therefore the above becomes

$$V_1 = EI\ \frac{d}{dx}\left(\frac{1}{\rho}\right)$$

Since

$$\frac{1}{\rho} = \frac{\frac{d^2u}{dx^2}}{(1+\left(\frac{du}{dx}\right)^2)^{3/2}}$$

and for a small angle of deflection

$du/\, dx \ll 1$

then

$\frac{1}{\rho} = (\frac{d^2u}{dx^2})$

and the above now becomes

$$V_1 = EI\frac{d^3u(x)}{dx^3}$$

Before continuing, the following diagram fig. (18) illustrates the above derivation. This comes from beam theory.

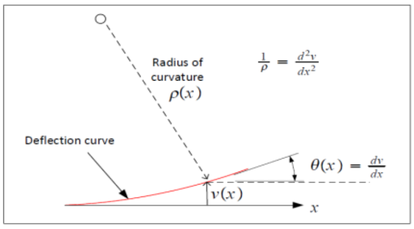

Fig. (18) radius of curvature

Now M_1 is obtained. M_1 is in the opposite sense of the bending moment M_{B1} hence a negative sign is added giving $M_1 = -M_{B1}$

But $M_{B1} = -\sigma(x)\frac{I}{y}$ therefore

$M_1 = \sigma(x)\frac{I}{y} = E\varepsilon(x)\frac{I}{y} = E(\frac{-y}{\rho})\frac{I}{y}$

$= -EI\,(\frac{1}{\rho}) = -EI\frac{d^2w}{dx^2}$

V_2 is now found. It is in the opposite sense of the shear force V_{B2}, hence a negative sign is added giving $V_2 = -V_{B2} = -\frac{dM_{B2}}{dx}$

Since $M_{B2} = -\sigma(x)\frac{I}{y}$, the above becomes

$$V_2 = \frac{I}{y}\frac{d\sigma(x)}{dx}$$

But $\sigma(x) = E\varepsilon(x)\ and\ \varepsilon(x) = \frac{-y}{\rho}$

where ρ is the radius of curvature. The above becomes

$$V_2 = EI\frac{d}{dx}(\frac{1}{\rho})$$

But

$$\frac{1}{\rho} = \frac{\frac{d^2w}{dx^2}}{(1+\left(\frac{dw}{dx}\right)^2)^{3/2}}$$

and for small angle of deflection

$$\frac{dw}{dx} \ll 1$$

hence $\frac{1}{\rho} = \left(\frac{d^2u}{dx^2}\right)$

then the above becomes

$$V_2 = -EI\frac{d^3u(x)}{dx^3}$$

Finally, $M_2$is in the same direction as M_{B2}so no significant change is needed.

$M_{B2} = -\sigma(x)\frac{I}{y}$

Therefore

$$M_2 = -\sigma(x)\frac{I}{y} = -E\varepsilon(x)\frac{I}{y}$$

$$= -E\left(\frac{-y}{\rho}\right)\frac{I}{y} = EI\left(\frac{1}{\rho}\right) = EI\frac{d^2u}{dx^2}$$

The following is a summary of what was found so far. Notice that the above expressions are evaluated at

x = 0 and at x = L. Accordingly, to obtain the nodal end forces vector $\{p\}$

$$\{p\} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} EI\frac{d^3u(x)}{dx^3}|_{x=0} \\ -EI\frac{d^2u}{dx^2}|_{x=0} \\ -EI\frac{d^3u(x)}{dx^3}|_{x=L} \\ EI\frac{d^2u}{dx^2}|_{x=L} \end{Bmatrix}$$

The RHS of the above is now expressed as a function of the nodal displacements $v1$, $\theta1$, $v2$, $\theta2$.

To do that, the field displacement $v\,(x)$ which is the transverse displacement of the beam is assumed to be a polynomial in x of degree 3 or

$V(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\theta(x) = \frac{dv(x)}{dx} a_1 + 2a_2 x + 3a_3 x^2$

$$v_1 = v(x)_{x=0} = a_0$$

And

$$\theta_1 = \theta_{(x)}|_{x=0} = a_1$$

Assuming the length of the beam is L, then

$$v_2 = v(x)|_{x=L}$$
$$= a_0 + a_1 L + a_2 L^2 + a_3 L^3 L3$$

$$\{d\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} a_0 \\ a_1 \\ a_0 + a_1 L + a_2 L^2 + a_3 L^3 \\ a_1 + 2a_2 L + 3a_3 L^2 \end{Bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Solving for ai gives

The polynomial of degree 3 is used since there are 4 degrees of freedom, and a minimum of 4 free parameters is needed. Hence

