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ELASTIC STABILITY OF STEEL STRUCTURAL SYSTEMS RESTRAINED BY INITIALLY CURVED MEMBERS

الاتزان المرن للمنشآت الحديدية والمسنودة باعضاء ملوية

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الملخص : لقد تم دراسة وتعيين الجساءة المحورية للاعضاء الملوية باستخدام نظرية الانحرافات الصغيرة وتم وصف واستخدام هذه الجساءة في حالة عمود مسنود جانبيا عند منتصف ارتفاعه بعضو ملوى . وكذلك تم دراسة انبعاج واتزان هذا العمود في مستوى العضو الملوى الساند له . وتم لحص وتعيين الاحمال الحرجة المؤثرة على هذا النوع من المنشآت وذلك باستخدام نظام المصفوفات لتعميم طريقة الحساب واستخدام الحاسب الالي في حلها.

ABSTRACT

The axial stiffness of elastic, initially curved, members is considered using small-deformation theory. Simple relations are obtained for the tangent axial stiffness. The use of this equivalent spring stiffness is described in an attempt to provide a unified description of the restraint given to a main column member by the provision of initially curved members at mid-height. The buckling of a column element in the plane of the nonlinear restraints is investigated. Further the purpose of the analysis is to describe precise values to the spring stiffness in the light of the analysis referred to the above. Further, the matrix technique illustrated in the present paper is capable of generalization, especially using automatic computational procedures. Finally the critical load of the structural system is investigated with respect to the presence of imperfection in the lateral restraints.

INTRODUCTION

The nonlinear analysis of structural systems consisting of main individual columns stiffened laterally by initially curved members are considered. Some unavoidable imperfection are taken into account and a critical state, still in the elastic range, is examined. The basic equations of the axial stiffness of elastic, initially curved, members are reviewed especially to provide information on the equivalent spring stiffness of this type of members. Simple relations are obtained for the tangent axial stiffness in terms of the slenderness ratio, initial curvature and axial thrust in the member. Then the use of this equivalent spring stiffness is described in an attempt to provide a unified description of the restraint given to a main column member by the provision of initially curved members at mid-height. The reduction of the critical load is investigated with respect to the presence of imperfection in the lateral restraints.

1: THE AXIAL STIFFNESS OF IMPERFECT CURVED, COMPRESSION, MEMBER.

Considering a compression member that has initial curvature of sine wave function as shown in Fig.(1) and given by;

$$y_o = a_o \sin(\frac{\pi x}{l}) \quad \ldots\ldots\ldots\ldots : \quad 0 < x < l \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where; y_o is the initial deflection of the member from its chord-line position

a_o is the amplitude and l is the chord line length at the initial curved position

Further, the effect of compressive axial load upon this initial shape is to amplify the initial deflections such that the final deflected shape "y" is easily obtained [6].

$$y = \frac{a_o}{1-P/P_E} \sin \frac{\pi x}{l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which; $P_E = \pi^2 E I / l^2$ is the Euler buckling load when the member is perfectly straight ,and

l is the chord line length of the initially curved member.

Now Eq.(2) may be used to evaluate the shortening "Δl" of the chord-line of the curved member due to the compression applied load "P"

$$\Delta_l = \Delta_B + \Delta_A \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where, Δ_B is the chord-line shortening due to bowing and,

Δ_A is the axial shortening of the member due to direct stress (i.e due to the axial force P) in which;

$$\Delta_B = \frac{1}{2} \int_o^l (\frac{dy}{dx})^2 dx - \frac{1}{2} \int_o^l (\frac{dy_o}{dx})^2 dx$$

using Eqs.(1) and (2) for the values of y_o and y, then;

$$\Delta_B = \frac{\pi^2 a_o^2}{2 l^2} \left(\frac{1}{(1-\mu)^2} - 1\right) \int_o^l \cos^2 \frac{\pi x}{l} dx \qquad \text{where } \mu = P/P_E$$

$$\Delta_B = \frac{\pi^2 a_o^2 \mu (2-\mu)}{4 l^2 (1-\mu)^2} \int_o^l (\cos \frac{2\pi x}{l} + 1)\, dx$$

or $\quad \Delta_B = \pi^2 a_o^2 \mu(2-\mu) \;/\; 4l(1-\mu)^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$

and $\quad \Delta_A \cong \frac{P}{AE} \left(l + \frac{\pi^2 a_o^2}{4l}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$

Since the actual length of the unloaded member is approximately equal to $(l + \pi^2 a_o^2 / 4l)$, therefore

$$\Delta_l = \frac{\pi^2 a_o^2 \mu (2-\mu)}{4l(1-\mu)^2} + \frac{P}{AE} \left(l + \frac{\pi^2 a_o^2}{4l}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

It is convenient to express Eq.(6) in non-dimensional form as follows;

let $\bar{\Delta}_t = \Delta_t/l$ $\qquad e = a/l \qquad$ and $\qquad P/AE = \mu\,\pi^2/\lambda^2$

where, λ is the slenderness ratio (l/r) of the element, and
r is the radius of gyration
Then Eq.(6) may be re-written as follows;

$$\bar{\Delta}_t = \frac{e^2\pi^2\mu(2-\mu)}{4(1-\mu)^2} + \mu\frac{\pi^2}{\lambda^2}\left(1 + \frac{e^2\pi^2}{4}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The axial stiffness of a curved member K^* can be defined as the load, applied along the chord-line of the member, which will produce a total axial shortening of the chord-line equal to unity ie.

$$P = K^*.\Delta_t$$

where; P is the axial load on the column
and Δ_t is the total shortening, or total elongation, of the chord-line.

In general K^* will be a function of P, but will reduce to the "Hooks low" stiffness AE/l when the effect of bowing is ignored. therefore;

$$K^* = P/\Delta_t \qquad \text{or in non-dimensional form;}$$

$$\frac{K^* l}{P_E} = \frac{\mu}{\bar{\Delta}_t} = \frac{4\lambda^2\mu(1-\mu)^2}{\pi^2(e^2\lambda^2\mu(2-\mu) + \mu(1-\mu)^2(4+e^2\pi^2))} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Eq.(8) gives the non-dimensional tangent axial stiffness of the curved member at any load level μ in terms of the initial amplitude of the curved shape, e, and the geometrical properties of the member. The initial stiffness of the unloaded curved member (μ=0) follows directly from Eq.(8) by differentiation of the numerator and denominator with respect to μ, and substitution of μ=0 then,

$$\left.\frac{K^* l}{P_E}\right|_{\mu \to 0} = \frac{4\lambda^2}{\pi^2(2e^2\lambda^2 + e^2\pi^2 + 4)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

if the initial curvature "e" is zero , then $(K^* l/P_E)_{\mu \to 0} = \lambda^2/\pi^2$
i.e .K^* reduces to EA/l "the Hokes low" stiffness.

Eq.(9) yields the equivalent axial spring stiffness of an initially curved member when used as a lateral restraint against buckling of a main column leg. By inspection in Eq.(9) it is seen that, the limiting value of $K^* l/P_E$ i.e. the tangent axial stiffness in non-dimensional form at μ=0, reduces while the amplitude of the curved shape, e, increases,at limiting value of λ.

The effect of axial load "P" on the non-dimensional tangent axial stiffness $K^* l/P_E$ for various values of λ, when e= 0.003 by way of example, is shown in Fig.(2).

The chart was obtained by plotting, on horizontal axis, values of the non-dimensional tangent axial stiffness $K^* l/P_E$ and, on vertical axis, values of the slenderness ratio λ. It is seen that, from the diagrams shown in Fig.(2), the axial stiffness K^* reduces while the μ value (i.e the applied load P) increases at limiting value of curvature e.