And

$$\theta_2 = \theta(x)|_{x=L}$$
$$= a_1 + 2a_2 L + 3a_3 L^2$$

Equations (2-5) gives

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{pmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} v_1 \\ \theta_1 \\ \frac{3}{L^2} v_{2_2} - \frac{1}{L}\theta_2 - \frac{3}{L^2} v_1 - \frac{2}{L}\theta_1 \\ \frac{1}{L^2}\theta_1 + \frac{1}{L^2}\theta_2 + \frac{2}{L^3} v_1 - \frac{2}{L^3} v_2 \end{Bmatrix}$$

$v\,(x)$, the field displacement function from Eq. (A) can now be written as a function of the nodal displacements

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad \text{(A)}$$
$$= v_1 + \theta_1 x + \left(\frac{3}{L^2} v_2 - \frac{1}{L}\theta_2 - \frac{3}{L^2} v_1 - \frac{2}{L}\theta_1\right) x^2 + \left(\frac{1}{L^2}\theta_1 + \frac{2}{L^3} v_1 - \frac{2}{L^3} v^2\right) x^3$$

Or in matrix form

$$v(x) = \left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \quad x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \quad 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \quad -\frac{x^2}{L} + \frac{x^3}{L^2}\right)$$

$$= \left(\frac{1}{L^3}(L^3 - 3Lx^2 + 2x^3 \quad \grave{} \frac{1}{L^2}(L^2 x - 2Lx^2 + x^3) \quad \frac{1}{L^3}(3Lx^2 - 2x^3) \quad \frac{1}{L^2}(-Lx^2 + x^3)\right) \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$v(x) = (N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)) \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \grave{}$$

The above can be written as

$$v(x) = [N]\{d\}$$

Where Ni are called the shape functions. The shape functions are

$$\begin{pmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{pmatrix} = \begin{bmatrix} \frac{1}{L^3}(L^3 - 3Lx^2 + 2x^3 \\ \frac{1}{L^2}(L^2 x - 2Lx^2 + x^3) \\ \frac{1}{L^3}(3Lx^2 - 2x^3) \\ \frac{1}{L^2}(-Lx^2 + x^3) \end{bmatrix}$$

We

notice that $N_1(0) = 1,$ and $N_1(L)$=0, as expected. Also

$$\frac{dN_{2(x)}}{dx}|_{x=0} = \frac{1}{L^2}(L^2 - 4Lx + 3x^2)|_{x=0} = 1$$

And

$$\frac{dN_{2(x)}}{dx}|_{x=L} = \frac{1}{L^2}(L^2 - 4Lx - 3x^2)|_{x=L} = 1$$

Also $N_{3(0)} = 0$ and $N_{3(L)} = 1$ and

$$\frac{dN_{4(x)}}{dx}|_{x=0} = \frac{1}{L^2}(-2Lx + 3x^2)|_{x=0} = 0$$

And

$$\frac{dN_{4(x)}}{dx}|_{x=L} = \frac{1}{L^2}(-2Lx - 3x^2)|_{x=L} = 1$$

The shape functions have thus been verified. The stiffness matrix is now found by substituting

Eq. (5A) into Eq. (1), repeated below

$$\{p\} = \begin{Bmatrix} v_1 \\ M_1 \\ v_2 \\ M_1 \end{Bmatrix} = \begin{pmatrix} EI\frac{d^3v(x)}{dx^3}|_{x=0} \\ -EI\frac{d^2v}{dx^2}|_{x=0} \\ -EI\frac{d^3v(x)}{dx^3}|_{x=L} \\ EI\frac{d^2v}{dx^2}|_{x=L} \end{pmatrix}$$

Hence

$$\{p\} = \begin{Bmatrix} v_1 \\ M_1 \\ v_2 \\ M_1 \end{Bmatrix}$$

$$= \begin{pmatrix} EI\frac{d^3}{dx^3}(N_1v_1 + N_2\theta_2 + N_3v_{2+}N_4\theta_2) \\ -EI\frac{d^2}{dx^2}(N_1v_1 + N_2\theta_1 + N_3v_{2+}N_4\theta_2) \\ -EI\frac{d^3v(x)}{dx^3}(N_1v_1 + N_2\theta_2 + N_3v_{2+}N_4\theta_2) \\ EI\frac{d^2}{dx^2}(N_1v_1 + N_2\theta_1 + N_3v_{2+}N_4\theta_2) \end{pmatrix}$$

Hence Eq. (6) becomes

$$\{p\} = \begin{Bmatrix} v_1 \\ M_1 \\ v_2 \\ M_1 \end{Bmatrix}$$

But

$$\frac{d^3}{dx^3}N_{1(x)} = \frac{1}{L^3}\frac{d^3}{dx^3}(L^3 - 3Lx^2 + 2x^3) = \frac{12}{L^3}$$

And

$$\frac{d^3}{dx^3}N_{2(x)} = \frac{1}{L^2}\frac{d^3}{dx^3}(L^2x - 2Lx^2 + x^3) = \frac{6}{L^2}$$

And

$$\frac{d^3}{dx^3}N_{3(x)} = \frac{1}{L^3}\frac{d^3}{dx^3}(3Lx^2 + 2x^3) = \frac{-12}{L^3}$$

And

$$\frac{d^3}{dx^3}N_{4(x)} = \frac{1}{L^2}\frac{d^3}{dx^3}(-Lx^2 + x^3) = \frac{6}{L^2}$$

For the second derivatives

$$\frac{d^2}{dx^2}N_{1(x)} = \frac{1}{L^3}\frac{d^2}{dx^2}(L^3 - 3Lx^2 + 2x^3) = \frac{1}{L^3}(12x - 6L)$$

$$\frac{d^2}{dx^2}N_{2(x)} = \frac{1}{L^2}\frac{d^2}{dx^2}(L^2x - 2Lx^2 + x^3) = \frac{1}{L^2}(6x - 4L)$$

And

$$\frac{d^2}{dx^2}N_{3(x)} = \frac{1}{L^3}\frac{d^2}{dx^2}(3Lx^2 - 2x^3) = \frac{1}{L^3}(6L - 12x)$$

And

$$\frac{d^2}{dx^2}N_{(x)} = \frac{1}{L^2}\frac{d^2}{dx^2}(-Lx^2 + x^3) = \frac{1}{L^3}(6x - 2L)$$

$$= \begin{pmatrix} EI\dfrac{d^3}{dx^3}(N_1v_1 + N_2\theta_{1+}N_3v_2 + N_4\theta_2)|_{x=0} \\ -EI\dfrac{d^2}{dx^2}(N_1v_1 + N_2\theta_{1+}N_3v_2 + N_4\theta_2)|_{x=0} \\ -EI\dfrac{d^3v}{dx^3}(N_1v_1 + N_2\theta_{1+}N_3v_2 + N_4\theta_2)|_{x=L} \\ EI\dfrac{d^2}{dx^2}(N_1v_1 + N_2\theta_{1+}N_3v_2 + N_4\theta_2)|_{x=L} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} EI\left(\frac{12}{L^3}v_1 + \frac{6}{L^2}\theta_1 - \frac{12}{L^3}v_2 + \frac{6}{L^2}\theta_2\right)|_{x=0} \\ -EI\left(\frac{1}{L^3}(12x-6L)v_1 + \frac{1}{L^2}(6x-4L)\theta_{1+}\frac{1}{L^3}(6L-12x)v_2 + \frac{1}{L^2}(6x-2L)\theta_2\right)|_{x=0} \\ -EI\left(\frac{12}{L^3}v_1 + \frac{6}{L^2}\theta_1 - \frac{12}{L^3}v_2 + \frac{6}{L^2}\theta_2\right)|_{x=L} \\ EI\left(\frac{1}{L^3}(12x-6L)v_1 + \frac{1}{L^2}(6x-4L)\theta_{1+}\frac{1}{L^3}(6L-12x)v_2 + \frac{1}{L^2}(6x-2L)\theta_2\right)|_{x=0} \end{pmatrix}$$

$$= \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ -(12x-6L)_{x=0} & -L(6x-4L)_{x=0} & -(6L-12x)_{x=0} & -L(6x-2L)_{x=0} \\ -12 & -6L & 12 & -6L \\ (12x-6L)_{x=L} & L(6x-4L)_{x=L} & (6L-12x)_{x=L} & L(6x-2L)_{x=L} \end{pmatrix}$$

Or in matrix form, after evaluating the expressions above for $x = L$ and $x = 0$ as

$$\begin{Bmatrix} v_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} == \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -\dfrac{12}{L^3} & -\dfrac{6}{L^2} & \dfrac{12}{L^3} & -\dfrac{6}{L^2} \\ (12L-6L) & L(6L-4L) & (6L-12L) & L(6L-2L) \end{pmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_1 \end{Bmatrix}$$

$$= \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & * 6L & 4L^2 \end{pmatrix}$$

The above now is in the form

$$\{p\} = [k]\{d\}$$

Hence the stiffness matrix is

$$[k] = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & * 6L & 4L^2 \end{pmatrix}$$

Knowing the stiffness matrix means knowing the nodal displacements $\{d\}$ when given the forces at the nodes. The power of the finite element method now comes after all the nodal displacements $v1$, $\theta1$, $v2$, $\theta2$ are calculated by solving

$$\{p\} = [k]\{d\}$$