2: THE ELASTIC BUCKLING LOADS OF A STRUCTURAL SYSTEM RESTRAINED LATERALLY BY INITIALLY CURVED MEMBER

Consider a simple problem of a main column leg pinned at its ends, and restrained laterally by an elastic curved member pinned to the column at mid-height and pinned to a rigid support at the other end, as shown in Fig.(3). The problem is easily daalt with a matrix technique similar to that employed in a general buckling analysis.

Now let $\{R\}=\{R_x\ R_y\ M\}$ be the small set of exciting forces, applied to the node "1" Fig.(3), and $\{r\}=\{r_x\ r_y\ \theta\}$ be the corresponding set of node displacements. Then, the governing equation for the behaviour of the system may be written as:

$$\{R\} = [K]\{r\} \qquad \text{...............(10)}$$

where; $[K]$ is the overall stiffness matrix of the spring-supported column, and in this case is an assembly of the member stiffness matrices for the elements "a" and "b" together with the contribution from the equivalent spring stiffness of the curved member.

Eq.(10) is an entirely general results in which the vector $\{R\}$ is a column matrix of all relevant node forces and $\{r\}$ is the corresponding vector of node displacements. The presence of spring supports can always be incorporated by addition of appropriate spring stiffness to the corresponding elements of the leading diagonal of $[K]$. Ignoring this contribution from the spring constraint, and referring to the elements "a" and "b" using the conventional notation of end 1 and end 2, as defined in Fig.(2) it follows that, in the simple structure under consideration

$$[K] = [k_{22a}] + [k_{11b}] \qquad \text{...............(11)}$$

where: $[k_{22}]$ and $[k_{11}]$ are the elements stiffness sub-matrices abstracted from a general statement of an element force-displacement relationship of the form ;

$$\{F'\} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \{u'\} \qquad \text{...............(12)}$$

where; F', u' are member forces and member displacements respectively

It should be noted that, in this case, $[k_{22a}]$ and $[k_{11b}]$ are member stiffness sub-matrices appropriate to a beam-column element pinned to a rigid support at end 1 and 2 respectively, and allow for the presence of axial load P in each element.

The overall stiffness matrix [K] for this particular structure can be obtained and derived as follows:

$$\begin{bmatrix} \dfrac{4EA_c}{L_c} & 0 & 0 \\ 0 & \dfrac{16\alpha_1 EI}{L_c^3} + K^* & 0 \\ 0 & 0 & \dfrac{4\alpha_2 EI}{L_c} \end{bmatrix} \qquad \text{...............(13)}$$

where; K^* is the equivalent spring stiffness of the curved member
EI is the flexural rigidity of the main column

A_c, l_c are the cross sectional area and the overall length of the column respectively, and
α_1, α_2 are stability functions dependent upon the axial load in the column and are defined as follows;

$$\left.\begin{aligned} \alpha_1 &= s(1 - c^2) - \pi^2 \mu_m \\ \alpha_2 &= s(1 - c^2) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (14)$$

where; $\mu_m = P/P_E = Pl_c^2/4\pi^2 EI$ is the ratio of the axial load in a column element to the Euler load of the column element.

and $$S = \frac{\beta(1 - 2\beta \cot 2\beta)}{\tan \beta - \beta},$$

$$C = \frac{2\beta - \sin 2\beta}{\sin 2\beta - 2\beta \cos 2\beta}$$

which $\beta = \pi/2 . \sqrt{\mu_m}$

The general condition for structural instability is that $|K|=0$. from which the following buckling criteria results;

$$\alpha_2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots (15)$$

$$\alpha_1 = - \frac{K^* . l_c^3}{16EI} \qquad \ldots\ldots\ldots\ldots\ldots (16)$$

This result is identical in form to that derived by Timoshinko and Gere [12]. The only merit of the analysis shown above is in that the technique is entirely general for any elastic spring-supported assemblage of element and may be automated for a computer solution.

The investigation is typical of that which could be carried out when the column is constrained laterally by a number of curved members pinned to it at various points along its length

Equations (15) and (16) define completely the occurrence of all in-plane buckling loads. There will be an infinite number of such loads and generally it is the first buckling load which is of practical significance. Examination of these equations gives the full account of the buckling behaviour, interpreted in terms of the geometrical properties of the side curved member and main column. Firstly examining Eq.(15) and writing α_2 in terms of β using the results quoted above for S and C, the equation may be re-written in the form;

$$\alpha_2 = \frac{4\beta^2 (\sin 2\beta - \beta \cos 2\beta - \beta)}{(1 - \beta \cot \beta)(\sin 2\beta - 2\beta \cos 2\beta)} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (17)$$

Inspection of Eq.(17) it is seen that;
$\alpha_2 = 0$ when $\beta = \pi/2$ i.e.
$\alpha_2 = 0$ when $\mu_m = 1$ and the initially buckled shape is the anti-symmetric one shown in fig (4-a). This may or may not be the first critical buckling load dependent upon the value of K^* resulted from Eq.(16).

Now proceed to an examination of Eq.(16), and substitute for K^* from

Eq.(9), after re-arrangement of the algebra, making use of the non-dimensional quantities λ_c and λ_s, the buckling criteria of Eq.(16) may be written in the following form,

The buckling occurs when;

$$\alpha_1 = \frac{A_s L_c \lambda_c^2}{4 A_c L_s (2e^2\lambda_s^2 + e^2\pi^2 + 4)} \qquad \ldots\ldots\ldots\ldots\ldots(18)$$

and this criterion is conveniently written, entirely non-dimensionally, as;

$$\alpha_1 = \frac{\nu}{2e^2\lambda_s^2 + e^2\pi^2 + 4} \qquad \ldots\ldots\ldots\ldots\ldots(19)$$

where, $$\nu = \frac{A_s L_c \lambda_c^2}{4 A_c L_s} \qquad \ldots\ldots\ldots\ldots\ldots(20)$$

λ_s is the slenderness ratio of the side curved member
λ_c is the slenderness ratio of the column

The slenderness ratios are defined as the actual length of the element divided by the appropriate radius of gyration.

It should be noted that α_1 is a function of the axial load on the main column and Eq.(19) may yield either the first or the second buckling mode. The mode shape is shown in Fig.(4-b) where the lateral displacement of node 1 depends upon the spring stiffness 'K'.

The results of the above analysis can be conveniently represented graphically in chart form giving an automatic means of determining the critical buckling loads from Eq.(19), at a glance.

The behaviour is summarised in Fig.(5) for a range of laterally curved member slenderness ratio 'λ_s' from 100 to 350 and a selection of values of the non-dimensional parameter ν. Two values of the initial curvature, e, of the lateral restraint are shown viz. e = 0.003 and e = 0.01. Similar charts could, of course, be plotted for other initially curved shapes. The diagram was obtained by plotting, on a horizontal axis, values of the stability function α_1, for various values of μ_c and, on a vertical axis, the ratio of the axial load in the column to the Euler load of the complete column ($\mu_c = 4\mu_m$), and the right hand side of Eq.(19) for various values of ν, e and λ_s.

To use the chart in Fig.(5), locate the slenderness ratio value of the lateral restraint on the right-hand side vertical axis. Knowing the value of curvature e, then ν value can be calculated from Eq.(20). The chart is traversed horizontally to meet the appropriate ν curve, thence vertically to intercept the single α_1 plot, and therefore the horizontal traverse is completed to intercept the left-hand side vertical axis at the critical value of μ_c.

It is seen that the chart in Fig.(5) is constructed specifically in terms of geometrical properties and imperfections of the side curved member. Also this chart predominantly yields values of μ_c <4.0 and indicates that Eq.(19) picks out the first critical buckling mode.

Finally, the relative importance of Eq.(15) and (16) in picking out the first critical buckling mode may be summarised in one simple interaction diagram between ν and e, for any value of λ_*, by substituting the value of α_1 at $\mu_m = 1.0$ in Eq.(19) and plotting the resulting equation in ν and e.

Since $\alpha_1 = s(1 - c^2) - \pi^2\mu_m$ and it may be shown that $s(1 - c^2) = 0$ when $\mu_m = 1.0$, then this condition, Eq.(19) reduces to:

$$\pi^2 = \nu \,/(2e^2\lambda_s^2 + e^2\pi^2 + 4)$$

$$\nu = \pi^2(e^2(2\lambda_s^2 + \pi^2) + 4\ ) \qquad \ldots\ldots\ldots\ldots\ldots(21)$$

This result is plotted in Fig.(6) , for a range of slenderness ratio of the side rail between 100 and 350. To determine whether the first critical buckling load of the column lies below or above $\mu_c = 4$, ($\mu_m = 1.0$), the interaction diagram is entered with pre-determined values of ν and e (which are functions of the geometry of the system only). If the entry point lies to the left of the curve corresponding to the appropriate value of λ_* then the first critical buckling load μ_c is less than 4.0 and will be given by Fig.(5). Other wise the first critical buckling load will be $\mu_c = 4.0$. The second critical buckling load, which is, of course, largely of theoretical interest only, can never exceed $\mu_c = 8.184$ since this corresponds to the mode shown in Fig.(4-c), and requires an infinitely stiff constraint at mid-height of the column.

CONCLUSIONS

An attempt has been made to assess the axial spring stiffness of curved elements and to apply the results to one fairly simple constraint problem. Whilst the problem chosen has fair resemblance to a particular practical condition, the main purpose in presenting the above work has been to provide a unified view of the elastic buckling behaviour of the system considered. The presentation of information in chart form has the merit that the results of the analysis can be ascertained with a minimum of involvement with the algebra, and does completely describe the fundamental behaviour concisely in a situation where there are many design parameters.

There seems to be no overwhelming difficulty in presenting similar charts for more complex situations containing a number of spring constraints and even including the effects of lateral-torsional buckling of elasto-plastic behaviour.

REFERENCES

1. **Ajmani , J.L.** "Stability of Columns Restrained by Side-Rails ". Journal of the Structural Engrg.No.8. , Volume 44 . August 1971

2. **Allen,H.G. and Bulson,P.S.** "Background to Buckling ".McGraw-Hill Book Company , U.K.,ltd. 1980

3. **Bolton, A.** "A Simple Understanding of Elastic Critical Loads".Journal of The Structural Engineering vol. s4. no. 6, PP-213-218,Sume. 1976.

4. **Bolton, A.** "The Critical Load of Portal Frames When Side Sway is Permitted" J of The Structural Engineering ,August,1965.

5. Horne, M.R.and Merchant, W. "The stability of frames".London pergamon, 1965.

6. Horne, M.R. "An Approximate Method for Calculating The Elastic Critical Loads of Multi-Story Plane Frames". The Structural Engineer , Vol.53, No.6, 242/248 June1975.

7. Livesley, R.K. "Matrix method of Structural Analysis." London Pergamon Press. 1975.

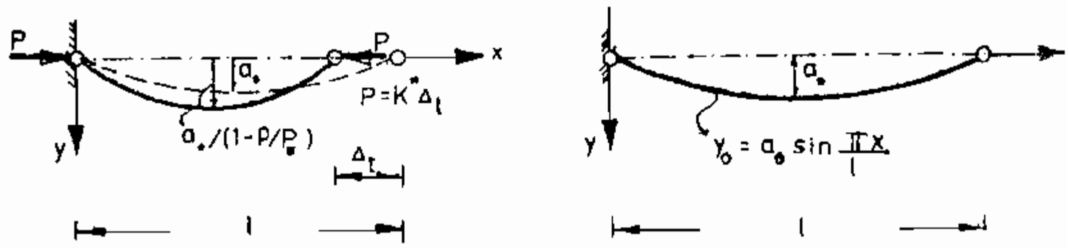
8. Livesley, R.K.and Chandler,D.B."Stability Functions for Structural Frame Works " . Manchester univ. Press,1956.

9. Majid, K.I. "Non- linear structures " Butterworths london 1972.

10. Merchant, W. "The Buckling of Pin-ended Struts Under Axial Load'. Journal of The Institution of Structural Engineers, September, 1949.

11. Stevens, L.K. "Elastic Stability of Practical Multistorey Frames" Proceedings Institution of Civil Engrg., vol. 36, 99/117. Jan. 1967.

12. Timoshenko,S. and Gere,J.M."Theory of Elastic Stability" New York, McGraw-Hill, 1961.



(a) Loaded strut

(b) Initially unloaded curved member

FIG.(1)

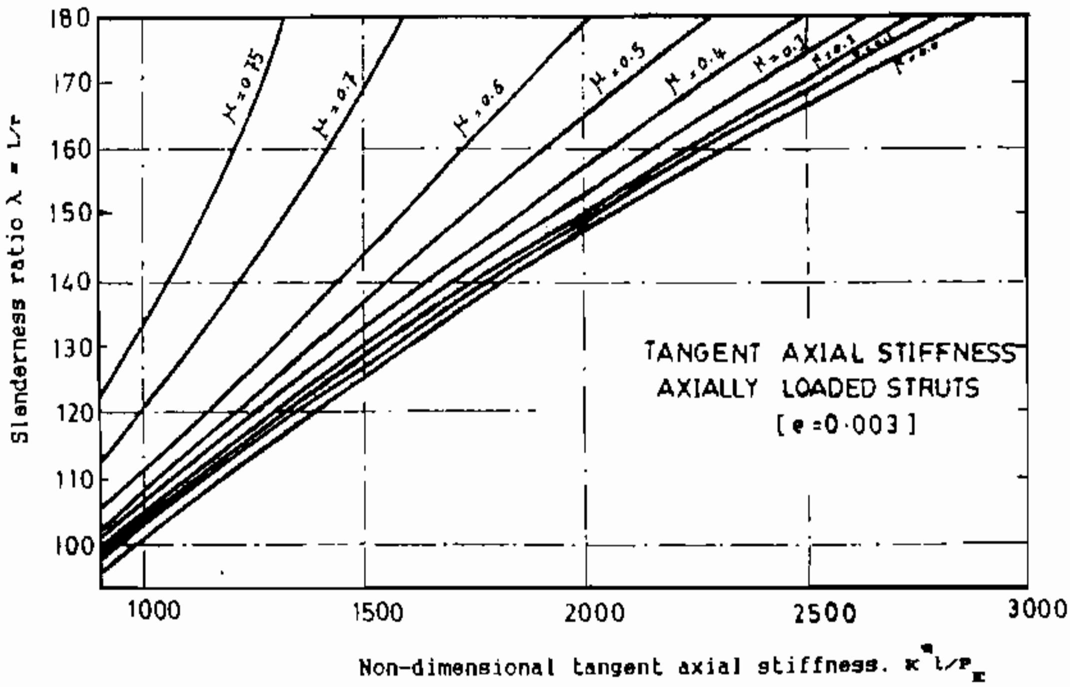
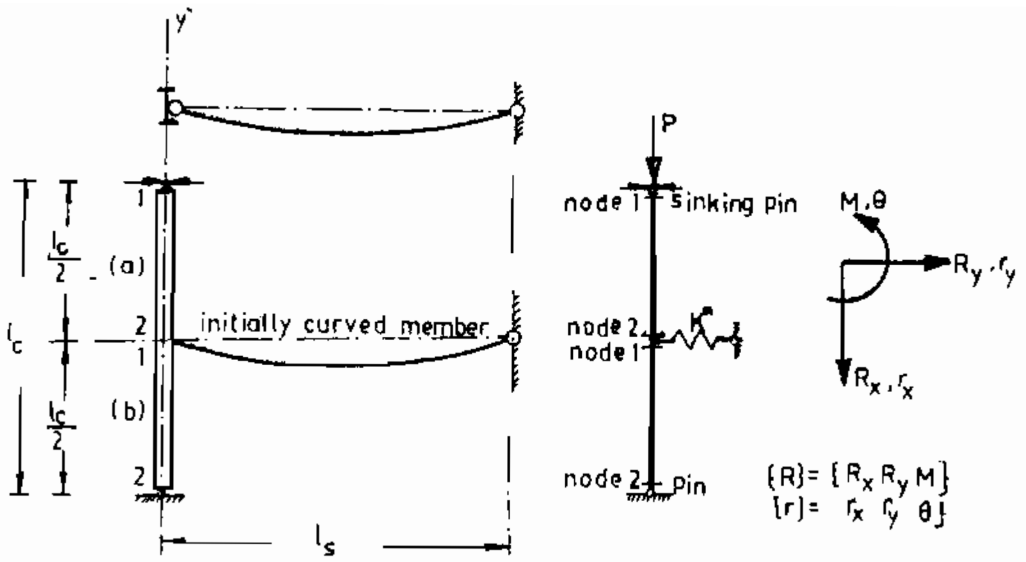
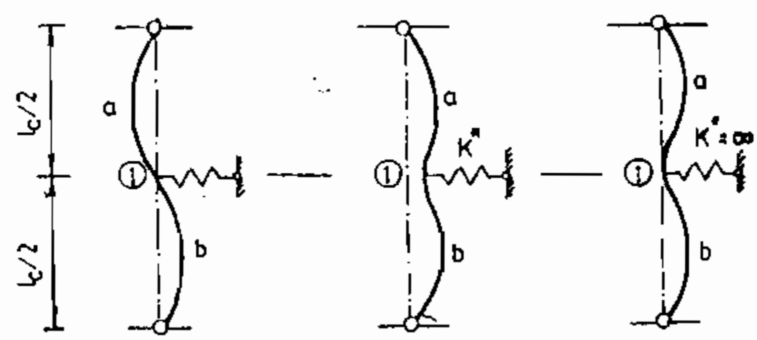


FIG.(2). THE EFFECT OF AXIAL LOAD "P" ON THE NON-DIMENSIONAL TANGENT AXIAL STIFFNESS



(a) Schematic Description of Problem (b) Mathematical Model

FIG.(3)



(a) $\mu_c = 4.0$ (b) $\mu_c = - \frac{K^* l_c^3}{48EI}$ (c) $\mu_c = 8.184$

FIG.(4) CRITICAL BUCKLING MODES

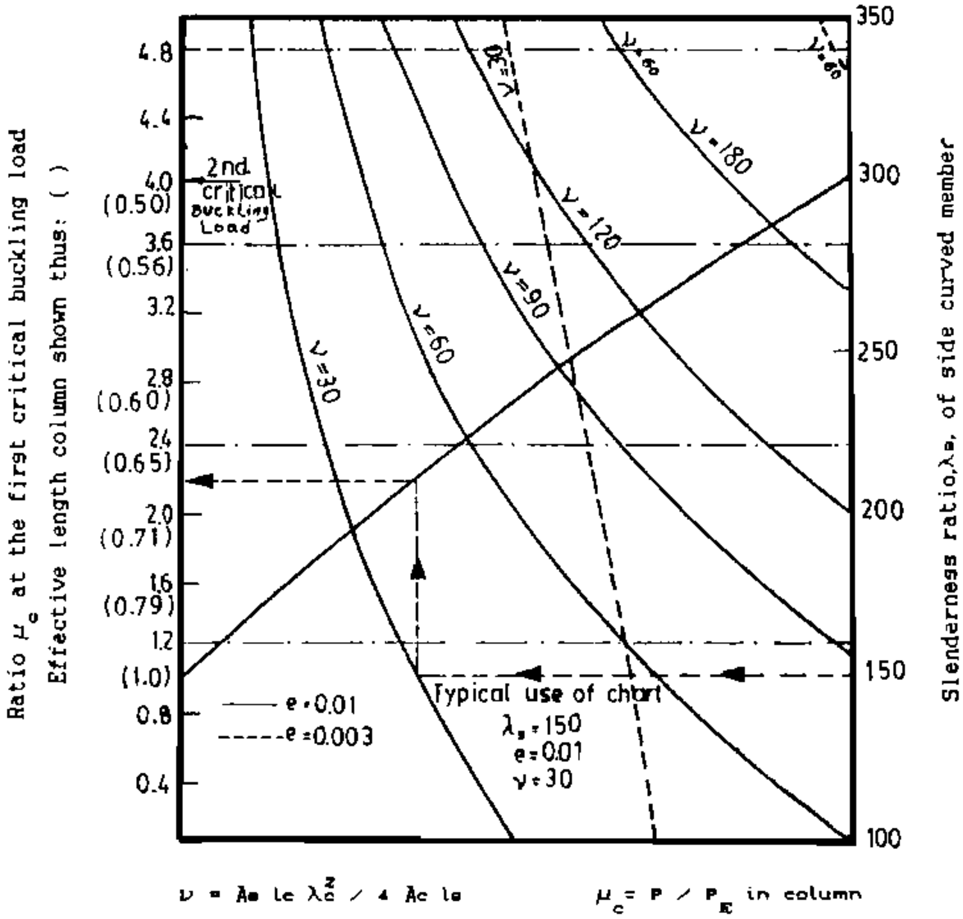


FIG.(5). TYPICAL CHART GIVING FIRST CRITICAL BUCKLING LOADS OF MAIN COLUMNS

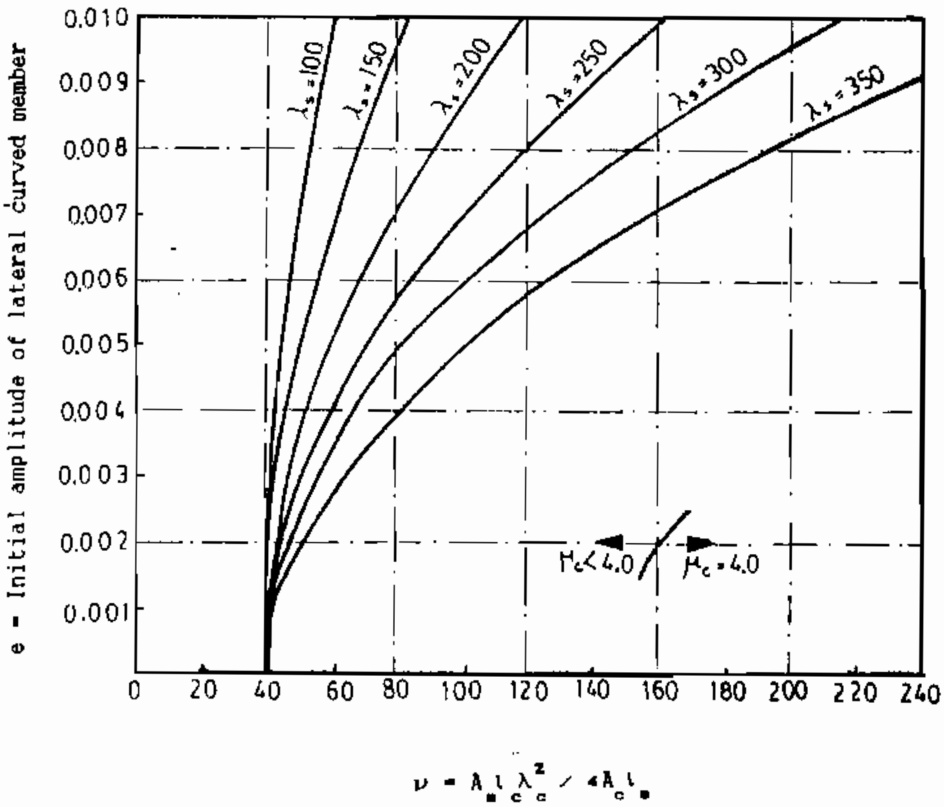

e - Initial amplitude of lateral curved member
0.010
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
20
40
60
80
100
120
140
160
180
200
220
240
λs=100
λs=150
λs=200
λs=250
λs=300
λs=350
μc<4.0
μc=4.0

Fig.(6)